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On Norm Estimates for Derivations in Norm-Attainable Classes

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Abstract. In this note, we provide detailed characterization of operators in terms of norm-attainability and norm estimates in Banach algebras. In particular, we establish the necessary and sufficient conditions for norm-attainability of the derivations and also give their norm bounds in the norm-attainable classes.

1. Introduction

The norm of a derivation was first introduced by Stampfli [49], who determined the inner derivation $\delta_{T_0} : A_0 \to T_0A_0 - A_0T_0$ which acts on $B(H)$, the algebra of all bounded linear operators on a complex Hilbert space H. Further, $\|\delta_{T_0}\| = \inf \ 2\|T_0 - \lambda I_0\|$, for every complex λ was shown. For a normal operator T, $\|\delta_{T_0}\|$ can be expressed as the geometry of the spectrum of T_0. Johnson [21] established methods which apply to a uniformly convex spaces with a large class, i.e the formula $\|\delta_T\|$ is false in l^p and $L^p(0,1)$ $1 < p < \infty$, $p \neq 2$. For L^1 space the formula is true for a real case and not for a complex case whose space dimension is 3 or more. Johnson [20] found that a derivation on $B(H)$ is a mapping $\Delta : B(H) \to B(H)$ with $\Delta(AS) = A\Delta(S) + \Delta(A)S$, where $A, S \in B(H)$. Such derivations are necessarily continuous and if $S \in B(H)$ then $\Delta_S(A) = AS - SA$ is a derivation on $B(H)$. Gajendragadka [18] was concerned with the Von Neumann algebra and computed the norm of a derivation. Specifically, it was proved that the Von Neumann algebra acts on a separable Hilbert space H, whereby if T is in $\mathfrak{U}$ and δ_T is the derivation induced by T, then $\|\delta_T|\mathfrak{U}\| = 2 \ \inf \|T - Z\|$, where Z is the centre of $\mathfrak{U}$. Therefore, Anderson [3] in his investigation on normal derivations with the operators $A, C \in B(H)$ proved if A is normal and AC commute, for every $X \in B(H)$, $\|\delta_A(X) + C\| \geq \|C\|$. Therefore, the inequality showed that the kernel and the range of δ_A are orthogonal to δ_A which is the commutation of $\{A\}'$ of A. Kyle [24] examined the relationship

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of the numerical range of inner derivation and that of the implementing element. Kyle [25] studied norms of inner derivations and used their properties and concluded that a closed subset of all derivations on a C^*-algebra, forms the set of inner derivations and obtained the result which is a converse of Stampfli [49]. Charles and Steve [11] answered the question when $X = T$ by structure characterization of compact derivations of C^*-algebras. Moreover, the structure of weak compact derivations of C^*-algebras was determined and as immediate corollaries of these results, conditions that were necessary and sufficient were obtained so that C^*-algebras can admit a non-zero compact or weakly compact derivation. Stampfli [50] studied operators on Hilbert spaces and their properties inducing a derivation whose closure is self-adjoint after the range of such operators are termed D-symmetric and then characterized compact D-symmetric operators. Erik [16] established that any operator T on a Hilbert space H with a cyclic vector has a property with a finite spectrum. Mecheri [31] established that $T(X)$ is linear for any m-linear derivation and hence, the topology of Von Neumann algebra X of type I is automatically continuous in measure with center m and the semi-finite trace τ which is normal is faithful. Therefore, $T(X)$ is the algebra of all τ-measurable operators affiliated with X. Mathieu [29] proved that for non-zero derivations, the product of two prime C^*-algebras are bounded if both of them are bounded. In [51], two automatic continuity problems for derivations on commutating Banach algebras were discussed, that is, derivation on a commutative algebra is mapped onto the radical, and Banach algebras are continuous on semiprime derivations. Bresar, Zalar [9] showed that a Jordan $*$-derivation is the map $\delta_a(x) = ax - x^*a$ for fixed $a \in \mathfrak{U}$; hence, the derivation is inner. Douglas [15] continued the study of $W_s(Y)$ which was considerably more amenable where Archbold [1] defined the smallest numbers to be $[0, \infty]$ and introduced two constants $W(Y)$ and $W_t(Y)$ such that $d(y, Z(Y)) \leq W(Y)\|D(y, Y)\|$, for all $y \in Y$ and $d(y, Z(Y)) \leq W_s(Y)\|D(y, Y)\|$, for all $y = y^* \in Y$. The author in [26] showed that for the nth order commutator $[[[k(B), Y], Y], ..., Y]$, a formula was obtained in terms of the Frechet derivatives $S^m k(B)$ in which the formula illustrated was used to obtain bounds for norms of a generalized commutator $k(B)Y - Yk(B)$ and their higher order analogues. In [17], numerical ranges of 2 x 2 matrices were determined and the convex of the numerical range for any Hilbert space operator was established in Toeplitz-Hausdorff theorem and the relation of the numerical range to that of spectrum was discussed. Further, the closure of the numerical range is contained in the spectrum and the intersection of closures of the numerical range of all operators were asserted by Hildebrandt's theorem. Considering results on special cases [10], established that $\|PXQ + QXP\| \geq \|P\|\|Q\|$. Chi-Kwong [13] established that for an n x n matrix X, the numerical range $W(X)$ has many properties which can be used to locate eigenvalues to obtain norm bounds. Algebraic and analytic properties were deduced which help in finding the dilations of simple structures. Let the linear operators X_i and Y_i, $1 \leq i \leq n$ act on separate Hilbert space H, therefore, Hong-Ke, Yue-qing [19] proved that $\sup\{\|\sum_{i=1}^{n} P_i X Q_i\| : X \in B(H), \|X\| \leq 1\} = \sup\{\|\sum_{i=1}^{n} P_i T Q_i\| : UU^* = T^*U =$

