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Numerical Stabilities of Vasicek and Geometric Brownian Motion Models

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Abstract. Stochastic differential equations (SDEs) are very often used as models for a large number of phenomena in the physical, economic and management sciences. They generalize the notion of ordinary differential equations, taking into account a white additive and multiplicative noise term, to model random trajectories such as stock market prices or particles movements, on the quantum scale, subject to diffusion phenomena. In rare cases, it is generally impossible to have explicit solution to these equations. In this case, the numerical approach, presenting itself under various aspects, is the only favorable outcome. However, the stability of numerical schemes for stochastic differential equations solution is much more significant. In this paper, we establish and make a classical proof of the mean and mean-square stabilities of the numerical SDEs schemes for Vasicek and Geometric Brownian motion models.

1. Introduction

Stochastic differential equations (SDEs) can be seen as ordinary differential equations, or as integral equations in which integrals occur with respect to Brownian motion. They were presented by Ito, with the aim of building continuous and strongly Markovian processes whose generators are second-order differential operators called diffusions [6]. In general, solving explicitly stochastic differential equations (SDEs), except for cases where the diffusion and drift coefficients are linears, seems difficult or impossible [8]. This is why the numerical approach is relevant because there are numerical methods allowing to predict the qualitative behavior such as the stability of the solutions.

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The choice of a suitable numerical scheme is based on the understanding and manipulation of certain qualitative properties as stability, consistency etc. The qualitative property like stability of stochastic differential equations solutions, introduced by I.Kats and N.Krasovskii [2] and perfected by I.I. Gikhman, A.V. Skorokhold [3] and A. Friedman [4] plays a major role in the study of SDEs and the numerical schemes associated. Thus, looking for numerical schemes that preserve qualitative properties as the stability of solutions constitutes and remains a very widespread problem in numerical analysis of SDEs.

In this article we establish and prove the conditions of numerical schemes stabilities in Mean and Mean-square. We apply the approach described by Y.Saito [5] to defined and demonstrate the stabilities of numericals SDEs schemes as: Euler-Maruyama, Milshtein and Implicit Euler-Maruyama for Vasicek and geometric Brownian motion models. To begin, let present some elementary notions relative to SDEs and the numerical schemes adapted to the SDEs.

2. Preliminary notions

2.1. **Stochastic differential equation and stabilities.** In this section,we present some definitions in connection with stochastic differential equation and stabilities of solutions of SDEs.

Definition 2.1. *(**Stochastic differential equation (SDE)** [13]) Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space, $(B_t)_{t\geq 0}$ a standard Brownian motion on $\mathbb{R}^d$ defines in a filtered probability space. A stochastic differential equation (SDE) on $\mathbb{R}^d$ with the drift coefficient:*

$$b\left(t, X_t\right) \in [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and the diffusion:

$$\sigma\left(t, X_t\right) \in [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$$

when X_o is random variable independent of $(B_t)_{t\geq 0}$ is an equation of the form:

$$\begin{cases} dX_t = & b\left(t, X_t\right)\ dt + \sigma\left(t, X_t\right)\ dB_t \\ X\left(o\right) = & X_o \end{cases} \tag{2.1}$$

The white noise $\sigma\left(t, X_t\right)$ can be additive or multiplicative, depending on whether it does not influence or does influence the state of the system.

Theorem 2.1. *(**Existence and uniqueness** [14]) We assume that there is a positive constant K such that $\forall \quad t \geq 0,\ X, Y \in \mathbb{R}^d$*

(1) ***Lipschitz condition:***

$$|b\left(t, X\right) - b\left(t, Y\right)| + |\sigma\left(t, X\right) - \sigma\left(t, Y\right)| \leq K|X - Y|$$

(2) ***Linear growth condition***:

$$|b(t,X)| \leq K(1+|X|),$$

$$|\sigma(t,X)| \leq K(1+|X|)$$

So the SDE (2.1) *admits, for any initial condition* X_o *of square integrable* $\left(E\left[|X_o|^2\right] < \infty\right)$ *the strong solution* $(X_t)_{t\in[0,T]}$*,unique, almost surely continuous and satisfying the following condition:*

$$E\left(\underset{0\leq t\leq T}{Sup}|X_t^2|\right) < \infty$$

Definition 2.2. *(**Asymptotic stability in probability in large sense** [1], [24]) The solution is said to be asymptotically and stochastically stable in the large sense if*

$$\forall \quad X_o \in \mathcal{L}^2_{\mathcal{F}_t}([-T,0],\mathbb{R}^n),$$

then

$$\mathbb{P}\left\{\lim_{t\longrightarrow\infty} X(t) = 0\right\} = 1.$$

Definition 2.3. *(**Stability of** p^{th} **moment** [23], [25])*

(1) *Let* $p \geq 2$ *we say that a solution of* (2.1) *is stable in* p^{th} *moment if* $\forall\epsilon > 0$ *it exists* $\delta > 0$ *such as*

$$E\left[\underset{t>0}{Sup}|X(t)|^p\right] < \epsilon \text{ avec } |X_o| < \delta$$

(2) *Let* $p \geq 2$*, we say that a solution of* (2.1) *is stable asymptoticaly in* p^{th} *moment if it is stable from* p^{eme} *moment*

$$\forall \quad X_o \in \mathcal{L}^2_{\mathcal{F}_{t_o}}([-T,0],\mathbb{R}^n)$$

then we have :

$$\lim_{T\longrightarrow\infty} E\left[\underset{t>T}{Sup}|X(t)|^p\right] = 0$$

2.2. **Stochastic numerical schemes.** In this section we present three numerical schemes as Euler-Maruyama, Implicit Euler-Maruyama and Milshtein schemes.

Definition 2.4. *(**Euler-Maruyama scheme** [10] , [11]) Let* $\{X_t\}$ *the diffusion solution of the SDE* (2.1)*. Let consider the interval* $[0,T]$ *and a regular subdivision*

$$t_0 = 0 < t_1 < t_2 < t_0 < \cdots < t_k = T$$

with step $\Delta t = \frac{T}{N} = \frac{T}{k}$*, the Euler-Maruyama scheme of* (2.1) *is defined like:*

$$\begin{cases} X_{k+1}^{EM} = X_k + b(t_k,X_k)(t_{k+1}-t_k) + \sigma(t_k,X_k)(B_{k+1}-B_k) \\ X(0) = X_0 \end{cases} \tag{2.2}$$

Definition 2.5. *(**Implicit Euler-Maruyama scheme** [10]) The implicit Euler-Maruyama scheme is a convergent scheme like the Euler-Maruyama scheme. To be reassured of the existence of the solutions of this scheme, only the term of the drift is implicit.*

For this fact: $b(X_k)\Delta t_k$ which is in the Euler-Maruyama scheme is replaced by $b(X_{k+1})\Delta t_k$ and the diffusion term: $\sigma(X_k)\Delta B_k$ remains unchanged.

The implicit Euler-Maruyama scheme of the EDS (2.1) has given by:

$$X_{k+1}^{IEM} = X_k + b(X_{k+1})\Delta t_k + \sigma(X_k)\Delta B_k \tag{2.3}$$

Definition 2.6. *(**Milshtein scheme** [7]) Let consider the SDE (2.1) and a regular subdivision of the intervalle et une subdivision of the interval $[0, T]$:*

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T \quad de \quad [0, T]$$

The Milshtein scheme is defined like:

$$\begin{cases} X_{k+1}^{M} = X_k + b(X_k)\Delta t_k + \sigma(X_k)\Delta B_k + \dfrac{1}{2}\sigma(X_k)\sigma'(X_k)(\Delta B_k - \Delta t_k) \\ X(0) = X_0 \end{cases} \tag{2.4}$$

Remark 2.1. *It should be noted that the Euler-Maruyama scheme converges strongly up to the order $\frac{1}{2}$ while that of Milshtein converges up to the order* 1.

