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Coordination of Classical and Dynamic Inequalities Complying on Time Scales

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Abstract. In this research article, we present extensions of some classical inequalities such as Schweitzer, Pólya–Szegö, Kantorovich and Greub–Rheinboldt inequalities of fractional calculus on time scales. To investigate generalizations of such types of classical inequalities, we use the time scales Riemann–Liouville type fractional integrals. We explore dynamic inequalities on delta calculus and their symmetric nabla versions. A time scale is an arbitrary nonempty closed subset of the real numbers. The theory of time scales is applied to combine results in one comprehensive form. The calculus of time scales unifies and extends continuous versions and their discrete and quantum analogues. By using the calculus of time scales, results are presented in more general form. This hybrid theory is also widely applied on dynamic inequalities.

1. Introduction

The calculus of time scales was initiated by Stefan Hilger [11]. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The time scales calculus is studied as delta calculus, nabla calculus and diamond-α calculus. During the last two decades, many researchers investigated several dynamic inequalities [1–4, 7, 16–18]. The basic work on dynamic inequalities is done by Ravi Agarwal, George Anastassiou, Martin Bohner, Allan Peterson, Donal O'Regan, Samir Saker and many other authors.

There have been recent achievements of the theory and applications of dynamic inequalities on time scales. From the theoretical point of view, the study provides a harmonious reconciliation and extension of commonly known differential, difference and quantum equations. Moreover, it is an important tool in many computational, biological, economical and numerical applications.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

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2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adopted from monographs [7,8].

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^Δ is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \epsilon |\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t, and $f^\Delta(t)$ is called the delta derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [7,8].

Definition 2.1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [6–8].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^\nabla(t)$, if there exists $f^\nabla(t) \in \mathbb{R}$ such that given any $\epsilon > 0$, there is a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \le \epsilon |\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [6–8].

Definition 2.2. A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

The following definition is taken from [2, 4].

Definition 2.3. For $\alpha \geq 1$, the time scale Δ-Riemann–Liouville type fractional integral for a function $f \in C_{rd}$ is defined by

$$\mathcal{I}_a^{\alpha} f(t) = \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau)\Delta\tau, \tag{1}$$

which is an integral on $[a, t)_{\mathbb{T}}$, see [9] and $h_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise rd-continuous functions, such that $h_0(t, s) = 1$,

$$h_{\alpha+1}(t, s) = \int_s^t h_{\alpha}(\tau, s)\Delta\tau, \ \forall s, t \in \mathbb{T}. \tag{2}$$

Notice that

$$\mathcal{I}_a^1 f(t) = \int_a^t f(\tau)\Delta\tau,$$

which is absolutely continuous in $t \in [a, b]_{\mathbb{T}}$, see [9].

The following definition is taken from [3, 4].

Definition 2.4. For $\alpha \geq 1$, the time scale ∇-Riemann–Liouville type fractional integral for a function $f \in C_{ld}$ is defined by

$$\mathcal{J}_a^{\alpha} f(t) = \int_a^t \hat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau)\nabla\tau, \tag{3}$$

which is an integral on $(a, t]_{\mathbb{T}}$, see [9] and $\hat{h}_{\alpha} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise ld-continuous functions, such that $\hat{h}_0(t, s) = 1$,

$$\hat{h}_{\alpha+1}(t, s) = \int_s^t \hat{h}_{\alpha}(\tau, s)\nabla\tau, \ \forall s, t \in \mathbb{T}. \tag{4}$$

Notice that

$$\mathcal{J}_a^1 f(t) = \int_a^t f(\tau)\nabla\tau,$$

which is absolutely continuous in $t \in [a, b]_{\mathbb{T}}$, see [9].

We will generalize the following classical inequalities [13] by using the calculus of time scales. First we consider the inequality given by Schweitzer [19] such that

$$\left(\frac{1}{p}\sum_{k=1}^{p} x_k\right)\left(\frac{1}{p}\sum_{k=1}^{p}\frac{1}{x_k}\right) \leq \frac{(M+m)^2}{4Mm}, \tag{5}$$

where $0 < m \leq x_k \leq M$ for $k = 1, \ldots, p$.