$I, U \in B(H)\}$. In addition, Okelo, Agure and Ambogo [35] established the norm of Jordan elementary operator $\mathcal{U}_{A,B} : B(H) \to B(H)$ which is given by $\mathcal{U}_{A,B} = AYB + BYA$, $\forall Y \in B(H)$ and A, B fixed in $B(H)$ and showed that $\|\mathcal{U}_{A,B}\| \geq \|A\|\|B\|$ and then characterized the norm-attainable operators using this norm. Inner derivations implemented by norm-attainable elements of a C^*-algebra has relation to those of ideals and primitive ideals. Since there is a relationship between the constants $A(\xi)$ and $A_s\xi$ of C^*-algebras to the ideals and primitive ideals then related results have been given in general Banch settins. Okelo, Agure and Oleche [38] gave results on necessary and sufficient conditions for norm-attainable operators and also studied norm-attainable operators and generalized derivations. Okelo [37] extended the work by presenting new results on conditions that are necessary and sufficient for norm-attainability for operators in Hilbert space, elementary operators and generalized derivations. Further, Okelo [37] established that a unit vector exists $\lambda \in H$, $\|\lambda\| = 1$ such that $\|S\lambda\| = \|S\|$ with $\langle S\lambda, \lambda\rangle = \eta$. Results from [23] showed that every Jordan derivation of the trivial extension of $\mathcal{A}$ by $\mathcal{M}$, under certain conditions, is the sum of a derivation and antiderivation. In [10], the author studied norm-attainable operators that are convergent and established norm-attainability of operators via projective tensor norm. Wickstead [52] showed that if an atomic Banach lattice Z with a continuous norm order, $X, Y \in \mathcal{T}^r$ and $M_{X,Y}$ is the operator on $\mathcal{T}^r(Z)$ defined by $M_{X,Y}(A) = XAY$, then $\|M_{X,Y}\|_r = \|X\|_r\|Y\|_r$ but there is no real $\beta > 0$ such that $\|M_{X,Y}\|_r = \beta\|X\|_r\|Y\|_r$. Okelo [36] outlined the theory of normal, self-adjoint and norm-attainable operators then presented norms of operators in Hilbert spaces. In [8] the author proved that for a linear map $\Delta : \mathfrak{U} \to \mathfrak{U}$, $\Delta(XY) = \Delta(X)Y + \Delta X(Y)$ for each $X, Y \in \mathfrak{U}$ is a derivation, and for any two derivations Δ and Δ' on a C^*-algebra $\mathfrak{U}$ there exists a derivation $\delta \in \mathfrak{U}$ such that $\Delta\Delta' = \delta^2$ if and only if either $\Delta' = 0$ or $\Delta = f\Delta'$ for any $f \in \mathbb{C}$. Clifford [12] studied hypercyclic generalized derivations acting on separable ideals of operators and also identified concrete examples and established some conditions that are necessary and sufficient for their hypercyclicity. Okelo [36] considered orthogonal and norm-attainable operators in Banach spaces, gave in details the characterization and generalizations of norm-attainability and orthogonality. The conditions that are sufficient and necessary for norm-attainability of operators on a Hilbert space, the result on orthogonal range and the kernel of elementary operators implemented by norm-attainable operators in Banach spaces were also given. Okelo [34] characterized norm-attainable classes in terms of orthogonality by giving norm-attainability conditions that were necessary and sufficient for Hilbert space operators first and the orthogonality result on the range and kernel of elementary operators when implemented by norm-attainable operators in norm-attainable classes were also given. Okelo [38] gave conditions for norm-attainability for linear functionals in Banach spaces, non-power operators on H and elementary operators and also gave a new notion of norm-attainability for power operators then characterized norm-attainable operators in normed spaces. In [51] determined the norm of the inner Jordan $*$-derivation $\delta_S : X \to SX - X^*S$ acting on the