3. Numerical stabilities of Vasicek model

3.1. **Explicit solution.** The Vasicek model (1977) is one of the first stochastic interest rate models. It is a Gaussian process generalizing the Ornstein-Unlenbeck model and explains the observed empirical mean reversion effect on interest rate curves [15], This model looks like:

$$\begin{cases} dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 dB_t \\ X(0) = X_0 \end{cases} \qquad \forall \theta_1, \theta_2 \text{ et } \theta_3 > 0 \tag{3.1}$$

With X_t: the instant interest rate; θ_2: mean reversion rate; θ_1: the long-term average and θ_3: the volatility.

The analytical solution of (3.1) model is:

$$X_t = \frac{\theta_1}{\theta_2} + \left(X_0 - \frac{\theta_1}{\theta_2}\right) e^{-\theta_2 t} + \theta_3 \int_0^{+\infty} e^{-\theta_2(t-u)} dB_u \tag{3.2}$$

The model (3.1) is equivalent to the model:

$$\begin{cases} dX_t = \theta(\mu - X_t)dt + \sigma dB_t \\ X(0) = X_0 \end{cases} \tag{3.3}$$

The solution of (3.3) has given by:

$$X_t = \mu + (X_0 - \mu)\, e^{-\theta t} + \theta \int_0^t e^{-\theta_2 (t-u)} dB_u \tag{3.4}$$

Considering the solution of (3.2), the mean and the mean-square give respectively:

$$E[X_t] = \frac{\theta_1}{\theta_2} \quad \forall \quad \theta_2 > 0$$

and

$$V(X_t) = \frac{\theta_3^2}{2\theta_2} \quad \forall \quad \theta_2 > 0$$

Which means that the stochastic process $X_t \simeq \mathcal{N}\left(\frac{\theta_1}{\theta_2}, \frac{\theta_3^2}{2\theta_2}\right)$

By using some properties of Brownian motion, the solution of the model (3.2) can be written as follows:

$$X_t = \frac{\theta_1}{\theta_2} + \frac{\theta_3 e^{-2\theta_2 t}}{\sqrt{2\theta_2}} B(e^{2\theta_2 t}) \tag{3.5}$$

Now, we present some numerical stabilities conditions for the system (3.1) of some numerical schemes (Euler-Maruyama, Implicit Euler-Maruyama and Milshtein) and the proofs of these based on the approach described in [5].

3.2. **Euler-Maruyama scheme stabilities.** The Euler-Maruyama scheme associated to the system (3.1) is :

$$X_{k+1}^{EM} = X_k + (\theta_1 - \theta_2 X_k)\,\Delta t + \theta_3 \Delta B_k$$

$$X_{k+1}^{EM} = \theta_1 \Delta t + (1 - \theta_2 \Delta t) X_k + \theta_3 \sqrt{\Delta t} Z_k \tag{3.6}$$

3.2.1. *Mean stability of Euler-Maruyama scheme.*

Theorem 3.1. ***(Mean stability of Euler-Maruyama scheme)*** *The Euler-Maruyama scheme (3.6) of the Vasicek model (3.1) is Mean asymptotically stable if:*

$$E\left[X_{k+1}^{EM}\right] = (1 - \theta_2 \Delta t)^{k+1} E[X_0] + \theta_1 \Delta t \left[\sum_{i=0}^{k+1} (1 - \theta_2 \Delta t)^i\right] \tag{3.7}$$

with $|1 - \theta_2 \Delta t| < 1$ *and*

$$\lim_{\Delta t \to 0} \left(\lim_{k \to +\infty} E\left[X_{k+1}^{EM}\right] \right) = \frac{\theta_1}{\theta_2}$$

Proof. To prove the theorem, we start by evaluating the mean of the (3.1) equation using the approach defined in [5]. In effect,

$$\begin{aligned}
E\left[X_{k+1}^{EM}\right] &= E\left[\theta_1\Delta t + X_k\left(1-\theta_2\Delta t\right) + \theta_3\sqrt{\Delta t}Z_k\right] \\
&= E\left[\theta_1\Delta t\right] + E\left[X_k\left(1-\theta_2\Delta t\right)\right] + E\left[\theta_3\sqrt{\Delta t}Z_k\right] \\
&= E\left[\theta_1\Delta t\right] + \left(1-\theta_2\Delta t\right)E\left[X_k\right] + 0] \quad \text{with} \quad Z_k \simeq \mathcal{N}(0,1) \\
&= \theta_1\Delta t + \left(1-\theta_2\Delta t\right)E[X_k] \\
&= \theta_1\Delta t + \left(1-\theta_2\Delta t\right)\left\{\left(1-\theta_2\Delta t\right)E[X_{k-1}] + \theta_1\Delta t\right\} \\
&= \theta_1\Delta t + \theta_1\Delta t(1-\theta_2\Delta t) + (1-\theta_2\Delta t)^2E[X_{k-1}] \\
&= \theta_1\Delta t(1+(1-\theta_2\Delta t)) + (1-\theta_2\Delta t)^2E[X_{k-1}] \\
&= \theta_1\Delta t(1+(1-\theta_2\Delta t)) + (1-\theta_2\Delta t)^2\left\{\left(1-\theta_2\Delta t\right)E[X_{k-2}] + \theta_1\Delta t\right\} \\
&= \theta_1\Delta t(1+(1-\theta_2\Delta t)) + \theta_1\Delta(1-\theta_2\Delta t)^2 + (1-\theta_2\Delta t)^3E[X_{k-2}] \\
&= \theta_1\Delta t(1+(1-\theta_2\Delta t) + (1-\theta_2\Delta t)^2) + (1-\theta_2\Delta t)^3E[X_{k-2}] \\
&= \theta_1\Delta t(1+(1-\theta_2\Delta t) + (1-\theta_2\Delta t)^2 + \cdots + (1-\theta_2\Delta t))^{k+1} + (1-\theta_2\Delta t)^{k+1}E[X_0] \\
&= (1-\theta_2\Delta t)^{k+1}E[X_0] + \theta_1\Delta t\left[\sum_{i=0}^{k+1}(1-\theta_2\Delta t)^i\right]
\end{aligned}$$

Using the theory of geometric sequences and series, we get:

$$E\left[X_{k+1}^{EM}\right] = (1-\theta_2\Delta t)^{k+1}E[X_0] + \theta_1\Delta t\left(\frac{(1-(1-\theta_2\Delta t)^{k+1})}{1-(1-\theta_2\Delta t)}\right) \tag{3.8}$$

As the identity (3.8) represents a geometric sequence, we have that it converges if

$$|1-\theta_2\Delta t| < 1$$

By calculating the limit of the (3.8), for $\Delta t \to 0$ and $k \to +\infty$, we get:

$$\lim_{\Delta t\to 0}\left(\lim_{k\to+\infty} E\left[X_{k+1}^{EM}\right]\right) = \frac{\theta_1}{\theta_2}$$

□

3.2.2. *Mean-square stability of Euler-Maruyama scheme.*

Theorem 3.2. *(**Mean-square stability of Euler-Maruyama scheme**) The Euler-Maruyama scheme (3.6) of the Vasicek model (3.1) is mean-square asymptotically stable if:*

$$E\left[\left|X_{k+1}^{EM}\right|^2\right] = (1-\theta_2\Delta t)^{2(k+1)}E\left(|X_0|^2\right) + \left(\theta_3^2 + \theta_1^2\Delta t\right)\Delta t\sum_{i=0}^{k+1}(1-\theta_2\Delta t)^{2i}$$

and that the following two conditions are satisfied simultaneously:

$$(1)\; |1-\theta_2\Delta t| < 1$$

$$(2)\; \lim_{\Delta t\to 0}\left(\lim_{k\to+\infty} E\left[X_{k+1}^{EM}\right]^2\right) = \frac{\theta_3^2}{2\theta_2}$$