In the same paper, Schweitzer has also shown that if functions $y \mapsto f(y)$ and $y \mapsto \frac{1}{f(y)}$ are integrable on $[a, b]$ and $0 < m \leq f(y) \leq M$ on $[a, b]$, then

$$\int_a^b f(y)dy \int_a^b \frac{1}{f(y)}dy \leq \frac{(M+m)^2}{4Mm}(b-a)^2. \tag{6}$$

Pólya and Szegö [15] proved that

$$\frac{\left(\sum\limits_{k=1}^{p} x_k^2\right)\left(\sum\limits_{k=1}^{p} y_k^2\right)}{\left(\sum\limits_{k=1}^{p} x_k y_k\right)^2} \leq \left(\frac{\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}}{2}\right)^2, \tag{7}$$

where $0 < m \leq x_k \leq M$ and $0 < n \leq y_k \leq N$ for $k = 1, \ldots, p$.

Kantorovich [12] proved that

$$\left(\sum_{k=1}^{p} x_k y_k^2\right)\left(\sum_{k=1}^{p}\frac{1}{x_k}y_k^2\right) \leq \frac{1}{4}\left(\sqrt{\frac{M}{m}}+\sqrt{\frac{m}{M}}\right)^2\left(\sum_{k=1}^{p} y_k^2\right)^2, \tag{8}$$

where $0 < m \leq x_k \leq M$ and $y_k \in \mathbb{R}$ for $k = 1, \ldots, p$, and he pointed out that inequality (8) is a particular case of (7).

Greub and Rheinboldt [10] proved that

$$\left(\sum_{k=1}^{p} x_k^2 z_k^2\right)\left(\sum_{k=1}^{p} y_k^2 z_k^2\right) \leq \frac{(MN+mn)^2}{4MNmn}\left(\sum_{k=1}^{p} x_k y_k z_k^2\right)^2, \tag{9}$$

where $0 < m \leq x_k \leq M$, $0 < n \leq y_k \leq N$ and $z_k \in \mathbb{R}$ for $k = 1, \ldots, p$ with $\sum\limits_{k=1}^{p} z_k^2 < \infty$.

3. Main Results

In order to present our main results, first we give a simple proof for an extension of Pólya–Szegö's inequality by using the time scale Δ-Riemann–Liouville type fractional integral.

Theorem 3.1. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be Δ-integrable functions. Assume that there exist four positive Δ-integrable functions f_1, f_2, g_1 and g_2 such that:*

$$0 < f_1(y) \leq |f(y)| \leq f_2(y) < \infty \textit{ and } 0 < g_1(y) \leq |g(y)| \leq g_2(y) < \infty, \ (y \in [a,x]_{\mathbb{T}}, \forall x \in [a,b]_{\mathbb{T}}).$$

Let $\alpha, \beta \geq 1$ and $h_{\alpha-1}(.,.), h_{\beta-1}(.,.) > 0$. Then we have the following inequality

$$\frac{\mathcal{I}_a^\alpha\left((f_1f_2)(x)|w(x)|\right)\mathcal{I}_a^\beta\left((g_1g_2)(x)|w(x)|\right)\mathcal{I}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{I}_a^\beta\left(|w(x)||g(x)|^2\right)}{\left\{\mathcal{I}_a^\alpha\left(f_1(x)|(wf)(x)|\right)\mathcal{I}_a^\beta\left(g_1(x)|(wg)(x)|\right)+\mathcal{I}_a^\alpha\left(f_2(x)|(wf)(x)|\right)\mathcal{I}_a^\beta\left(g_2(x)|(wg)(x)|\right)\right\}^2} \leq \frac{1}{4}. \tag{10}$$

Proof. Using the given conditions, for $y, z \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, we have

$$\left(\frac{f_2(y)}{g_1(z)} - \frac{|f(y)|}{|g(z)|}\right) \geq 0,$$

and

$$\left(\frac{|f(y)|}{|g(z)|} - \frac{f_1(y)}{g_2(z)}\right) \geq 0,$$

which imply that

$$\left(\frac{f_1(y)}{g_2(z)} + \frac{f_2(y)}{g_1(z)}\right)\frac{|f(y)|}{|g(z)|} \geq \frac{|f(y)|^2}{|g(z)|^2} + \frac{f_1(y)f_2(y)}{g_1(z)g_2(z)}.$$