Banach algebra $B(H)$. It was shown that $\|\delta_S\| \geq 2\sup_{\lambda \in W_0(S)} |\Im\lambda|$ in which $W_0(S)$ is the maximal numerical range of operator S. The work of [1] obtained precisely when zero belongs to maximal numerical range of composition operators on H and then characterized the norm-attainability of derivations on $B(H)$. In Okelo [41] norm-attainability for hyponormal operators that are compact were characterized, sufficient conditions for a compact hyponormal operator that is linear and bounded on an infinite dimension for a complex Hilbert space to be norm attainable were given. Further, the structure and other properties of compact hyponormal operators when they are self-adjoint, normal and norm attainable with their commutators were discussed in general. Lumer [27] obtained a sharp estimate not only from $|sp(R)|$ equal to spectral radius of R but indeed for $|sp(R)|$ in terms of $\sup(|X(R)|,\ |X(R^n)|^{1/n})$, n being any positive even integer. In [18] the author studied the algebra of functions that are continuous on $[0,1]$ and are $\|.\|_w$-approximate polynomial; i.e point-wise functions of limits of $\|.\|_w$-Cauchy sequence of polynomial. Archbold [1] investigated whether the simple triangle inequality $\|T(a,A)\| \leq 2t(a,Z)$ if applied holds. $D(A)$ was defined to be a minimum value D in $[0,\infty]$ such that $t(a,Z) \leq D\|T(a,A)\|$. The behaviour of D in ideals and quotients were discussed which proved that $D_s(A) \leq 1$ for a weakly central C^*-algebra A and considered a class of n-homogeneous C^*-algebras that are special. D and D_s were investigated and approximated finite-dimension $(AF)C^*$-algebra in that context and an example was given to show certain estimates. The results of [44] showed that for a certain Von Neumann algebra $\mathfrak{U}$, a constant F existed such that $\text{dist}(T,\mathfrak{U}) \leq F\sup_{P\in lat\mathfrak{U}} \|P^{\perp}TP\|\, \forall T \in B(H)$. The work was extended to a Von Neumann algebra $\mathfrak{U}$ and showed that there exists a constant $G \in B(H)$, $\text{dist}(T,\mathfrak{U}) \leq G\|\Delta_T|\mathfrak{U}'\|$ where δ_T is the derivation $\delta_T(S) = ST - TS$ thus proving that the inequality holds for large classes of Von Neumann algebras. In [14] the researcher considered $\lambda(M)$ defined as the smallest number $\|Z\|^2$ of Z that satisfy $[Z^*,Z] = M$ and showed that $1 \leq \lambda(M) \leq 2$. Matej [28] estimated the distance of d_1 and d_2 to the generalized derivations and the normed algebra of P and considered the cases when P is an ultraprime, when $d_1 = d_2$ and P are ultrasemiprime and when P is a Von Neumann algebra we have the equation $\|P+Q\| = \|P\| + \|Q\|$, $P,Q \in B(H)$. Further, a constructive proof was provided that a minimum bound is not valid and a relevant method to analyze the problem on estimation of eigenvalues such an interpolation matrix was commented on. The norm property was done by Cabrera, Rodriguez [10] for basic elementary operators and obtained $\|M_{a,b}\| \leq 2\|a\|\|b\|$, for Jordan elementary operator $\|U\| = \|M_{a,b}\| + \|M_{a,b}\|$, $\|M_{a,b}\| + \|M_{a,b}\| \leq 2\|a\|\|b\|$ for the upper estimates. In fact, [30] gave an estimate on matrix-valued function that is regular and showed that for normal matrices it is attainable and investigated their stability. Kittaneh [26] established the orthogonality, kernel and the range of a normal derivation associated with norm ideals of operators with respect to the unitarily invariant norms. Results related to orthorgonality of some derivation that are not normal were also obtained. Stacho and Zalar [48] established the lower estimates for elementary operators of Jordan type in standard Banach algebras. Danko [14] established that for all unitarily invariant norms and for bounded Hilbert space operators there exist $\{x_n\}_n \subseteq H$ which is a unit sequence such that $\lim_n \|C - \omega\|_{x_n} = 0$. From [11], $\|A\| \in \sigma(A)$ if and only if $\|A\| \in \sigma_{ap}(A)$ also $\sigma(A) \subseteq \overline{W(A)}$ (spectral inclusion) and if $\omega(A) = \|A\|$, then $\gamma(A) = \|A\|$. Therefore, the result implied that $\|A\| \subseteq \overline{W(A)}$ if and only if $\|A\| \in \sigma(A)$. In fact, Megginson [32] established that for all $Y \in \mathfrak{K}$, then $\delta_B(Y) \in \mathfrak{J}$ and $\|BY - YB\|_{\mathfrak{K}} = \|(B-\lambda)Y - Y(B-\alpha)\|_{\mathfrak{J}} \leq 2\|B-\alpha\|\|Y\|_{\mathfrak{K}}$ for all $\alpha \in \mathbb{C}$. Hence, $\|\delta_B(Y)\|_{\mathfrak{K}} \leq 2d(B)\|Y\|_{\mathfrak{K}}$, implying that $\|\delta_B|\mathfrak{K}\| \leq 2d(B)$. Further, the notion of R-universal operators was introduced and that R-universal is an operator $A \in B(H)$ if $\|\delta_B|\mathfrak{K}\| = 2d(B)$ for every norm ideal $\mathfrak{K} \in B(H)$. Landsman [23] proved that for a standard operator algebra on H $\|M_{a,b}\| + \|M_{a,b}\| \geq 2(\sqrt{2}-1)\|a\|\|b\|$. Therefore, both the lower norm and upper norm bounds have been established for normally represented elementary operators. The work of [3] had an estimate on transfer functions of stable linear time-invariant systems on stochastic assumptions. The approach of nonparametric minimax was adopted to measure the estimate accurately, an estimator of quality was measured over a family of transfer functions by its worst case error. In [32] the author established that for a holomorphic functions f with $Re\{gf'(g)\} > \alpha$ and $Re\{gf''(g)/f'(g)\} > \alpha - 1, (0 \leq \alpha < 1)$ respectively in $\{|g| < 1\}$, estimates of $\sup_{|g|<1}(1-|g|^2)|f''(g)/f'(g)|$ were given and functions Gelfer-convex of exponential order α, β was also considered. Milos, Dragoljub [33] considered elementary operators $x \to \sum_{j=1}^{n} v_j x w_j$ that acts on a Banach algebra. The ascent estimation and lower bound estimation of an operator was given. Barraa and Boumazgour [4] showed that the norm of bounded operators more than one on a Hilbert space is the same as the sum of the norms and showed that $\delta_{S,A,B}$ is convexoid with the convex hull of its spectrum if and only if A and B are convexoid. Richard [44] established the CB-norms of elementary operators and the lower bounds for norms on $B(H)$. The result was concerned with the operator $\mathcal{U}_{A,B}X = AXB + BXA$ which showed that $\|\mathcal{U}_{A,B}\| \geq \|A\|\|B\|$ which proved a conjecture of Mathieu, other results and formula of $\|\mathcal{U}_{A,B}\|_{CB}$ and $\|\mathcal{U}_{A,B}\|$ were established. Richard [45] provided the Haagerup estimation on the norm of elementary operators that are completely bounded. Seddik [46] proved that lower estimate bound $\|T_{M,N}\| \geq 2(\sqrt{2}-1)\|M\|\|N\|$ holds, if it is either a standard operator algebra or a norm ideal on $B(H)$ and $M, N \in B(H)$. Florin, Alexandra [17] estimated the norm of operator $H_{\theta,\lambda} = U_\theta + U_\theta^* + (\lambda/2)(V_\theta + V_\theta^*)$ which is an element on a C^*-algebra $A_\theta = C^*(U_\theta, V_\theta$ unitaries : $U_\theta V_\theta = e^{2\pi i\theta} V_\theta U_\theta)$, and proved that for every $\lambda \in \mathbb{C}$ and $\theta \in [\frac{1}{4}, \frac{1}{2}]$ the inequality $\|H_{\theta,\lambda}\| \leq \sqrt{4 + \lambda^2 - (1 - \frac{1}{\tan\theta,\lambda})(1 - \sqrt{\frac{1+\cos^2 4\pi\theta}{2}})min\{4, \lambda^2\}}$ holds. This significantly improved the inequality $\|H_{\theta,2}\| \leq 2\sqrt{2}, \theta \in [\frac{1}{4}, \frac{1}{2}]$, conjectured by [18]. The author in [31] considered commuting matrices of matrix valued analytic function and established a norm estimate, in particular, two matrices of matrix valued functions on a tensor product in a Euclidean space were explored. In [5] the research communicated results on complex symmetric operator theory and showed that two non-trivial examples were of great use in studying Schrödinger operators. The work of [43] showed that triangle inequality served an upper norm bound for the sum operators that is $\sup\{\|T^*RT+V^*SV\| : T \text{and} V\}$ are unitaries. The result discussed had relationship to normal dilations, spectral sets and the Von Neumann inequality. Yong, Toshiyuki [53] gave a norm estimate on pre-Schwarzian derivatives of a specific type of convex functions by introducing a maximal operator of independent interest of a given kind. The relationship between the convex functions and the Hardy spaces was discussed. In [16] the author analyzed the structure of the set $\mathcal{D} = \{y \in D(\delta) : \lim_{n\to\infty} \Delta_n(y) = \Delta(y)\}$ for convergence of the generators that are pointwise where α is an approximate inner flow on a C^*-algebra T with generator Δ and Δ_n for bounded generators of the approximate flows α^n. In fact, the relationship of $\mathcal{D}$ and various cores related to spectral subspaces were examined. Seddik [47] showed that Q is a normal operator which is invertible in $B(H)$ if the estimate $\|Q \otimes Q^{-1} + Q^{-1} \otimes Q\|_\lambda \leq \|Q\|\|Q^{-1}\| + \frac{1}{\|Q\|\|Q^{-1}\|}$ holds, such that $\|.\|_\lambda$ is the injective norm on the tensor product $B(H) \otimes B(H)$, when Q is invertible self-adjoint then the equation becomes an equality. Bonyo and Agure [7] characterized the norm of inner derivation on norm ideal to be equal to the quotient algebra and investigated them when they are implemented by normal and hyponormal operators on norm ideals. A hyponormal X is a bounded linear operator on a Hilbert space H if $X^*X - XX^* \geq 0$ and is normal if $X^*X = XX^*$. Bonyo and Agure [8] investigated the relation of the diameter of the numerical range of an operator $B \in B(H)$ and the norm of inner derivation implemented by B on a norm ideal J and considered the application of S-universality to the relation. Bonyo and Agure [6] defined inner derivations implemented by A, B respectively on $B(H)$ by $\delta_A(Y) = AY - YA$, $\delta_B(Y) = BY - YB$ and generalized derivation by $\delta_{A,B}(Y) = AY - YB \; \forall \; Y \in B(H)$. Further, a relationship between the norms of δ_A, δ_B and $\delta_{A,B}$ on $B(H)$ was established, specifically when the operators A, B are S-universal. Ber, Sukochev [5] showed that for every self-adjoint element $b \in S(N)$ a scalar $\lambda_0 \in \mathbb{R}$ exists such that $\forall \; \varepsilon > 0$, then there exists a unital element u_ε from N satisfy $|[b, u_\varepsilon]| \geq (1-\varepsilon)|b - \lambda_0 1|$. From this result a consequence is that for any derivation δ on N with the range on an ideal $I \subseteq N$ the derivation δ is inner i.e $\delta(.) = \delta_a(.) = [a, .]$ and $a \in I$. Pablo, Jussi, Mikael [42] provided theoretic estimate of two functions for the essential norm as a composition operator C_φ that acts on the space $BMOA$; one in terms of the n-th power φ^n denoted by φ and the other involved the Nevanlinna counting function. The research of [20] introduced a new type of norm for semimartangles, the defined norm of quasimartangales and then characterized the square integrable semimartangales. In [4] the author gave the result on lower bound of the norms for finite dimensional operators. The work of [14] determined the norm of two-sided symmetric operator in an algebra. More precisely, the lower bound of the operator using injective tensor norm was investigated. Further, the inner derivation norm on irreducible C^*-algebra was determined and Stampfli's [49] result for these algebras was confirmed.