Proof. As in the previous theorem, we start by calculating the expression:

$$E\left[\left|X_{k+1}^{EM}\right|^2\right]$$

of Vasicek model of the equation (3.1). In effect,

$$\begin{aligned}
E\left[\left|X_{k+1}^{EM}\right|^2\right] &= |\theta_1\Delta t|^2 + E\left(|X_k(1-\theta_2\Delta t)|\right)^2 + \theta_3^2\Delta t \quad Z_k \simeq \mathbb{N}(0,1)\\
&= (1-\theta_2\Delta t)^2 E\left(|X_k|^2\right) + (\theta_1^2\Delta t + \theta_3^2)\Delta t\\
&= (1-\theta_2\Delta t)^2 E\left(|X_k|^2\right)\left\{E\left(|X_{k+1}|^2\right)(1-\theta_2\Delta t)^2 + (\theta_3^2+\theta_1^2\Delta t)\Delta t\right\} + (\theta_3^2+\theta_1^2\Delta t)\Delta t\\
&= (1-\theta_2\Delta t)^4 E\left(|X_{k-1}|^2\right) + (\theta_3^2+\theta_1^2\Delta t)\Delta t\left[(1-\theta_2\Delta t)^2+1\right]\\
&= (1-\theta_2\Delta t)^6 E\left(|X_{k-2}|^2\right) + (\theta_3^2+\theta_1^2\Delta t)\Delta t[(1-\theta_2\Delta t)^4+(1-\theta_2\Delta t)^2+1]\\
&= (1-\theta_2\Delta t)^8 E\left(|X_{k-3}|^2\right) + (\theta_3^2+\theta_1^2\Delta t)\Delta t\left[(1-\theta_2\Delta t)^6+(1-\theta_2\Delta t)^4+(1-\theta_2\Delta t)^2+1\right]\\
&= (1-\theta_2\Delta t)^{2k+1} E\left(|X_0|^2\right) + \left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t\left[(1-\theta_2\Delta t)^{2k}+\ldots+(1-\theta_2\Delta t)^4\right.\\
&\quad \left.+(1-\theta_2\Delta t)^2+(1-\theta_2\Delta t)^0\right]\\
&= (1-\theta_2\Delta t)^{2(k+1)} E\left(|X_0|^2\right) + \left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t\sum_{i=0}^{k+1}(1-\theta_2\Delta t)^{2i}\\
&= (1-\theta_2\Delta t)^{2k+2} E\left(|X_0|^2\right) + \left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t\left[\frac{1-|1-\theta_2\Delta t|^{2k+2}}{1-|1-\theta_2\Delta t|^2}\right]
\end{aligned}$$

We get:

$$E\left[\left|X_{k+1}^{EM}\right|^2\right] = (1-\theta_2\Delta t)^{2k+2} E\left(|X_0|^2\right) + \left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t\left[\frac{1}{1-|1-\theta_2\Delta t|^2}\right] \tag{3.9}$$

The expression (3.9) as the geometric sequence, we have that it converges when

$$|1-\theta_2\Delta t| < 1$$

Passing to the limit of the equation (3.9), for $\Delta t \to 0$ and $k \to +\infty$, we find the desired result i.e:

$$\lim_{\Delta t\to 0}\left(\lim_{k\to+\infty} E\left[X_{k+1}^{EM}\right]\right) = \frac{\theta_3^2}{2\theta_2}$$

□

3.3. **Milshtein's scheme stabilities.** The Milshtein scheme associated to the system (3.1) is:

$$X_{k+1}^{M} = \theta_1 \Delta t + (1 - \theta_2 \Delta t) X_k + \theta_3 \sqrt{\Delta t} Z_k \tag{3.10}$$

with

$$\sigma = \theta_3 \quad \sigma' = 0$$

Then mean and mean-square stabilities gives the same results as in the Euler-Maruyama scheme i.e

Theorem 3.3. *(**Mean stability of Milshtein scheme**) The Milshtein scheme (3.10) of Vasicek model (3.1) is Mean asymptotically stable if:*

$$E\left[X_{k+1}^{M}\right] = \theta_1 \Delta t \left[\sum_{i=0}^{k+1} (1 - \theta_2 \Delta t)^i\right] + (1 - \theta_2 \Delta t)^{k+1} E[X_0]$$

and that the following two conditions are satisfied simultaneously:

$$(1)\ |1 - \theta_2 \Delta t| < 1$$

$$(2)\ \lim_{\Delta t \to 0} \left(\lim_{k \to +\infty} E\left[X_{k+1}^{M}\right] \right) = \frac{\theta_1}{\theta_2}$$

Theorem 3.4. *(**Mean-square stability of Milshtein scheme**)*
The Milshtein scheme (3.10) of Vasicek model (3.1) is mean-square asymptotically stable if:

$$E\left[\left|X_{k+1}^{M}\right|^2\right] = \left(\theta_3^2 + \theta_1^2 \Delta t\right) \Delta t \sum_{i=0}^{k+1} (1 - \theta_2 \Delta t)^{2i} + (1 - \theta_2 \Delta t)^{2(k+1)} E\left(|X_0|^2\right)$$

and that the following two conditions are satisfied simultaneously:

$$(1)\ |1 - \theta_2 \Delta t| < 1$$

$$(2)\ \lim_{\Delta t \to 0} \left(\lim_{k \to +\infty} E\left[X_{k+1}^{M}\right]^2 \right) = \frac{\theta_3^2}{2\theta_2}$$

Proof. The proofs of these theorems above is done in the same way as the result theorems of the Euler-Maruyama scheme for Vasicek model. □

3.4. **Implicit Euler-Maruyama scheme stabilities.** The implicit Euler-Maruyama scheme associated to the system (3.1) is :

$$X_{k+1}^{IEM} = X_k + (\theta_1 - \theta_2 X_{k+1}) \Delta t + \theta_3 \sqrt{\Delta t} Z_k$$

$$X_{k+1} + \theta_2 \Delta t X_{k+1} = X_k + \theta_1 \Delta t + \theta_3 \sqrt{\Delta t} Z_k$$

$$X_{k+1} (1 + \theta_2 \Delta t) = \theta_1 \Delta t + X_k + \theta_3 \sqrt{\Delta t} Z_k$$

$$X_{k+1} = \frac{\theta_1 \Delta t}{1 + \theta_2 \Delta t} + \frac{1}{1 + \theta_2 \Delta t} X_k + \frac{\theta_3 \sqrt{\Delta t}}{1 + \theta_2 \Delta t} Z_k$$

We get:

$$X_{k+1}^{IEM} = \frac{\theta_1 \Delta t}{1+\theta_2 \Delta t} + \frac{1}{1+\theta_2 \Delta t} X_k + \frac{\theta_3 \sqrt{\Delta t}}{1+\theta_2 \Delta t} Z_k \tag{3.11}$$

3.4.1. *Mean stability of implicit Euler-Maruyama scheme.*

Theorem 3.5. *(**Mean stability of implicit Euler-Maruyama scheme**) The implicit Euler-Maruyama scheme (3.11) of Vasicek model (3.1) is Mean asymptotically stable if:*

$$E\left(X_{k+1}^{IEM}\right) = \left(\frac{1}{1+\theta_2 \Delta t}\right)^{k+1} E\left(X_0\right) + \theta_1 \Delta t \sum_{i=0}^{k+1} \left(\frac{1}{1+\theta_2 \Delta t}\right)^{i}$$

and that the following two conditions are satisfied simultaneously:

$$(1)\ |1+\theta_2 \Delta t| > 1$$

$$(2)\ \lim_{\Delta t \to 0} \left(\lim_{k \to \infty} E\left[X_{k+1}^{IEM}\right] \right) = \frac{\theta_1}{\theta_2}$$