Multiplying both sides by $g_1(z)g_2(z)|g(z)|^2$, we have

$$f_1(y)g_1(z)|f(y)g(z)| + f_2(y)g_2(z)|f(y)g(z)| \geq g_1(z)g_2(z)|f(y)|^2 + f_1(y)f_2(y)|g(z)|^2. \tag{11}$$

Multiplying both sides of (11) by $h_{\alpha-1}(x,\sigma(y))|w(y)|h_{\beta-1}(x,\sigma(z))|w(z)|$ and double integrating over y and z from a to x, respectively, we have

$$\begin{aligned}&\mathcal{I}_a^\alpha\left(f_1(x)|w(x)f(x)|\right)\mathcal{I}_a^\beta\left(g_1(x)|w(x)g(x)|\right)+\mathcal{I}_a^\alpha\left(f_2(x)|w(x)f(x)|\right)\mathcal{I}_a^\beta\left(g_2(x)|w(x)g(x)|\right)\\&\geq \mathcal{I}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{I}_a^\beta\left(g_1(x)g_2(x)|w(x)|\right)+\mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)|\right)\mathcal{I}_a^\beta\left(|w(x)||g(x)|^2\right).\end{aligned} \tag{12}$$

Applying the AM-GM inequality $\sqrt{\zeta\eta} \leq \frac{\zeta+\eta}{2}$, $\zeta \geq 0$, $\eta \geq 0$, the inequality (12) takes the form

$$\begin{aligned}&\mathcal{I}_a^\alpha\left(f_1(x)|w(x)f(x)|\right)\mathcal{I}_a^\beta\left(g_1(x)|w(x)g(x)|\right)+\mathcal{I}_a^\alpha\left(f_2(x)|w(x)f(x)|\right)\mathcal{I}_a^\beta\left(g_2(x)|w(x)g(x)|\right)\\&\geq 2\sqrt{\mathcal{I}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{I}_a^\beta\left(g_1(x)g_2(x)|w(x)|\right)\mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)|\right)\mathcal{I}_a^\beta\left(|w(x)||g(x)|^2\right)}.\end{aligned} \tag{13}$$

Inequality (13) directly yields inequality (10). The proof of Theorem 3.1 is completed. □

Now, we give an extension of Pólya–Szegö's inequality by using the time scale ∇-Riemann–Liouville type fractional integral.

Theorem 3.2. *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be ∇-integrable functions. Assume that there exist four positive ∇-integrable functions f_1, f_2, g_1 and g_2 such that:*

$$0 < f_1(y) \leq |f(y)| \leq f_2(y) < \infty \text{ and } 0 < g_1(y) \leq |g(y)| \leq g_2(y) < \infty, \ (y \in [a,x]_{\mathbb{T}}, \forall x \in [a,b]_{\mathbb{T}}).$$

Let $\alpha, \beta \geq 1$ and $\hat{h}_{\alpha-1}(.,.), \hat{h}_{\beta-1}(.,.) > 0$. Then we have the following inequality

$$\frac{\mathcal{J}_a^\alpha\left((f_1f_2)(x)|w(x)|\right)\mathcal{J}_a^\beta\left((g_1g_2)(x)|w(x)|\right)\mathcal{J}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{J}_a^\beta\left(|w(x)||g(x)|^2\right)}{\left\{\mathcal{J}_a^\alpha\left(f_1(x)|(wf)(x)|\right)\mathcal{J}_a^\beta\left(g_1(x)|(wg)(x)|\right)+\mathcal{J}_a^\alpha\left(f_2(x)|(wf)(x)|\right)\mathcal{J}_a^\beta\left(g_2(x)|(wg)(x)|\right)\right\}^2} \leq \frac{1}{4}. \tag{14}$$