2. Preliminaries

This section provides the basic concepts which are useful in the sequel.

Definition 1 ([1], Definition 1.5). *A Banach $*$-algebra $\mathcal{T}$ is called C^*-algebra if* $\|tt^*\| = \|t\|^2,\ \forall\ t \in \mathcal{T}$.

Definition 2 ([37], Definition 2.1). *Elementary operator $T : B(H) \to B(H)$ is defined by $T_{D_i,E_i}(X) = \sum_{i=1}^{n} D_i\ X\ E_i\ \forall\ X \in B(H)$ and $\forall\ D_i, E_i$ fixed in $B(H)$ where $i = 1, ..., n$. For $B(H)$, we define the particular elementary operators as below:*

(i). *Left multiplication operator $L_D : B(H) \to B(H)$ by $L_D(X) = DX,\ \forall\ X \in B(H)$.*

(ii). *Right multiplication operator $R_E : B(H) \to B(H)$ by $R_E(X) = XE,\ \forall\ X \in B(H)$.*

(iii). *Generalized derivation (implemented by D, E) by $\delta_{D,E} = L_D - R_E$.*

(iv). *Inner derivation (implemented by D) by $\delta_D(X) = DX - XD$.*

(v). *Basic elementary operator (implemented by D, E) by $M_{D,E}(X) = DXE,\ \forall\ X \in B(H)$.*

(vi). *Jordan elementary operator (implemented by D, E) by $U_{D,E}(X) = DXE + EXD,\ \forall\ X \in B(H)$.*

Definition 3 ([49], Definition 2.3). *A derivation is a map $D : U \to U$ satisfying $D(fg) = fD(g) + D(f)g$ for all $f, g \in U$.*

Definition 4 ([39], Definition 1.2). *The maximal numerical range of an operator S is defined by: $W_0(S) = \{\beta : \langle St, t\rangle \to \beta$, where $\|t\| = 1$ and $\|St\| \to \|S\|\}$.*

Definition 5 ([35], Definition 2.1). *An operator K is norm-attainable if $t \in H$ exists which is a unit vector such that $\|Kt\| = \|K\|$. Moreover, it is self-adjoint if $K = K^*$.*

3. Main Results

In this section, we give results on norm-attainability conditions an norm estimates for derivations. We begin with the following proposition.

Proposition 6. *Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. $A \in B(H)$ is norm-attainable if and only if its adjoint $A^* \in B(H)$ is norm-attainable.*

Proof. Given $A \in B(H)$ is norm-attainable then we need to show that $A^* \in B(H)$ is norm-attainable. If $A \in B(H)$ is norm-attainable then by definition of norm-attainability there exists a unit vector $x \in H$ with $\|x\| = 1$ such that $\|Ax\| = \|A\|$. That is, $\|AA^*x\| = \|A^2x\|$. Let $\eta = \frac{Ax}{\|A\|}$, then η is a unit vector such that $\|\eta\| = 1$ this implies that $\|A^*\eta\| = \|A\| = \|A^*\|$. Hence, A^* is norm-attainable. □

The next result gives norm-attainability conditions for operators via the essential numerical range. An analogy of the same can be found in [37].

Proposition 7. *Let $A \in B(H)$, $\lambda \in W_{ess}(A)$ and $\eta > 0$. Then there exists $A_0 \in B(H)$ such that $\|A\| = \|A_0\|$ with $\|A - A_0\| > \eta$.*

Proof. See [37] for the proof. □

Remark 8. *The set of all norm-attainable operators is denoted by $NA(H)$, the set of all norm-attainable self adjoint operators is denoted by $NA^*(H)$ and the set of all norm-attainable elementary operators is denoted by $\mathcal{E}_{NA}[B(H)]$.*

At this point, we consider norm-attainability in a general set up. We begin with the following proposition.

Proposition 9. *Let D be the unit disc of a complex Hilbert space H and $A : H \to H$ be compact and self adjoint. Then there exists $x \in D$ such that $\|Ax\| = \|A\|$.*

Proof. By the definition of usual norm, we have $\|A\| = \sup_{x \in D} \|Ax\|$. So, there exists a sequence $x_1, x_2, ..., x_n$ in D such that $\|Ax_n\| = \|A\|$. But A is compact so let $y_0 = \lim_{n\to\infty} Ax_n$ exist in H. Suppose $Y = span\{x_1, x_2\}$, then it is a closed subspace of H. If we pick a subsequence x_{n_k} of x_n, then it converges weakly to x and we have done $\langle x, x\rangle = \lim_{k\to\infty}\langle x_{n_k}, x\rangle$ and $|\langle x_{n_k}, x\rangle| \leq \|x_{n_k}\|\|x\| = 1$ for all k. Therefore, $\|x\| \leq 1$ but we cannot have $\|x\| < 1$ since then $\|Ax\| = \|A\|\|x\| < \|T\|$ which is a contradiction. Thus, $\|x\| = 1$ i.e $x \in D$. Hence, the existence of x is shown and thus completes the proof. □

At this point, we consider q-normality and q-norm-attainability.

Lemma 10. *Let $A \in NA(H)$ then A is q-norm-attainable if it is q-normal.*

Proof. Let $A \in NA(H)$ be q-normal i.e $A^qA^* = A^*A^q$. Raising A^* to power q and using it to replace A^* we have $A^q(A^*)^q = (A^*)^qA^q$. This shows that A^q is normal. Now $A^qA^* = A^*A^q$ by Fuglede property. Therefore, A is q-normal. However, $A \in NA(H)$ and A^q is normal so it follows that there exists a unit vector $x \in H$ such that $\|A^qx\| = \|A^q\|$, for any $q \in \mathbb{N}$. Hence, A^q is norm-attainable. □

Remark 11. *Every norm-attainable operator and every self adjoint operator is q-norm-attainable and q-normal for any $q \in \mathbb{N}$. However, the converse need not be true in general see* [66].