Proof. We start by evaluating the mean of the Implicit Euler-Maruyama scheme of the expression defined in (3.11), In effect:

$$\begin{aligned}
E\left(X_{k+1}^{IEM}\right) &= E\left(\frac{\theta_1 \Delta t}{1+\theta_2 \Delta t}\right) + E\left(\frac{1}{1+\theta_2 \Delta t} X_k\right) + E\left(\frac{\theta_3 \sqrt{\Delta t}}{1+\theta_2 \Delta t} Z_k\right) \\
&= \frac{\theta_1 \Delta t}{1+\theta_2 \Delta t} + \frac{1}{1+\theta_2 \Delta t} E\left(X_k\right) \\
&= \frac{\theta_1 \Delta t}{1+\theta_2 \Delta t} + \frac{1}{1+\theta_2 \Delta t} \left\{ \left(\frac{1}{1+\theta_2 \Delta t}\right) E\left(X_{k-1}\right) + \frac{\theta_1 \Delta t}{1+\theta_2 \Delta t} \right\} \\
&= \frac{\theta_1 \Delta t}{1+\theta_2 \Delta t} + \frac{\theta_1 \Delta t}{(1+\theta_2 \Delta t)^2} + \frac{1}{(1+\theta_2 \Delta t)^2} E\left(X_{k-1}\right) \\
&= \left(\frac{1}{1+\theta_2 \Delta t}\right)^2 E\left(X_{k-1}\right) + \theta_1 \Delta t \left(\frac{1}{(1+\theta_2 \Delta t)^2} + \frac{1}{(1+\theta_2 \Delta t)}\right) \\
&= \left(\frac{1}{1+\theta_2 \Delta t}\right)^3 E\left(X_{k-2}\right) + \theta_1 \Delta t \left(\left(\frac{1}{1+\theta_2 \Delta t}\right)^3 + \left(\frac{1}{1+\theta_2 \Delta t}\right)^2 + \left(\frac{1}{1+\theta_2 \Delta t}\right) \right) \\
&= \left(\frac{1}{1+\theta_2 \Delta t}\right)^4 E\left(X_{k-3}\right) + \theta_1 \Delta t \left(\left(\frac{1}{1+\theta_2 \Delta t}\right)^4 + \left(\frac{1}{1+\theta_2 \Delta t}\right)^3 + \cdots + 1 \right)
\end{aligned}$$

By continuing the iterations until $k+1$, we obtain:

$$E\left(X_{k+1}^{IEM}\right) = \left(\frac{1}{1+\theta_2 \Delta t}\right)^{k+1} E\left(X_0\right) + \theta_1 \Delta t \sum_{i=0}^{k+1} \left(\frac{1}{1+\theta_2 \Delta t}\right)^{i} \tag{3.12}$$

The equation (3.12) is the geometric sum of geometric sequence and geometric series, the expression:

$$\left(\frac{1}{1+\theta_2 \Delta t}\right) < 1$$

or

$$|1+\theta_2\Delta t|>1$$

By using limit of (3.12), for $\Delta t\to 0$ and $k\to+\infty$, we get:

$$\lim_{\Delta t\to 0}\left(\lim_{k\to+\infty}E\left[X_{k+1}^{IEM}\right]\right)=\frac{\theta_1}{\theta_2}$$

□

3.4.2. *Mean-square stability of Implicit Euler-Maruyama scheme.*

Theorem 3.6. ***(Mean-square stability of Implicit Euler-Maruyama scheme)*** *The Implicit Euler-Maruyama of Vasicek model (3.1) is mean-square asymptotically stable if:*

$$E\left(\left|X_{k+1}^{IEM}\right|^2\right)=\left(\frac{1}{1+\theta_2\Delta t}\right)^{2(k+1)}E\left(|X_0|\right)^2+\left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t\sum_{i=0}^{k+1}\left(\frac{1}{1+\theta_2\Delta t}\right)^{2i}$$

, with $|1+\theta_2\Delta t|>1$ *and*

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty}E\left(|X_{k+1}|^2\right)\right)=\frac{\theta_3^2}{2\theta_2}$$

Proof. Let us evaluate the mean-square of the implicit Euler-Maruyama scheme (3.11), in effect:

$$\begin{aligned}
E\left(\left|X_{k+1}^{IEM}\right|^2\right)&=E\left(\left|\frac{\theta_1\Delta t}{1+\theta_2\Delta t}\right|^2\right)+E\left(\left|\frac{1}{1+\theta_2\Delta t}X_k\right|\right)+\left|\frac{\theta_3^2\Delta t}{1+\theta_2\Delta t}\right|^2\\
&=\frac{\theta_1^2(\Delta t)^2}{(1+\theta_2\Delta t)^2}+\frac{1}{(1+\theta_2\Delta t)^2}E\left(|X_k|^2\right)+\frac{\theta_3^2\Delta t}{(1+\theta_2\Delta t)^2}\\
&=\frac{\theta_1^2(\Delta t)^2+\theta_3^2\Delta t}{(1+\theta_2\Delta t)^2}+\frac{1}{(1+\theta_2\Delta t)^2}E\left(|X_k|^2\right)\\
&=\frac{\left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t}{(1+\theta_2\Delta t)^2}+\left(\frac{1}{1+\theta_2\Delta t}\right)^2\left\{\left(\frac{1}{1+\theta_2\Delta t}\right)^2E\left(|X_{k-1}|\right)^2+\frac{\left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t}{(1+\theta_2\Delta t)^2}\right\}\\
&=\left(\frac{1}{1+\theta_2\Delta t}\right)^4E\left(|X_{k-1}|\right)^2+\left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t\left[\left(\frac{1}{1+\theta_2\Delta t}\right)^4+\left(\frac{1}{1+\theta_2\Delta t}\right)^2\right]\\
&\ \ \vdots\\
&=\left(\frac{1}{1+\theta_2\Delta t}\right)^{2(k+1)}E\left(|X_0|\right)^2+\left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t\sum_{i=1}^{k+1}\left(\frac{1}{1+\theta_2\Delta t}\right)^{2i}
\end{aligned}$$

By using the geometrical sequence and geometrical series, we get:

$$E\left(\left|X_{k+1}^{IEM}\right|^2\right)=\left(\frac{1}{1+\theta_2\Delta t}\right)^{2(k+1)}E\left(|X_0|\right)^2+\left(\theta_3^2+\theta_1^2\Delta t\right)\Delta t\sum_{i=1}^{k+1}\left(\frac{1}{1+\theta_2\Delta t}\right)^{2i}$$

we have the geometric sequence and series, converging when $|1+\theta_2\Delta t|>1$, by calculating limit of the equation below , for $\Delta t\to 0$ and $k\to+\infty$, we get:

$$\lim_{\Delta t\to 0}\left(\lim_{k\to+\infty}E\left[X_{k+1}^{IEM}\right]\right)=\frac{\theta_3^2}{2\theta_2}$$

□

4. Numerical stabilities of geometric Brownian motion

4.1. **Explicit solution of the model.** Geometric Brownian motion known as exponential Brownian motion is a continuous stochastic process whose logarithm follows a Brownian motion. it is applied in the mathematical modeling of certain courses in the financial markets [26]. It represents a reasonable approximation of the evolution of stock market prices, because a quantity which follows a geometric Brownian motion takes all strictly positive values and only the elementary changes undergone by the random variable are significant.

The geometric Brownian motion X_t is a process which is written in the form [13]:

$$\begin{cases} dX_t = \theta_1 X_t dt + \theta_2 X_t dB_t \\ X(0) = X_0 \quad \forall \theta_1, \theta_2 \in \mathbb{R} \end{cases} \tag{4.1}$$

This process admits as an explicit solution:

$$X_t = X_0 e^{\{(\theta_1 - \frac{1}{2}\theta_2^2)t + \theta_2 B_t\}} \tag{4.2}$$

The variable on the right hand follows a normal distribution, it can also be written in the form:

$$X_t = X_s e^{\left\{\left(\theta_1 - \frac{1}{2}\theta_2^2\right)t + \theta_2 (B_t - B_s)\right\}} \tag{4.3}$$

the conditionnal mean is:

$$E\left(X_t | X_s\right) = X_s e^{\theta_1 (t-s)} \tag{4.4}$$

The (4.3) process is often widely used to model the price of a financial asset the return on the asset between two dates is measured by the difference in the logarithms of the prices and is given by the Gaussian variable below:

$$\left\{\theta_1 - \frac{1}{2}\theta_2^2\right\}(t-s) + \theta_2 \left(B_t - B_s\right)$$

The mean and the mean-square give respectively:

$$E(X_t) = X_0 e^{\theta_1 t} \quad E(X_t^2) = X_0^2 e^{\left(2\theta_1 + \theta_2^2\right)t} \tag{4.5}$$

Remark 4.1. *It should be noted that:*

(1) *For mean if $t \to \infty$ and $\theta_1 < 0$ we have:*

$$\lim_{t\to\infty} E(X_t) = \lim_{k\to\infty} X_0 e^{\theta_1 t} = 0$$

(2) *For Mean-square if $\left(2\theta_1 + \theta_2^2\right) < 0$ and $t \to \infty$ i.e*

$$\lim_{t\to\infty} E(X_t^2) = 0 \text{ with } \left(2\theta_1 + \theta_2^2\right) < 0.$$

Now, let's analyze the stabilities of some numerical schemes (Euler-Maruyama, Milshtein and Implicit Euler-Maruyama) in mean and mean-square.