Proof. Similar to the proof of Theorem 3.1. □

Corollary 3.3. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be Δ-integrable functions such that $0 < m \leq |f(y)| \leq M < \infty$ and $0 < n \leq |g(y)| \leq N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha, \beta \geq 1$ and $h_{\alpha-1}(.,.), h_{\beta-1}(.,.) > 0$. Then we have the following inequality*

$$\frac{\mathcal{I}_a^\alpha(|w(x)|)\,\mathcal{I}_a^\beta(|w(x)|)\,\mathcal{I}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{I}_a^\beta\left(|w(x)||g(x)|^2\right)}{\left\{\mathcal{I}_a^\alpha(|(wf)(x)|)\,\mathcal{I}_a^\beta(|(wg)(x)|)\right\}^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right)^2. \quad (15)$$

Proof. Putting $f_1 = m$, $f_2 = M$, $g_1 = n$ and $g_2 = N$ in Theorem 3.1, we get the inequality (15). □

Corollary 3.4. *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be ∇-integrable functions such that $0 < m \leq |f(y)| \leq M < \infty$ and $0 < n \leq |g(y)| \leq N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha, \beta \geq 1$ and $\hat{h}_{\alpha-1}(.,.), \hat{h}_{\beta-1}(.,.) > 0$. Then we have the following inequality*

$$\frac{\mathcal{J}_a^\alpha(|w(x)|)\,\mathcal{J}_a^\beta(|w(x)|)\,\mathcal{J}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{J}_a^\beta\left(|w(x)||g(x)|^2\right)}{\left\{\mathcal{J}_a^\alpha(|(wf)(x)|)\,\mathcal{J}_a^\beta(|(wg)(x)|)\right\}^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right)^2. \quad (16)$$

Proof. Similar to the proof of Corollary 3.3. □

Remark 3.1. Let $\mathbb{T} = \mathbb{R}$, $\alpha, \beta > 0$, $a = 0$, $x > 0$, $w \equiv 1$, $f > 0$ and $g > 0$. Then (10) reduces to

$$\frac{\mathcal{I}_0^\alpha((f_1f_2)(x))\,\mathcal{I}_0^\beta((g_1g_2)(x))\,\mathcal{I}_0^\alpha\left(f^2(x)\right)\mathcal{I}_0^\beta\left(g^2(x)\right)}{\left\{\mathcal{I}_0^\alpha((f_1f)(x))\,\mathcal{I}_0^\beta((g_1g)(x))+\mathcal{I}_0^\alpha((f_2f)(x))\,\mathcal{I}_0^\beta((g_2g)(x))\right\}^2} \leq \frac{1}{4}, \quad (17)$$

as given in [14].

Theorem 3.5. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be Δ-integrable functions. Assume that there exist four positive Δ-integrable functions f_1, f_2, g_1 and g_2 such that:*

$$0 < f_1(y) \leq |f(y)| \leq f_2(y) < \infty \text{ and } 0 < g_1(y) \leq |g(y)| \leq g_2(y) < \infty, \ (y \in [a,x]_{\mathbb{T}}, \forall x \in [a,b]_{\mathbb{T}}).$$

Let $\alpha, \beta \geq 1$ and $h_{\alpha-1}(.,.), h_{\beta-1}(.,.) > 0$. Then we have the following inequality

$$\mathcal{I}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{I}_a^\beta\left(|w(x)||g(x)|^2\right) \leq \mathcal{I}_a^\alpha\left(\frac{f_2(x)}{g_1(x)}|(wfg)(x)|\right)\mathcal{I}_a^\beta\left(\frac{g_2(x)}{f_1(x)}|(wfg)(x)|\right). \quad (18)$$

Proof. Using the given condition, for $y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, we have

$$|f(y)|^2 \leq \frac{f_2(y)}{g_1(y)}|f(y)g(y)|.$$

Multiplying both sides of the last inequality by $h_{\alpha-1}(x,\sigma(y))|w(y)|$ and integrating over y from a to x, we have

$$\int_a^x h_{\alpha-1}(x,\sigma(y))|w(y)||f(y)|^2\Delta y \leq \int_a^x h_{\alpha-1}(x,\sigma(y))\frac{f_2(y)}{g_1(y)}|w(y)||f(y)g(y)|\Delta y. \quad (19)$$