Lemma 12. *Let $NA_q(H)$ be the set of all q-norm-attainable operators on H. Then $NA_q(H)$is a closed subset of $NA(H)$ which is algebraic if and only if for any $A \in NA(H)$, A is q-normal.*

Proof. Let A be q-normal and pick $\lambda \in \mathbb{K}$. By premultiplying by λ and postmultiplying by q as a power on the normal A we have $(\lambda A)^q(\lambda A)^* = (\lambda A)^*(\lambda A)^q$. This proves the normality of λA. Now if $A \in NA(H)$ then the converse is true if we take limits over a sequence of vectors in H and also by Proposition 9. Therefore, A is a q-normal. □

Theorem 13. *Let $A \in NA_q(H)$. Then the following conditions are true.*

(i). A^* is q-norm-attainable.
(ii). VAV^* is q-normal, for a unitary operator $V \in NA_q(H)$.
(iii). A^{-1} is q-norm-attainable if it exists.
(iv). $A_0 = A/G$ is q-norm-attainable for some G which is a uniformly invariable subspace of H which reduces to A.
(v). A_0 is uniformly equivalent to A implies A_0 is norm-attainable.

Proof. (i). Since $A \in NA_q(H)$, then from Lemma 10, A^q is q-norm-attainable and so $(A^*)^q$ is norm-attainable. Consequently, A^* is q-norm-attainable.
(ii). Since V is unitary then $VV^* = V^*V = I$, where I is the identity operator. By definition of norm-attainability and Lemma 10 we obtain the desired results.
(iii). If A^{-1} exists then since A is q-norm-attainable, A^q is q-norm-attainable. Now since A is q-norm-attainable then by Lemma 10 A^q is q-norm-attainable. But $(A^q)^{-1} = (A^{-1})^q$ is q-norm-attainable. So A^{-1} is q-norm-attainable.
(iv). Follows from the fact that G invariant under A.
(v). Follows from (iii) since V is unitary.

□

Corollary 14. *Let $A^q, A_0^q \in NA_q(H)$ be commuting operators, then $A, A_0 \in NA_q(H)$.*

Proof. Since $A^q, A_0^q \in NA_q(H)$ are commuting then A, A_0 are commuting normal operators. By supraposinormality of operators in dense classes we have $A, A_0 \in NA_q(H)$ and hence are norm-attainable. Indeed, $A^qA_0^q = (AA_0)^q = (A_0A)^q$ which is normal and norm-attainable. Hence, $A, A_0 \in NA_q(H)$. □

Remark 15. *Not all q-norm-attainable operators are q-normal. Thus, the following example shows that the two commuting q-normal operators need not be q-normal.*

Example 16. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. *Now* $A + A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $(A + A_0)^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ *are not normal. So $A + A_0$ is not 2-normal. We note that A_0 is self-adjoint.*

Lemma 17. *The sum of norm-attainable operators is norm-attainable.*

Proof. Consider $A, B \in B(H)$. We need to show that the sum of A and B is norm-attainable. For A, B to be norm-attainable then there exists a unit vector $x \in H$ such that $\|x\| = 1$, $\|(A+B)x\| = \|Ax+Bx\| = \|A+B\| = \|A\|+\|B\|$. Since $\|Ax+Bx\| \leq \|Ax\|+\|Bx\| \leq \|A\|+\|Bx\| \leq \|A\|+\|B\|$ then for an orthonormal sequence $x_n \in H$ we have $\lim_{n\to\infty}(\|Ax_n + Bx_n\|) = \|Ax + Bx\|$. But since A and B are norm-attainable we have $\|Ax+Bx\| = \|(A+B)x\| = \|A+B\|$ is norm-attainable. □

Theorem 18. *A norm-attainable operator perturbed by an identity operators is norm-attainable.*

Proof. Let $B \in B(H)$ be norm-attainable. Since B is norm-attainable then there exists a unit vector $x_0 \in H$, an identity $I \in B(H)$ and for every $\varepsilon > 0$ we have $\|(BI)x_0\| \leq \|BIx_0\| + \varepsilon \leq \|B\|\|I\|\|x_0\| + \varepsilon$. Since ε is arbitrary then it follows that $\|(BI)x_0\| \leq \|B\|\|I\|\|x_0\| = \|B\|$. Hence, $\|(BI)x_0\| = \|B\|$. □

At this point, we consider norm-attainability for elementary operators. We begin with inner derivations.

Lemma 19. *Let $\delta_A \in \mathcal{E}[B(H)]$, then δ_A is norm-attainable if there exists a unit vector $x_0 \in H$, $A \in NA(H)$ and $\langle Ax_0, x_0\rangle \in W_{ess}(A)$.*

Proof. For an operator $A \in NA(H)$ we know that an operator is norm-attainable via essential numerical range from proposition 4.2. Now, we need to show that $\delta_A \in \mathcal{E}[B(H)]$ is norm-attainable. By the definition of inner derivation, $\delta_A = AY_0 - Y_0A$. Since A is norm-attainable then there exists a unit vector $x_0 \in H$ such that $\|x_0\| = 1$, $\|Ax_0\| = \|A\|$. By orthogonality let y_0 satisfy $y_0 \perp \{Ax_0, x_0\}$ and a contractive Y_0 be defined as a linear transformation $Y_0 : x_0 \to x_0$ with $Ax_0 \to -Ax_0$ as $y_0 \to 0$. Since Y_0 is a bounded linear operator on H, then by norm-attainability $\|Y_0x_0\| = \|Y_0\| = 1$ and

$$\|AY_0x_0 - Y_0Ax_0\| = \|Ax_0 - (-Ax_0)\| = 2\|A\|.$$

It follows from Lemma 3.1 in [49] that $\|\delta_A\| = 2\|A\|$. By the inner product $\langle Ax_0, x_0\rangle = 0 \in W_{ess}(A)$, it follows that $\|\delta_A\| = 2\|A\|$. Therefore, $\|AY_0 - Y_0A\| = 2\|A\| = \|\delta_A\|$. Hence, δ_A is norm-attainable. □

Lemma 20. *Let $A, A_0 \in B(H)$. If there exists unit vectors y and y_0 on H such that A, A_0 are norm-attainable then δ_{A,A_0} is also norm-attainable.*