4.2. **Euler-Maruyama scheme stabilities.** The Euler-Maruyama scheme associated to (4.1) is:

$$X_{k+1}^{EM} = X_k + \theta_1 X_k \Delta t + \theta_2 X_k \Delta B_k$$

$$X_{k+1}^{EM} = X_k\left(1 + \theta_1 \Delta t\right) + \theta_2 X_k \sqrt{\Delta t} Z_k \tag{4.6}$$

4.2.1. *Mean stability of Euler-Maruyama scheme.*

Theorem 4.1. *(**Mean stability of Euler-Maruyama scheme**) The Euler-Maruyama scheme* (4.6) *associated to* (4.1) *model is mean asymptotically stable if*

$$E\left[X_{k+1}^{EM}\right] = (1 + \theta_2 \Delta t)^{k+1} E\left(X_0\right)$$

with $|1 + \theta_1 \Delta t| < 1$ *and*

$$\lim_{\Delta t \to 0}\left(\lim_{k \to +\infty} E\left[X_{k+1}^{EM}\right]\right) = 0$$

Proof. By calculating the mean of the expression(4.6), we obtain:

$$\begin{aligned} E\left[X_{k+1}^{EM}\right] &= E\left[X_t(1 + \theta_1 \Delta t) + \theta_2 X_t \sqrt{\Delta t} Z_t\right] \\ &= E\left[X_t(1 + \theta_1 \Delta t)\right] + E\left[\theta_2 \sqrt{\Delta t} X_t Z_t\right] \\ &= E\left[(1 + \theta_1 \Delta t) X_t\right] + E\left[\theta_2 \sqrt{\Delta t}\right] E\left[X_t\right] E\left[Z_t\right] \end{aligned}$$

as $Z_k \simeq \mathcal{N}(0,1) \quad E(Z_k) = 0$

$$\begin{aligned} E\left[X_{k+1}\right] &= (1 + \theta_1 \Delta t) E(X_t) \\ &= (1 + \theta_1 \Delta t)\left((1 + \theta_2 \Delta t) E\left(X_{k-1}\right)\right) \\ &= (1 + \theta_2 \Delta t)^2 E\left(X_{k-1}\right) \\ &= (1 + \theta_1 \Delta t)^2\left((1 + \theta_2 \Delta t) E\left(X_{k-2}\right)\right) \\ &\vdots \\ &= (1 + \theta_2 \Delta t)^{k+1} E\left(X_0\right) \end{aligned}$$

We get the following geometric sequence:

$$E\left[X_{k+1}^{EM}\right] = (1 + \theta_2 \Delta t)^{k+1} E\left(X_0\right)$$

which converges if $|1 + \theta_2 \Delta t| < 1$ et nd passing to the limit for a $\Delta t \to 0$ and $k \to +\infty$, we obtain:

$$\lim_{\Delta t \to 0}\left(\lim_{k \to +\infty} E\left[X_{k+1}^{EM}\right]\right) = 0$$

□

4.2.2. *Mean-square stability of Euler-Maruyama scheme.*

Theorem 4.2. *(**Mean-square stability of Euler-Maruyama scheme**) The Euler-Maruyama scheme* (4.6) *associated to* (4.1) *model is mean-square asymptotically stable if:*

$$E\left(\left|X_{k+1}^{EM}\right|^2\right) = \left(|(1+\theta_1\Delta t)|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2\right)^{2k+2} E\left(|X_0|^2\right),$$

with $\left||1+\theta_1\Delta t|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2\right| < 1$ *and*

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty} E\left(|X_{k+1}|^2\right)\right) = 0$$

Proof. The mean-square of the expression (4.6) gave:

$$\begin{aligned}
E\left[\left|X_{k+1}^{EM}\right|^2\right] &= E\left[\left|X_k(1+\theta_1\Delta t) + \theta_2 X_k\sqrt{\Delta t}Z_k\right|^2\right]\\
&= E\left[|X_k(1+\theta_1\Delta t)|^2 + \left|\theta_2\sqrt{\Delta t}X_kZ_k\right|^2 + 2\left|X_k(1+\theta_1\Delta t)\theta_2\sqrt{\Delta t}X_kZ_k\right|\right]\\
&= E\left[|X_k(1+\theta_1\Delta t)|^2\right] + E\left[\left|\theta_2\sqrt{\Delta t}X_kZ_k\right|^2\right] + 2E\left[|X_k(1+\theta_1\Delta t)|\left|\theta_2\sqrt{\Delta t}X_kZ_k\right|\right]\\
&= |(1+\theta_1\Delta t)|^2 E\left[|X_k|^2\right] + \left|\theta_2\sqrt{\Delta t}\right|^2 E\left[|X_k|^2\right]\\
&= \left(|(1+\theta_1\Delta t)|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2\right) E\left[|X_t|^2\right]\\
&= \left(|(1+\theta_1\Delta t)|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2\right)\left(|(1+\theta_1\Delta t)|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2\right) E\left[|X_{k-1}|^2\right]\\
&\vdots\\
&= \left(|(1+\theta_1\Delta t)|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2\right)^{2k+2} E\left[|X_0|^2\right]
\end{aligned}$$

We get a geometric sequence:

$$E\left(\left|X_{k+1}^{EM}\right|^2\right) = \left(|(1+\theta_1\Delta t)|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2\right)^{2(k+1)} E\left(|X_0|^2\right)$$

For $\left||1+\theta_1\Delta t|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2\right| < 1$ the sequence converges, and passing to the limit, we obtain for a $\forall\Delta t \to 0$ and $k\to+\infty$,

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty} E\left(|X_{k+1}|^2\right)\right) = 0$$

□

4.3. **Milshtein's scheme stabilities.** The Milshtein Schema associated to the expression(4.1) is given by :

$$\begin{aligned}
X_{k+1}^M &= X_k + b(X_k)\Delta t + \sigma(X_k)\Delta B_k + \frac{1}{2}\sigma\sigma'(X_k)\left\{(\Delta B_k)^2 - \Delta t\right\}\\
&= X_k + \theta_1 X_k \Delta t + \theta_2 X_k \Delta B_k + \frac{1}{2}\theta_2 X_t \theta_2 \left\{(\Delta B_k)^2 - \Delta t\right\}\\
&= X_k + \theta_1 X_k \Delta t + \theta_2 X_k \sqrt{\Delta t} Z_k + \frac{1}{2}\theta_2^2 X_k \left(\Delta t Z_k^2 - \Delta t\right)\\
&= \left(1 + \theta_1 \Delta t - \frac{1}{2}\theta_2^2 \Delta t\right) X_k + \theta_2 X_k \sqrt{\Delta t} Z_k + \frac{1}{2}\theta_2^2 X_k \Delta t Z_k^2\\
&= \left(1 + \left(\theta_1 - \frac{1}{2}\theta_2^2\right)\Delta t\right) X_k + \theta_2 X_k \sqrt{\Delta t} Z_k + \frac{1}{2}\theta_2^2 X_k \Delta t Z_k^2
\end{aligned}$$

We have after calculation:

$$X_{k+1}^M = X_k \left(1 + \left(\theta_1 - \frac{1}{2}\theta_2^2\right)\Delta t\right) + \theta_2 X_k \sqrt{\Delta t} Z_k + \frac{1}{2}\theta_2^2 X_k \Delta t Z_k^2 \tag{4.7}$$

We now consider the same model of geometric Brownian motion, we state some results on the stabilities following Milshtein's scheme and we prove these results.