The inequality (19) takes the form

$$\mathcal{I}_a^\alpha\left(|w(x)||f(x)|^2\right) \leq \mathcal{I}_a^\alpha\left(\frac{f_2(x)}{g_1(x)}|w(x)||f(x)g(x)|\right). \tag{20}$$

Similarly, we have that

$$\mathcal{I}_a^\beta\left(|w(x)||g(x)|^2\right) \leq \mathcal{I}_a^\beta\left(\frac{g_2(x)}{f_1(x)}|w(x)||f(x)g(x)|\right). \tag{21}$$

Multiplying (20) and (21), we get the desired inequality (18). □

Theorem 3.6. *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be ∇-integrable functions. Assume that there exist four positive ∇-integrable functions f_1, f_2, g_1 and g_2 such that:*

$$0 < f_1(y) \leq |f(y)| \leq f_2(y) < \infty \text{ and } 0 < g_1(y) \leq |g(y)| \leq g_2(y) < \infty,\ (y \in [a,x]_{\mathbb{T}}, \forall x \in [a,b]_{\mathbb{T}}).$$

Let $\alpha, \beta \geq 1$ and $\hat{h}_{\alpha-1}(.,.), \hat{h}_{\beta-1}(.,.) > 0$. Then we have the following inequality

$$\mathcal{J}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{J}_a^\beta\left(|w(x)||g(x)|^2\right) \leq \mathcal{J}_a^\alpha\left(\frac{f_2(x)}{g_1(x)}|(wfg)(x)|\right)\mathcal{J}_a^\beta\left(\frac{g_2(x)}{f_1(x)}|(wfg)(x)|\right). \tag{22}$$

Proof. Similar to the proof of Theorem 3.5. □

Corollary 3.7. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be Δ-integrable functions. Assume that there exist four positive constants m, M, n and N such that $0 < m \leq |f(y)| \leq M < \infty$ and $0 < n \leq |g(y)| \leq N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha, \beta \geq 1$ and $h_{\alpha-1}(.,.), h_{\beta-1}(.,.) > 0$. Then we have the following inequality*

$$\frac{\mathcal{I}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{I}_a^\beta\left(|w(x)||g(x)|^2\right)}{\mathcal{I}_a^\alpha\left(|(wfg)(x)|\right)\mathcal{I}_a^\beta\left(|(wfg)(x)|\right)} \leq \frac{MN}{mn}. \tag{23}$$

Proof. Putting $f_1 = m$, $f_2 = M$, $g_1 = n$ and $g_2 = N$ in Theorem 3.5, we get the desired inequality. □

Corollary 3.8. *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be ∇-integrable functions. Assume that there exist four positive constants m, M, n and N such that $0 < m \leq |f(y)| \leq M < \infty$ and $0 < n \leq |g(y)| \leq N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha, \beta \geq 1$ and $\hat{h}_{\alpha-1}(.,.), \hat{h}_{\beta-1}(.,.) > 0$. Then we have the following inequality*

$$\frac{\mathcal{J}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{J}_a^\beta\left(|w(x)||g(x)|^2\right)}{\mathcal{J}_a^\alpha\left(|(wfg)(x)|\right)\mathcal{J}_a^\beta\left(|(wfg)(x)|\right)} \leq \frac{MN}{mn}. \tag{24}$$

Proof. Similar to the proof of Corollary 3.7. □

Remark 3.2. Let $\mathbb{T} = \mathbb{R}$, $\alpha > 0$, $\alpha = \beta$, $a = 0$, $x > 0$, $w \equiv 1$, $f > 0$ and $g > 0$. Then (23) reduces to

$$\frac{\mathcal{I}_0^\alpha\left(f^2(x)\right)\mathcal{I}_0^\alpha\left(g^2(x)\right)}{\left\{\mathcal{I}_0^\alpha\left(f(x)g(x)\right)\right\}^2} \leq \frac{MN}{mn}. \tag{25}$$

Inequality (25) may be found in [5].