Proof. Given the operators $A, A_0 \in B(H)$ are norm-attainable then we need to show that δ_{A,A_0} is also norm-attainable. We define the generalized derivation by $\delta_{A,A_0} = AY - YA_0$. Since A, A_0 are norm-attainable then there exists unit vectors y and y_0 on H such that $\|y\| = \|y_0\| = 1$, $\|Ay\| = \|A\|$ and $\|A_0y_0\| = \|A_0\|$. By linear dependence of vectors, if y and Ay are linearly dependent then we have $\|Ay\| = \eta\|A\|y$ where $|\eta| = 1$ and $|\langle Ay, y\rangle| = \|A\|$. It follows that $|\langle A_0y_0, y_0\rangle| = \|A_0\|$ which implies that $\|A_0y_0\| = \phi\|A_0\|y_0$ and $|\phi| = 1$. Therefore, $\langle \frac{A_0y_0}{\|A_0\|}, y_0\rangle = \phi = -\langle \frac{Ay}{\|A\|}, y\rangle = -\eta$. If Y is defined as $Y : y \to y_0$ and $y_0 \to 0$, $\|Y\| = 1$ then $(AY - YA_0)y_0 = \phi(\|A\| + \|A_0\|)y_0$ which implies $\|AY - YA_0\| = \|(AY - YA_0)y_0\| = \|A\| + \|A_0\| = \|\delta_{A,A_0}\|$. Hence, δ_{A,A_0} is norm-attainable. □

Lemma 21. *Every inner derivation is norm-attainable if and only if it is self-adjoint.*

Proof. Let $\delta_A \in B(H)$ be norm-attainable then we show that $\delta_A = \delta_A^*$. Now since $\delta_A \in B(H)$ is norm-attainable then there exists a contraction $Y \in B(H)$ such that $\|\delta_A Y\| = \|\delta_A\|$. That is, $\|\delta_A^*\delta_A Y\| = \|\delta_A^2 Y\|$. Let $\eta \in H$ be defined as $\eta = \frac{\delta_A}{\|\delta_A\|}$ then η is contractive such that $\|\delta_A^*\eta\| = \|\delta_A\| = \|\delta_A^*\|$. Hence, δ_A is self-adjoint. Conversely, let δ_A be self-adjoint. Now since δ_A^* is norm-attainable from the first part, then there exists a contractive $M \in B(H)$ such that $\|\delta_A^* M\| = \|\delta_A^*\|$, i.e $\|\delta_A\delta_A^* M\| = \|\delta_A^2 M\|$. Let ζ be denoted by $\zeta = \frac{\delta_A^*}{\|\delta_A^*\|}$ where $\|\zeta\| = 1$ such that $\|\delta_A\zeta\| = \|\delta_A^*\| = \|\delta_A\|$. Hence, δ_A is norm-attainable. □

Lemma 22. *Every generalized derivation is norm-attainable if and only if it is implemented by orthogonal projections.*

Proof. Let $A, A_0 \in B(H)$ be orthogonal projections. Indeed, to show that a generalized derivation is implemented by orthogonal projections A and A_0, it is enough to show that it is self-adjoint if and only if it is normal as proved in [22]. Let $\delta_{A,A_0} : B(H) \to B(H)$ be bounded linear operator on $B(H)$. Then exists a unique bounded linear operator $\delta_{A,A_0}^* : B(H) \to B(H)$ such that $\langle \delta_{A,A_0}X, Y\rangle = \langle X, \delta_{A,A_0}^* Y\rangle$, for all $X, Y \in B(H)$. Now,

$$\begin{aligned}\|\delta_{A,A_0}^* Y\| &= \sup_{\|X\|=1} \langle \delta_{A,A_0}X, Y\rangle \\ &\leq \sup_{\|X\|=\|Y\|=1} \|\delta_{A,A_0}\|\|X\|\|Y\| \\ &= \|\delta_{A,A_0}\|\end{aligned}$$

So, we conclude that δ_{A,A_0}^* is norm-attainable. Conversely, let δ_{A,A_0} be norm-attainable. We need to show that it is implemented by orthogonal projections. This follows immediately from [22] and this completes the proof. □

At this point, we give results on upper norm estimates for norm-attainable derivations. We consider both inner derivations and generalized derivations. We begin with the following proposition.

Proposition 23. *Let $A, B \in NA(H)$ and $\delta_{A,B}$ be bounded then $\|\delta_{A,B}\| \leq \|A\| + \|B\|$.*

Proof. Since $\delta_{A,B}$ is bounded then for fixed $A, B \in NA(H)$ we have $\|\delta_{A,B}(X)\| \leq \|AX - XB\| \leq \|AX\| + \|XB\| \leq \|A\|\|X\| + \|X\|\|B\|$. Let X be of norm 1 and take suprеmum over $X \in NA(H)$ then $\|\delta_{A,B}\| \leq \|A\| + \|B\|$. □

Remark 24. *If $A = B$ then $\|\delta_A\| \leq 2\|A\|$.*

Next, we consider upper bounds in the unit ball of $NA(H)$ denoted by $[NA(H)]_0$.

Lemma 25. *Let $[NA(H)]_0$ be the unit ball of $NA(H)$ and S be a fixed element of $NA(H)$. Let $X \in [NA(H)]_0$ then $\|\delta_S|_{[NA(H)]_0}\| \leq 2d(S)$.*

Proof. Since $X \in [NA(H)]_0$ has norm 1 then we have $\|\delta_S|_{[NA(H)]_0}(X)\| = \|SX - XS\|_{[NA(H)]_0} = \|(S-\lambda)X - X(S-\lambda)\|_{[NA(H)]_0} \leq \|S-\lambda\|\|X\|_{[NA(H)]_0} + \|X\|\|S-\lambda\|_{[NA(H)]_0}$. Taking the supremum over $[NA(H)]_0$, we obtain $\|\delta_S|_{[NA(H)]_0}\| \leq 2\|S-\lambda\|$ and considering the infimum over $\lambda \in \mathbb{C}$ we obtain $\|\delta_S|_{[NA(H)]_0}\| \leq 2\inf_{\lambda\in\mathbb{C}} \|S-\lambda\| = 2d(S)$. □

Remark 26. *The restriction of $\delta_A|_{[NA(H)]_0}$ i.e δ_A to $[NA(H)]_0$ is a bounded linear operator.*

Next we give an extension of Lemma 25 to a generalized derivation in the following theorem.

Theorem 27. *Let S, S_0 be fixed elements of $NA(H)$ then $\|\delta_{S,S_0}|_{[NA(H)]_0}\| \leq \|\delta_{S,S_0}\|$.*

Proof. Since $X \in [NA(H)]_0$ has norm 1 then we have $\|\delta_{S,S_0}|_{[NA(H)]_0}(X)\| = \|SX - XS_0\|$. Following proof of lemma 25 anologously we have

$$\|\delta_{S,S_0}|_{[NA(H)]_0}(X)\| \leq \|S-\lambda\|\|X\|_{[NA(H)]_0} + \|X\|\|S_0-\lambda\|_{[NA(H)]_0}.$$

Taking the supremum over $X \in [NA(H)]_0$ we obtain

$\|\delta_{S,S_0}|_{[NA(H)]_0}\| \leq \inf_{\lambda\in\mathbb{C}}(\|S-\lambda\| + \|S_0-\lambda\|) = \|\delta_{S,S_0}\|$. □

Corollary 28. *Every generalized derivation δ_{S,S_0} is norm-bounded.*

Proof. This follows immediately from [49] and from Theorem 27. This completes the proof. □

Now, we consider lower bounds for norms of derivations. We begin the following proposition on generalized derivation.