4.3.1. *Mean stability of Milshtein's scheme.*

Theorem 4.3. *(**Mean stability of Milshtein's scheme**) The Milshtein's scheme (4.7) assocated to (4.1) model is mean asymptotically stable if*

$$E\left(X_{k+1}^M\right) = [1 + \theta_1 \Delta t]^{k+1} E\left(X_0\right)$$

with

$$|1 + \theta_1 \Delta t| < 1$$

and

$$\lim_{\Delta t \to 0}\left(\lim_{k \to \infty} E\left(X_{k+1}^M\right)\right) = 0$$

Proof. Applying the usual approach, let us evaluate the mean of gives:

$$\begin{aligned}
E\left(X_{k+1}^M\right) &= E\left(X_k\left(1 + \left(\theta_1 - \frac{1}{2}\theta_2^2\right)\Delta t\right) + \theta_2 X_k \sqrt{\Delta t} Z_k + \frac{1}{2}\theta_2^2 X_k \Delta t Z_k^2\right)\\
&= E\left(X_k\left(1 + \left(\theta_1 - \frac{1}{2}\theta_2^2\right)\Delta t\right)\right) + E\left(\theta_2 X_k \sqrt{\Delta t} Z_k\right) + E\left(\frac{1}{2}\theta_2^2 X_k \Delta t Z_k^2\right)\\
&= \left(1 + \left(\theta_1 - \frac{1}{2}\theta_2^2\right)\Delta t + \frac{1}{2}\theta_2^2 \Delta t\right) E\left(X_k\right)\\
&= \left(1 + \theta_1 \Delta t\right) E\left(X_k\right)\\
&\vdots\\
&= \left(1 + \theta_1 \Delta t\right)^{k+1} E\left(X_0\right)
\end{aligned}$$

ultimately we get that:

$$E\left(X_{k+1}^M\right) = [1+\theta_1\Delta t]^{k+1}\,E\left(X_0\right)$$

As the previous expression has the form of a geometric sequence, we know that it converges if $|1+\theta_1\Delta t| < 1$, passing to the limit, for all $\Delta t \to 0$ and $k \to +\infty$ we find the results searched i.e:

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty} E\left(X_{k+1}^M\right)\right) = 0$$

□

4.3.2. *Mean-square stability of Milshtein's scheme.*

Theorem 4.4. *(**Mean-square stability of Milshtein's scheme**) The Milshtein scheme* (4.7) *associated to* (4.1) *model is mean-square asymptotically stable if*

$$E\left(\left|X_{k+1}^M\right|\right) = \left[\left|1+\left(\theta_1-\frac{1}{2}\theta_2^2\right)\Delta t\right|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2 + \left|\frac{1}{2}\theta_2^2\Delta t\right|^2\right]^{2(k+1)} E\left(|X_0|^2\right)$$

with $\left|\left|1+\left(\theta_1-\frac{1}{2}\theta_2^2\right)\Delta t\right|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2 + \left|\frac{1}{2}\theta_2^2\Delta t\right|^2\right| < 1$ *and*

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty} E\left(\left|X_{k+1}^M\right|^2\right)\right) = 0$$

Proof. Let's start by calculating the mean-sqaure of the model expression, ie:

$$\begin{aligned}
E\left(\left|X_{k+1}^M\right|^2\right) &= E\left(\left|X_k\left(1+\left(\theta_1-\frac{1}{2}\theta_2^2\right)\Delta t\right) + \theta_2 X_k\sqrt{\Delta t}Z_k + \frac{1}{2}\theta_2^2 X_k\Delta t Z_k^2\right|^2\right)\\
&= E\left(\left|X_k\left(1+\left(\theta_1-\frac{1}{2}\theta_2^2\right)\Delta t\right)\right|^2\right) + E\left(\left|\theta_2 X_k\sqrt{\Delta t}Z_k\right|^2\right) + E\left(\left|\frac{1}{2}\theta_2^2 X_k\Delta t Z_k^2\right|^2\right)\\
&= \left|1+\left(\theta_1-\frac{1}{2}\theta_2^2\right)\Delta t\right|^2 E\left(|X_k|^2\right) + \left|\theta_2\sqrt{\Delta t}\right|^2 E\left(|X_k|^2\right) + \left|\frac{1}{2}\theta_2^2\Delta t\right|^2 E\left(|X_k|^2\right)\\
&= \left[\left|1+\left(\theta_1-\frac{1}{2}\theta_2^2\right)\Delta t\right|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2 + \left|\frac{1}{2}\theta_2^2\Delta t\right|^2\right] E\left(|X_k|^2\right)\\
&\;\vdots\\
&= \left[\left|1+\left(\theta_1-\frac{1}{2}\theta_2^2\right)\Delta t\right|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2 + \left|\frac{1}{2}\theta_2^2\Delta t\right|^2\right]^{2(k+1)} E\left(|X_0|^2\right)
\end{aligned}$$

Continuing with the iterations, we get:

$$E\left(\left|X_{k+1}^M\right|\right) = \left[\left|1+\left(\theta_1-\frac{1}{2}\theta_2^2\right)\Delta t\right|^2 + \left|\theta_2\sqrt{\Delta t}\right|^2 + \left|\frac{1}{2}\theta_2^2\Delta t\right|^2\right]^{2(k+1)} E\left(|X_0|^2\right)$$

passing to the limit with $\Delta t \to 0$ and $k \to +\infty$, we obtain the stated results, ie:

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty} E\left(\left|X_{k+1}^M\right|^2\right)\right) = 0$$

□

4.4. **Implicit Euler-Maruyama scheme stabilities.** The implicit Euler-Maruyama scheme (IEM) gives:

$$X_{k+1}^{IEM} = X_k + b(X_{k+1})\Delta t + \delta(X_t)\Delta B_k$$
$$X_{k+1} = X_k + \theta_1 X_{k+1}\Delta t + \theta_2 X_t \Delta B_k$$
$$X_{k+1} - \theta_1 X_{k+1}\Delta t = X_k + \theta_2 X_k \Delta B_k$$
$$X_{k+1}(1-\theta_1\Delta t) = X_k + \theta_2 X_k \Delta B_k$$

We obtain:

$$X_{k+1}^{IEM} = \frac{1}{1-\theta_1\Delta t}X_k + \frac{\theta_2\sqrt{\Delta t}}{1-\theta_1\Delta t}X_k Z_k \quad Z_k \simeq \mathbb{N}(0,1) \tag{4.8}$$

4.4.1. *Mean stability of implicit Euler-Maruyama scheme.*

Theorem 4.5. *(**Mean stability of implicit Euler-Maruyama scheme**) The implicit Euler-Maruyama scheme (IEM)* (4.8) *associated to* (4.1) *model is mean asymptotically stable if*

$$E\left(X_{k+1}^{IEM}\right) = \left(\frac{1}{1-\theta_1\Delta t}\right)^{k+1} E(X_0) \tag{4.9}$$

with $|1-\theta_1\Delta t| > 1$ *then,*

$$\lim_{\Delta t\to 0}\left(\lim_{k\to\infty} E\left(\left|X_{k+1}^{IEM}\right|^2\right)\right) = 0$$

Proof. Let's evaluate the mean associated to the implicit Euler-Maruyama scheme

$$\begin{aligned}
E\left(X_{k+1}^{IEM}\right) &= E\left(\frac{1}{1-\theta_1\Delta t}X_k + \frac{\theta_2}{1-\theta_1\Delta t}X_k\sqrt{\Delta t}Z_k\right)\\
&= E\left(\frac{1}{1-\theta_1\Delta t}X_k\right) + E\left(\frac{\theta_2}{1-\theta_1\Delta t}\sqrt{\Delta t}\right)(X_k)(Z_k)\\
&= E\left(\frac{1}{1-\theta_1\Delta t}X_k\right)\\
&= \frac{1}{1-\theta_1\Delta t}E(X_k)\\
&= \left(\frac{1}{1-\theta_1\Delta t}\right)^2 E(X_{k-1})\\
&= \left(\frac{1}{1-\theta_1\Delta t}\right)^3 E(X_{k-2})\\
&\vdots\\
&= \left(\frac{1}{1-\theta_1\Delta t}\right)^{k+1} E(X_0)
\end{aligned}$$

Continuing with the iterations we get:

$$E\left(X_{k+1}^{IEM}\right) = \left(\frac{1}{1-\theta_1\Delta t}\right)^{k+1} E(X_0)$$

passing to the limit with $\Delta t \to 0$ and $k \to +\infty$, we obtain the stated results, ie

$$\lim_{\Delta t \to 0} \left(\lim_{k \to \infty} E\left(|X_{k+1}^{IEM}|^2 \right) \right) = 0$$