Theorem 3.9. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be Δ-integrable functions. Assume that there exist four positive Δ-integrable functions f_1, f_2, g_1 and g_2 such that:*

$$0 < f_1(y) \le |f(y)| \le f_2(y) < \infty \text{ and } 0 < g_1(y) \le |g(y)| \le g_2(y) < \infty, \ (y \in [a,x]_{\mathbb{T}}, \forall x \in [a,b]_{\mathbb{T}}).$$

Let $\alpha \ge 1$ and $h_{\alpha-1}(.,.) > 0$. Then we have the following inequality

$$\frac{\mathcal{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)}{\{\mathcal{I}_a^\alpha\left(\left(f_1(x)g_1(x)+f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right)\}^2} \le \frac{1}{4}. \tag{26}$$

Proof. Using the given conditions, for $y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, we have

$$\left(\frac{f_2(y)}{g_1(y)} - \frac{|f(y)|}{|g(y)|}\right) \ge 0,$$

and

$$\left(\frac{|f(y)|}{|g(y)|} - \frac{f_1(y)}{g_2(y)}\right) \ge 0.$$

Multiplying the last two inequalities, we have

$$\left(\frac{f_2(y)}{g_1(y)} - \frac{|f(y)|}{|g(y)|}\right)\left(\frac{|f(y)|}{|g(y)|} - \frac{f_1(y)}{g_2(y)}\right) \ge 0,$$

which implies

$$\left(\frac{f_1(y)}{g_2(y)} + \frac{f_2(y)}{g_1(y)}\right)\frac{|f(y)|}{|g(y)|} \ge \frac{|f(y)|^2}{|g(y)|^2} + \frac{f_1(y)f_2(y)}{g_1(y)g_2(y)}.$$

Multiplying both sides by $g_1(y)g_2(y)|g(y)|^2$, we have

$$f_1(y)g_1(y)|f(y)g(y)| + f_2(y)g_2(y)|f(y)g(y)| \ge g_1(y)g_2(y)|f(y)|^2 + f_1(y)f_2(y)|g(y)|^2. \tag{27}$$

Multiplying both sides of (27) by $h_{\alpha-1}(x,\sigma(y))|w(y)|$ and integrating over y from a to x, we have

$$\begin{aligned}&\mathcal{I}_a^\alpha\left(\left(f_1(x)g_1(x)+f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right)\\ &\qquad \ge \mathcal{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right) + \mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right).\end{aligned} \tag{28}$$

Applying the AM-GM inequality, we get

$$\begin{aligned}&\mathcal{I}_a^\alpha\left(\left(f_1(x)g_1(x)+f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right)\\ &\qquad \ge 2\sqrt{\mathcal{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)}.\end{aligned} \tag{29}$$

Analogously, we have that

$$\begin{aligned}&\mathcal{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)\\ &\qquad \le \frac{1}{4}\{\mathcal{I}_a^\alpha\left(\left(f_1(x)g_1(x)+f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right)\}^2.\end{aligned} \tag{30}$$

This directly yields the desired inequality (26). $\square$

Theorem 3.10. *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be ∇-integrable functions. Assume that there exist four positive ∇-integrable functions f_1, f_2, g_1 and g_2 such that:*

$$0 < f_1(y) \le |f(y)| \le f_2(y) < \infty \text{ and } 0 < g_1(y) \le |g(y)| \le g_2(y) < \infty, \ (y \in [a,x]_{\mathbb{T}}, \forall x \in [a,b]_{\mathbb{T}}).$$

Let $\alpha \ge 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$. Then we have the following inequality

$$\frac{\mathcal{J}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\mathcal{J}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)}{\left\{\mathcal{J}_a^\alpha\left(\left(f_1(x)g_1(x)+f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right)\right\}^2} \le \frac{1}{4}. \tag{31}$$

Proof. Similar to the proof of Theorem 3.9. □

Remark 3.3. Let $\mathbb{T}=\mathbb{R}$, $\alpha > 0$, $a = 0$, $x > 0$, $w \equiv 1$, $f > 0$ and $g > 0$. Then (26) reduces to

$$\frac{\mathcal{I}_0^\alpha\left(g_1(x)g_2(x)f^2(x)\right)\mathcal{I}_0^\alpha\left(f_1(x)f_2(x)g^2(x)\right)}{\left\{\mathcal{I}_0^\alpha\left(\left(f_1(x)g_1(x)+f_2(x)g_2(x)\right)f(x)g(x)\right)\right\}^2} \le \frac{1}{4}. \tag{32}$$

Inequality (32) may be found in [14].