Proposition 29. *Let S, S_0 be fixed elements of $NA(H)$ then*

$\|\delta_{S,S_0}|_{[NA(H)]_0}\| \geq \|S\| + \|S_0\|$.

Proof. Let η, ξ and x be unit vectors in H and ϕ, φ be positive linear functionals such that $\phi \otimes \eta : H \to \mathbb{C}$ and $\varphi \otimes \xi : H \to \mathbb{C}$ be of rank 1 defined as $(\phi \otimes \eta)x = \phi(x)\eta$ and $(\varphi \otimes \xi)x = \varphi(x)\xi, \forall x \in H, \|x\| = 1$. Now we have that $\|(\phi\otimes\eta)x\| = \sup\{\|(\phi\otimes\eta)x\|, \|x\| = 1\} = |\phi(x)| = |\phi|$. Similarly, we have $\|(\varphi \otimes \xi)x\| = \|\varphi\|$. Letting $S = \phi \otimes \eta$ and $S_0 = \varphi \otimes \xi$ then $\|S\| = \|\phi\|$ and $\|S_0\| = \|\varphi\|$. Now from Corollary 28 we have that every generalized derivation is norm-bounded this implies that $\|\delta_{S,S_0}|_{[NA(H)]_0}(X)\| \geq \|\delta_{S,S_0}(X)\|$ where $X \in [NA(H)]_0$. Therefore, $\|\delta_{S,S_0}|_{[NA(H)]_0}\|^2 \geq \|SX - XS_0\|^2$ implying that $\|\delta_{S,S_0}|_{[NA(H)]_0}\|^2 \geq [\|S\| + \|S_0\|]^2$. Taking positive square root on both sides we obtain $\|\delta_{S,S_0}|_{[NA(H)]_0}\| = \|\delta_{S,S_0}\| \geq \|S\| + \|S_0\|$. □

Remark 30. *If $S = S_0$ then $\|\delta_{S,S_0}\| = \|\delta_S\| \geq 2\|S\|$.*

Remark 31. *From Theorem 27 and Proposition 3 it is easy to see that $\|\delta_{S,S_0}\| = \|S\| + \|S_0\|$ and hence $\|\delta_S\| = 2\|S\|$.*

Theorem 32. *Let $S, S_0 \in NA(H)$ and $\alpha_1 \in W_0(S)$ and $\alpha_2 \in W_0(S_0)$. Then $\|\delta_{S,S_0}\| \geq (\|S\|^2 - |\alpha_1|^2)^{1/2} + (\|S_0\|^2 - |\alpha_2|^2)^{1/2}$.*

Proof. By definition of $W_0(S)$ we have $x_n \in H$ such that $\|Sx_n\| = \|S\|$ and $\langle Sx_n, x_n\rangle \to \alpha_1$ for $\alpha_1 \in W_0(S)$. This argument follows for $W_0(S_0)$ and $\alpha_2 \in W_0(S_0)$. Let $Sx_n = \delta_n x_n + \beta_n y_n$ so $S_0 x_n = \sigma_n x_n + \lambda_n y_n$ where $\langle x_n, y_n\rangle = 0, \|y_n\| = 1$. Take $U_n x_n = x_n$ and $U_n y_n = -y_n$ for $U_n = 0$ in $\{x_n, y_n\}$. Then $\|SU_n x_n - U_n S_0 x_n\| = \|\delta_n + \beta_n\| \leq |\delta_n| + |\beta_n|$. But $|\delta_n| + |\beta_n| \geq (\|S\|^2 - |\delta_n|^2)^{1/2} - \xi_n + (\|S_0\|^2 - |\beta_n|^2)^{1/2} - \xi_n)$. Since ξ_n is arbitrary and letting $n \to \infty$, so it follows that $\|\delta_{S,S_0}\| \geq \|(SU_n - U_n S_0)x_n\| = |\delta_n| + |\beta_n| = (\|S\|^2 - |\alpha_1|^2)^{1/2} + (\|S_0\|^2 - |\alpha_2|^2)^{1/2}$. □

Corollary 33. *Let $\langle x_n, y_n\rangle = 0$ then $0 \in W_0(S)$ and if $0 \in W_0(S_0)$ then $\|\delta_{S,S_0}\| \geq \|S\| + \|S_0\|$.*

Proof. Follows immediately from definition of $W_0(S)$ and the Theorem 32. □

4. Conclusion

In this paper, we have given a detailed characterization of operators in terms of norm-attainability conditions and norm estimates for in Banach algebras. In particular, we have established norm-attainability conditions for the derivations and also given the norm bounds in the norm-attainable classes.

References

[1] R.J. Archbold, On the norm of an inner derivation of a C^*-algebra, Math. Proc. Camb. Phil. Soc. 84 (1978) 273–291. https://doi.org/10.1017/s0305004100055109.

[2] N.M. Abolfazl, On the norm of Jordan ∗-derivations, Khaayyam J. Math. 6 (2020) 104-107. https://doi.org/10.22034/kjm.2019.97176.

[3] J. Anderson, On normal derivations, Proc. Amer. Math. Soc. 38 (1973) 135–140. https://doi.org/10.1090/s0002-9939-1973-0312313-6.

[4] M. Barraa, M. Boumazgour, Inner derivations and norm equality, Proc. Amer. Math. Soc. 130 (2001) 471-476. https://www.jstor.org/stable/2699643.

[5] A.F. Ber, F.A. Sukochev, Commutator estimates in W^*-factors, Trans. Amer. Math. Soc. 364 (2012) 5571–5587. https://doi.org/10.1090/s0002-9947-2012-05568-1.

[6] J.O. Bonyo, J.O. Agure, Norms of derivations implemented by S-universal operators, Int. J. Math. Anal. 5 (2011) 215-222.

[7] J.O. Bonyo, J.O. Agure, Norm of a derivation and hyponormal operators, Int. J. Math. Anal. 4 (2010) 687-693.

[8] J.O. Bonyo, J.O. Agure, Norms of Inner derivations on norm ideals, Int. J. Math. Anal. 4 (2010) 695-701.

[9] M. Bresar, B. Zalar, On the structure of Jordan ∗-derivations, Colloq. Math. 63 (1992) 163-171.

[10] M. Cabrera, A. Rodriguez, Nondegenerately ultraprint Jordan Banach algebras, Proc. London Math. Soc. 69 (1994) 576-604.

[11] A.A. Charles, W. Steve, Compact and weakly compact derivations of C∗-algebras, Pac. J. Math. 85 (1979) 79-96.

[12] G. Clifford, Dynamics of generalized derivations and elementary operators, (2017), arXiv:1605.07409v2 [math.FA]. https://arxiv.org/abs/1605.07409v2.

[13] C.K. Li, Lecture notes on numerical range, 2005. http://www.math.wm.edu/~ckli/nrnote.

[14] D.R. Jocić, Norm inequalities for self-adjoint derivations, J. Funct. Anal. 145 (1997) 24–34. https://doi.org/10.1006/jfan.1996.3004.