□

4.4.2. *Mean-square stability of Implicit Euler-Maruyama scheme.*

Theorem 4.6. *(**Mean-square stability of Implicit Euler-Maruyama scheme**) The implicit Euler-Maruyama scheme associated to the model (4.1) is asymptotically mean-square stable if*

$$E\left(|X_{k+1}^{IEM}|^2 \right) = \left[\frac{1 + |\theta_2 \sqrt{\Delta t}|}{1 - \theta_1 \Delta t} \right]^{2(k+1)} E\left(|X_0|^2 \right)$$

with $\left| \frac{1+|\theta_2\sqrt{\Delta t}|}{1-\theta_1 \Delta t} \right| < 1$ *and*

$$\lim_{\Delta t \to 0} \left(\lim_{k \to \infty} E\left(|X_{k+1}^{EMI}|^2 \right) \right) = 0$$

Proof. : Let us calculate the quadratic mean, in effect,

$$\begin{aligned}
E\left(|X_{k+1}^{M}|^2 \right) &= E\left(\left| \frac{1}{1 - \theta_1 \Delta t} X_t + \frac{\theta_2 \sqrt{\Delta t}}{1 - \theta_1 \Delta t} X_t Z_t \right|^2 \right) \\
&= \left(\frac{1}{1 - \theta_1 \Delta t} \right)^2 E\left(\left| X_k + \theta_2 \sqrt{\Delta t} X_k Z_k \right|^2 \right) \\
&= \left(\frac{1}{1 - \theta_1 \Delta t} \right)^2 \left[E\left(|X_k|^2 \right) + E\left(\left| +\theta_2 \sqrt{\Delta t} X_k Z_k \right|^2 \right) \right] \\
&= \left| \frac{1}{1 - \theta_1 \Delta t} \right|^2 \left[E\left(|X_k|^2 \right) + \left| +\theta_2 \sqrt{\Delta t} \right|^2 E\left(|X_k|^2 \right) \right] \\
&= \frac{\left(1 + |\theta_2 \sqrt{\Delta t}|^2 \right)}{(1 - \theta_1 \Delta t)^2} E\left(|X_k|^2 \right) \\
&= \frac{\left(1 + |\theta_2 \sqrt{\Delta t}|^2 \right)}{(1 - \theta_1 \Delta t)^2} \frac{\left(1 + |\theta_2 \sqrt{\Delta t}|^2 \right)}{(1 - \theta_1 \Delta t)^2} E\left(|X_{k-1}|^2 \right)
\end{aligned}$$

Continuing with the iterations, we get:

$$E\left(|X_{k+1}^{EMI}|^2 \right) = \left[\frac{1 + |\theta_2 \sqrt{\Delta t}|}{1 - \theta_1 \Delta t} \right]^{2(k+1)} E\left(|X_0|^2 \right)$$

passing to the limit with $\Delta t \to 0$ and $k \to +\infty$, we obtain the stated results, i.e :

$$\lim_{\Delta t \to 0} \left(\lim_{k \to \infty} E\left(|X_{k+1}^{M}|^2 \right) \right) = 0$$

□

5. Numerical simulations and residual calculations

In this section, we present some numerical simulations for Vasicek and geometric Brownian motion models using Matlab and we calculate the errors between the exact solution and that obtained by applying the numerical schemes of Euler-Maruyama, Milshtein and Implicit Euler-Maruyama.

5.1. **Numerical simulation of Vasicek and geometric Brownian motion models.** We present some simulations of Vasicek and Brownian geometric motion models in the increasing and decreasing cases.