Corollary 3.11. *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be Δ-integrable functions. Assume that there exist four positive constants m, M, n and N such that $0 < m \le |f(y)| \le M < \infty$ and $0 < n \le |g(y)| \le N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha \ge 1$ and $h_{\alpha-1}(.,.) > 0$. Then we have the following inequality*

$$\frac{\mathcal{I}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{I}_a^\alpha\left(|w(x)||g(x)|^2\right)}{\left\{\mathcal{I}_a^\alpha\left(|w(x)||f(x)g(x)|\right)\right\}^2} \le \frac{1}{4}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right)^2. \tag{33}$$

Proof. Putting $f_1 = m$, $f_2 = M$, $g_1 = n$ and $g_2 = N$ in Theorem 3.9, we get the desired inequality (33). □

Corollary 3.12. *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be ∇-integrable functions. Assume that there exist four positive constants m, M, n and N such that $0 < m \le |f(y)| \le M < \infty$ and $0 < n \le |g(y)| \le N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha \ge 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$. Then we have the following inequality*

$$\frac{\mathcal{J}_a^\alpha\left(|w(x)||f(x)|^2\right)\mathcal{J}_a^\alpha\left(|w(x)||g(x)|^2\right)}{\left\{\mathcal{J}_a^\alpha\left(|w(x)||f(x)g(x)|\right)\right\}^2} \le \frac{1}{4}\left(\sqrt{\frac{MN}{mn}}+\sqrt{\frac{mn}{MN}}\right)^2. \tag{34}$$

Proof. Similar to the proof of Corollary 3.11. □

Remark 3.4. We have the following:

(i) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $x_k > 0$, $w(k) = w_k = \frac{1}{x_k}$, $f(k) = x_k$ for $k = 1, \ldots, p$ and $n = g = N = 1$. Then inequality (33) reduces to inequality (5).

(ii) Let $\alpha = 1$, $\mathbb{T} = \mathbb{R}$, $x = b$, $0 < m \le f(y) \le M < \infty$, $w(y) = \frac{1}{f(y)}$ on $[a,b]$ and $n = g = N = 1$. Then inequality (33) reduces to inequality (6).

(iii) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $w \equiv 1$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$ for $k = 1, \ldots, p$. Then inequality (33) reduces to inequality (7).

(iv) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $x_k > 0$, $y_k \in \mathbb{R}$, $w(k) = w_k = \frac{1}{x_k} y_k^2$, $f(k) = x_k$ for $k = 1, \dots, p$ and $n = g = N = 1$. Then inequality (33) reduces to inequality (8).

(v) Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $z_k \in \mathbb{R}$, $w(k) = w_k = z_k^2$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$ for $k = 1, \dots, p$. Then inequality (33) reduces to inequality (9).

4. Conclusion

The subject of dynamic inequalities on time scales has become a crucial field of pure and applied mathematics. Many researchers developed interesting results concerning fractional calculus on time scales. Due to utility of dynamic inequalities in many branches of mathematics, this field is given a prominent importance. This field has a wide scope. Recently, interesting results have obtained by using Specht's ratio and Kantorovich's ratio on time scales as given in [18]. By using these ratios, we can explore further results.

Dynamic inequalities may be extended by applying other techniques such as diamond-α integral, which is defined as a linear operator of delta and nabla integrals on time scales. Quantum calculus, α, β-symmetric quantum calculus, functional generalization, fractional derivatives and n-tuple diamond-alpha integral are some other developed techniques and we will continue them to investigate other dynamic inequalities in future research.

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