[15] D.W.B. Somerset, The inner derivations and the primitive ideal space of a C∗-algebra, J. Oper. Theory, 29 (1993) 307-321. https://www.jstor.org/stable/24714573.

[16] C. Erik, Extensions of derivations II, Math. Scand. 50 (1982) 111-122.

[17] F.P. Boca, A. Zaharescu, Norm estimates of almost Mathieu operators, J. Funct. Anal. 220 (2005) 76–96. https://doi.org/10.1016/j.jfa.2004.09.013.

[18] P. Gajendragadkar, Norm of a derivation on a von Neumann algebra, Trans. Amer. Math. Soc. 170 (1972) 165–165. https://doi.org/10.1090/s0002-9947-1972-0305090-x.

[19] H.K. Du, Y.Q. Wang, G.B. Gao, Norms of elementary operators, Proc. Amer. Math. Soc. 136 (2008) 1337-1348. https://doi.org/10.1090/S0002-9939-07-09112-5.

[20] B. Johnson, Characterization and norms of derivations on von Neumann algebras, in: P. de la Harpe (Ed.), Algèbres d'Opérateurs, Springer Berlin Heidelberg, Berlin, Heidelberg, 1979: pp. 228–236. https://doi.org/10.1007/BFb0062619.

[21] B.E. Johnson, Norms of derivations on $\mathcal{L}(\mathcal{X})$, Pac. J. Math. 38 (1971) 465-469. https://doi.org/10.2140/pjm.1971.38.465.

[22] E. Kreyszig, Introduction Functional Analysis with Applications, Book.Canada publications, Toronto, 1978.

[23] N.P. Landsman, C∗-algebras and quantum mechanics. Lecture notes, 1998.

[24] J. Kyle, Numerical ranges of derivations, Proc. Edinburgh Math. Soc. 21 (1978) 33-39. https://doi.org/10.1017/S0013091500015856.

[25] J. Kyle, Norms of derivations, J. London Math. Soc. 16 (1977) 297-312. https://doi.org/10.1112/jlms/s2-16.2.297.

[26] F. Kittaneh, Normal derivations in norm ideals, Proc. Amer. Math. Soc. 123 (1995) 1779-1785. https://doi.org/10.2307/2160991.

[27] G. Lumer, Complex methods and the estimation of operator norms and spectra from real numerical ranges, J. Funct. Anal. 10 (1972) 482-495. https://doi.org/10.1016/0022-1236(72)90043-2.

[28] B. Matej, On distance of the composition of two derivations to the generalized derivations, Glasgow Math. J. 33 (1991) 89-93. https://doi.org/10.1017/S0017089500008077.

[29] M. Mathieu, More properties of the product of two derivations of a C^*-algebra, Bull. Austral. Math. Soc. 42 (1990) 115-120. https://doi.org/10.1017/S0004972700028203.

[30] M. Mathieu, Elementary operators on Calkin algebras, Irish Math. Soc. Bull. 46 (2001) 33-44.

[31] S. Mecheri, The Gateaux derivative orthogonality in C_∞. Lecture notes, 1991.

[32] R.E. Megginson, An introduction to Banach space theory, Springer-Verlag, New York, 1998.

[33] M. Arsenovic, D. Keckic, Elementary operators on Banach algebras and Fourier transform, Stud. Math. 173 (2006) 149-166.

[34] N.B. Okelo, On orthogonality of elementary operators in norm-attainable classes, Taiwan. J. Math. 24 (2020) 119-130. https://doi.org/10.11650/tjm/190502.

[35] N.B. Okelo, J.O. Agure, D.O. Ambogo, Norms of elementary operators and characterization of norm-attainable operators, Int. J. Math. Anal. 4 (2010) 1197-1204.

[36] N.B. Okelo, Norm-attainability and range-kernel orthogonality of elementary operators, Commun. Adv. Math. Sci. 1 (2018) 91–98. https://doi.org/10.33434/cams.442556.

[37] N.B. Okelo, The norm-attainability of some elementary operators, Appl. Math. E-Notes, 13 (2013) 1-7.

[38] N.B. Okelo, J.O. Agure, P.O. Oleche, Certain conditions for norm-attainability of elementary operators and derivations, Int. J. Math. Soft Comput. 3 (2013) 53-59.

[39] N.B. Okelo, J.O. Agure, A two-sided multiplication operator norm, Gen. Math. Notes, 2 (2011) 18-23.

[40] N.B. Okelo, Fixed points approximation for nonexpansive operators in Hilbert spaces, Int. J. Open Problems Comput. Math. 14 (2021) 1-5.

[41] N.B. Okelo, Characterization of absolutely norm attaining compact hyponormal operators, Proc. Int. Math. Sci. 2 (2020) 96-102. https://doi.org/10.47086/pims.689633.

[42] P. Galindo, J. Laitila, M. Lindström, Essential norm estimates for composition operators on BMOA, J. Funct. Anal. 265 (2013) 629–643. https://doi.org/10.1016/j.jfa.2013.05.002.

[43] A. Pinchuck, Functional analysis notes, Springer Verlag, New York, 2011.

[44] R.M. Timoney, Norms and CB norms of Jordan elementary operators, Bull. Sci. Math. 127 (2003) 597-609. https://doi.org/10.1016/S0007-4497(03)00046-0.

[45] R.M. Timoney, Computing the norms of elementary operators, Illinois J. Math. 47 (2003) 1207-1226. https://doi.org/10.1215/ijm/1258138100.

[46] A. Seddik, On the numerical range and norm of elementary operators, Linear Multilinear Algebra. 52 (2004) 293–302. https://doi.org/10.1080/0308108031000122515.

[47] A. Seddik, On the injective norm and characterization of some subclasses of normal operators by inequalities or equalities, J. Math. Anal. Appl. 351 (2009) 277–284. https://doi.org/10.1016/j.jmaa.2008.10.008.

[48] L.L. Stacho, B. Zalar, On the norm of Jordan elementary operators in standard operator algebra, Publ. Math. Debrecen, 49 (1996) 127-134.

[49] J. Stampfli, The norm of a derivation, Pac. J. Math. 33 (1970) 737–747. https://doi.org/10.2140/pjm.1970.33.737.

[50] J. Stampfli, On selfadjoint derivation ranges, Pac. J. Math. 82 (1979) 257–277. https://doi.org/10.2140/pjm.1979.82.257.

[51] V. Runde, Automatic continuity of derivations and epimorphisms, Pac. J. Math. 147 (1991) 365–374. https://doi.org/10.2140/pjm.1991.147.365.

[52] A.W. Wickstead, Norms of basic elementary operators on algebras of regular operators, Proc. Amer. Math. Soc. 143 (2015) 5275–5280. https://doi.org/10.1090/proc/12664.

[53] Y.C. Kim, T. Sugawa, Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions, Proc. Edinburgh Math. Soc. 49 (2006) 131-143. https://doi.org/10.1017/s0013091504000306.