Figure 1. Increasing Vasicek model

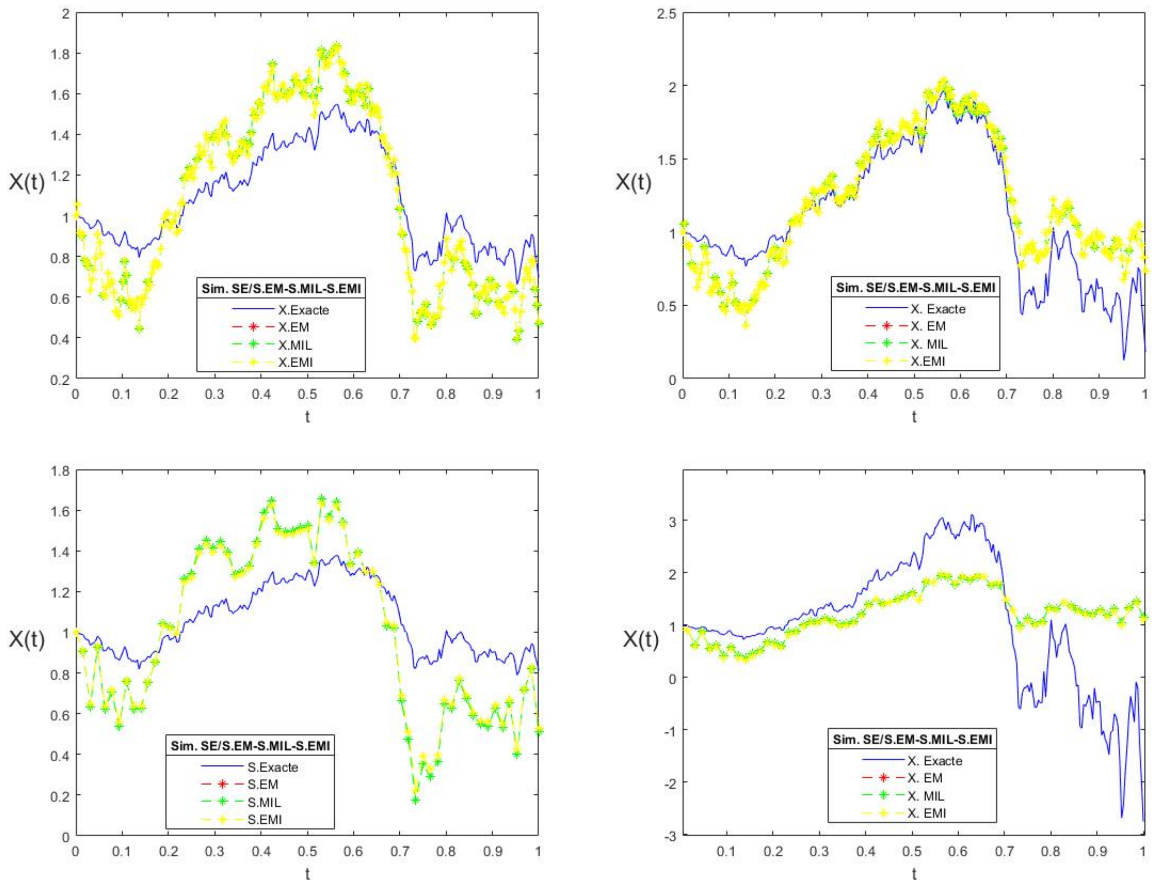


Figure 2. Decreasing Vasicek model

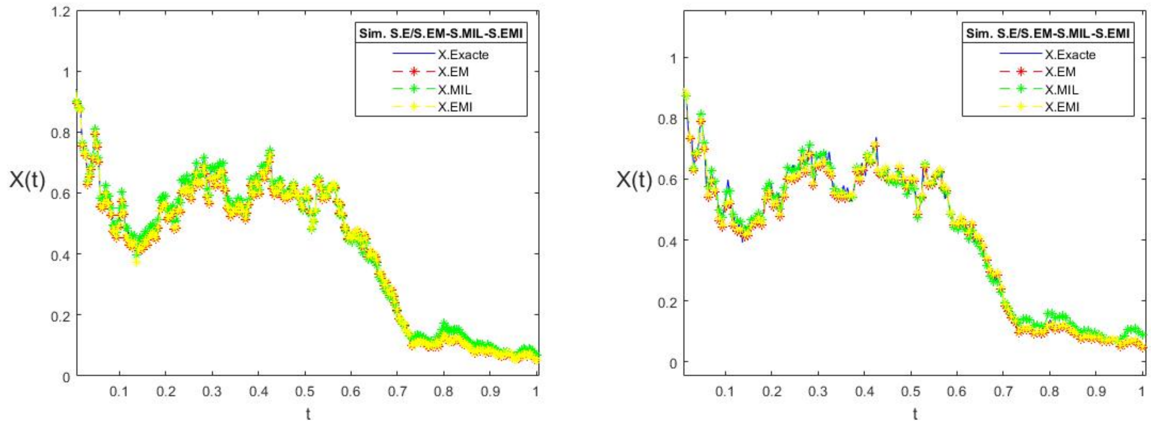


Figure 3. Increasing geometric Brownian motion model

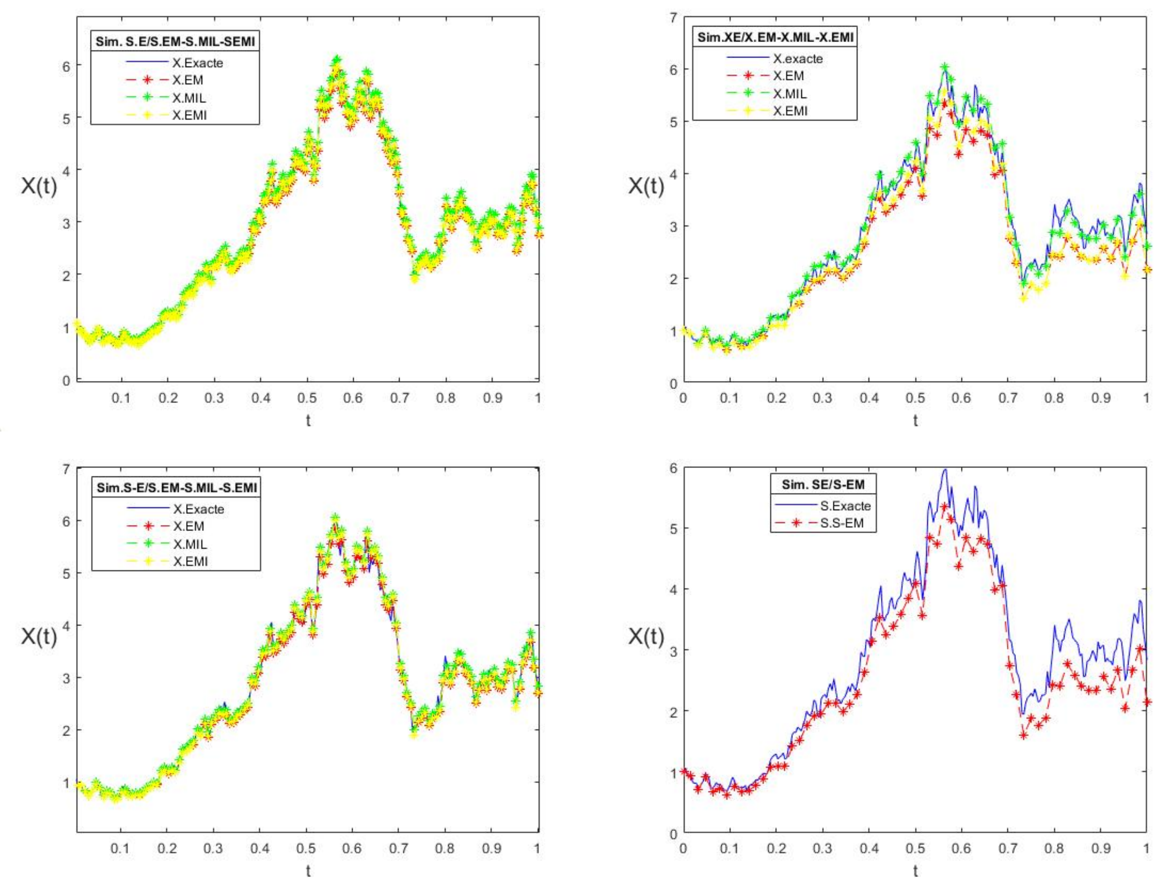
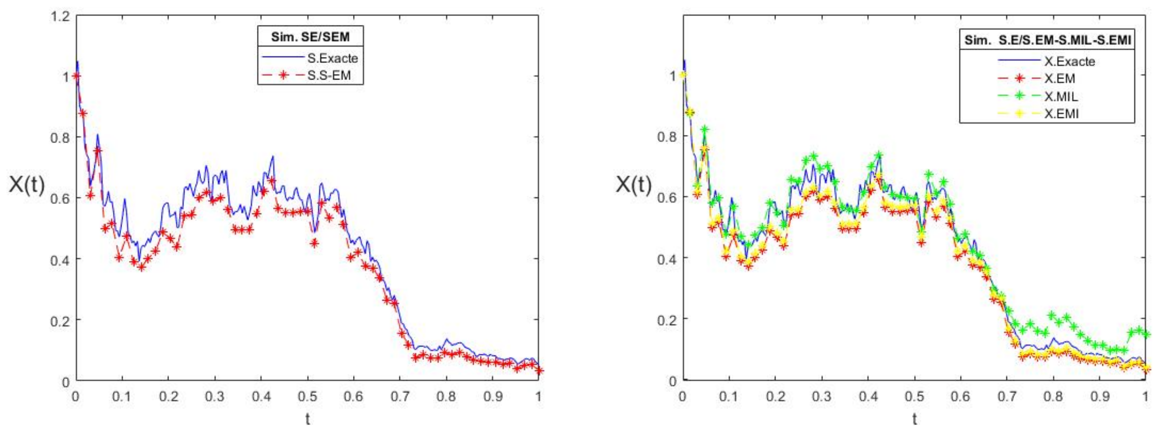


Figure 4. Decreasing geometric Brownian motion model



5.2. **Interpretation of results.**

5.2.1. *Vasicek model.* The figures 1 and 2 show the stability of the Vasicek model in the increasing and decreasing cases, in these figures we see that the Euler-Maruyama scheme coincides with that of Milshtein. We have in the first two figures of figure 1 the following errors: Emerr = 0.2280, Milerr = 0.2280 and Iemerr = 0.2258 In both figures of figure 1 Emerr = 0.3007, Milerr = 0.3007 and Iemerr = 0.2851

In both figures of figure 2 Emerr = 0.2268, Milerr = 0.2268 and Iemerr = 0.2237.

5.2.2. *Geometric Brownian motion model.* The figures figure 3 et figure 4 present the stability of geometric Brownian motion in the increasing and decreasing cases. Indeed, the first three figures in figure 3 present the increasing stability of geometric motion and the last figure in figure 3 and the two figures in figure 4 show the decreasing stability of the model. We have in the first two figures and figure 3 the following errors: Emerr = 0.0027, Milerr = 0.0011 and Iemerr = 0.0013 and for the third figure in figure 3: Emerr = 0.0177, Milerr = 0.0111 and Iemerr = 0.0128. In the both figures of figure 4: Emerr = 0.0054, Milerr = 0.0022 and Iemerr = 0.0026.

Remark 5.1. *From the results bellow, in the cases of increasing and decreasing stabilities of Vasicek et Geometric Brownian motion, we have that, the Milshtein scheme is the best scheme because it's the best approximates the exact solution.*

6. Conclusion

We have presented in this article the analysis of the stability in mean and mean-square for Vasicek and geometric Brownian motion models. In these models, we established the conditions of the numerical stabilities of Euler-Maruyama, implicit Euler-Maruyama and Milshtein schemes. these conditions have been proved by using classical manner and Y. Saito's approach. It should be noted that each case is different from the other depending on whether the models examined have additive (Vasicek model) or multiplicative (Geometric brownian motion) white noise type.

Finally, for these models, we found that the stability conditions of the Vasicek model coincides with the stability of the ODEs, on the other hand, for the stability conditions of the second model to coincide with the stability of the ODEs, it is necessary that $\theta_2 < 0$. To support these results, numerical simulations were made and the calculations of the residuals (errors) comes in support of the results found. In the next work we will analyze the numerical stabilities of these two models by using Non-standard Euler-Maruyama scheme.

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