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Woven $K-g-$Fusion Frames in Hilbert C^*-Modules

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Abstract. In this paper, we introduced the notion of woven $K-g-$fusion frames in Hilbert C^*-modules. We present necessary and sufficient conditions for these woven and also construct them by linear bounded operator. Finally we study perturbation of weaving $K-g-$fusion frames.

1. Introduction

Basis is one of the most important concepts in Vector Spaces study. However, Frames generalise orthonormal bases and were introduced by Duffin and Schaefer [3] in 1952 to analyse some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [5] for signal processing. In 2000, Frank-larson [4] introduced the concept of frames in Hilbet C^*-modules as a generalization of frames in Hilbert spaces. The basic idea was to consider modules over C^*-algebras of linear spaces and to allow the inner product to take values in the C^*-algebras [6]. Many generalizations of the concept of frame have been defined in Hilbert C^*-modules [7,9,11–16].

Throughout this paper, H is considered to be a countably generated Hilbert C^*-module. Let $\{H_j\}_{j\in\mathbb{J}}$ are the collection of Hilbert C^*-module and $\{W_j\}_{j\in\mathbb{J}}$ is a collection of closed orthogonally complemented submodules of H, where $\mathbb{J}$ be finite or countable index set. $End^*_{\mathcal{A}}(H,H_j)$ is a set of all adjointable operator from H to H_j. In particular $End^*_{\mathcal{A}}(H)$ denote the set of all adjointable operators on H. P_{W_j} denote the orthogonal projection onto the closed submodule orthogonally

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complemented W_j of H. Define the module

$$l^2(\{H_j\}_{j\in\mathbb{J}}) = \{\{f_j\}_{j\in\mathbb{J}} : f_j \in H_j, \|\sum_{j\in\mathbb{J}}\langle f_j, f_j\rangle\| < \infty\}$$

with $\mathcal{A}-$valued inner product $\langle f, g\rangle = \sum_{j\in\mathbb{J}}\langle f_j, g_j\rangle$, where $f = \{f_j\}_{j\in\mathbb{J}}$ and $g = \{g_j\}_{j\in\mathbb{J}}$, clearly $l^2(\{H_j\}_{j\in\mathbb{J}})$ is a Hilbert $\mathcal{A}-$module.

Definition 1.1. [8] Let $\mathcal{A}$ be a unital C^*-algebra and H be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and H are compatible. H is a pre-Hilbert $\mathcal{A}$-module if H is equipped with an $\mathcal{A}$-valued inner product $\langle ., .\rangle : H \times H \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle f, f\rangle \geq 0$ for all $f \in H$ and $\langle f, f\rangle = 0$ if and only if $f = 0$.
(ii) $\langle af + g, h\rangle = a\langle f, h\rangle + \langle g, h\rangle$ for all $a \in \mathcal{A}$ and $f, g, h \in H$.
(iii) $\langle f, g\rangle = \langle g, f\rangle^*$ for all $f, g \in H$.

For $f \in H$, we define $||f|| = ||\langle f, f\rangle||^{\frac{1}{2}}$. If H is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in a C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on H is defined by $|f| = \langle f, f\rangle^{\frac{1}{2}}$ for $f \in H$.

Lemma 1.2. *[10] Let $\{W_j\}_{j\in J}$ be a sequence of orthogonally complemented closed submodules of H and $T \in End^*_{\mathcal{A}}(H)$ invertible, if $T^*TW_j \subset W_j$ for each $j \in J$, then $\{TW_j\}_{j\in J}$ is a sequence of orthogonally complemented closed submodules and $P_{W_j}T^* = P_{W_j}T^*P_{TW_j}$.*

Lemma 1.3. *[2]. Let H and K two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(H, K)$. Then the following statements are equivalent:*

(i) *T is surjective.*
(ii) *T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in K$.*
(iii) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle$ for all $x \in K$.*

Lemma 1.4. *[1]. Let U and H two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(U, H)$. Then:*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Definition 1.5. [10] Let $\{W_i\}_{i\in I}$ be a sequence of closed orthogonally complemented submodules of H, $\{v_i\}_{i\in I}$ be a familly of positive weights in $\mathcal{A}$, i.e., each v_i is a positive invertible element from the center of the C^*-algebra $\mathcal{A}$ and $\Lambda_i \in End^*_{\mathcal{A}}(H, H_i)$ for all $i \in I$. We say that $\Lambda = \{W_i, \Lambda_i, v_i\}_{i\in I}$ is a $g-$fusion frame for H if and only if there exists two constants $0 < A \leq B < \infty$ such that

$$A\langle x, x\rangle \leq \sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H. \tag{1.1}$$

The constants A and B are called the lower and upper bounds of $g-$fusion frame, respectively. If $A = B$ then Λ is called tight g-fusion frame and if $A = B = 1$ then we say Λ is a Parseval $g-$fusion frame. If Λ satisfies the inequality

$$\sum_{i\in I} v_i^2 \langle \Lambda_i P_{W_i} x, \Lambda_i P_{W_i} x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in H.$$

then it is called a $g-$fusion bessel sequence with bound B in H.

Definition 1.6. [10]

let $\Lambda = \{W_j, \Lambda_j, v_j\}_{j\in\mathbb{J}}$ be a $g-$fusion bessel sequence for H. Then the operator $T_\Lambda : l^2(\{H_j\}_{j\in\mathbb{J}}) \to H$ defined by

$$T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j P_{W_j} \Lambda_j^* f_j, \qquad \forall \{f_j\}_{j\in\mathbb{J}} \in l^2(\{H_j\}_{j\in\mathbb{J}}).$$

Is called synthesis operator. We say the adjoint U_Λ of the synthesis operator the analysis operator and it is defined by $U_\Lambda : \mathcal{H} \to l^2(\{H_j\}_{j\in\mathbb{J}})$ such that

$$U_\Lambda(f) = \{v_j \Lambda_j P_{W_j}(f)\}_{j\in\mathbb{J}}, \qquad \forall f \in H.$$

The operator $S_\Lambda : H \to H$ defined by

$$S_\Lambda f = T_\Lambda U_\Lambda f = \sum_{j\in\mathbb{J}} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f), \qquad \forall f \in H.$$

Is called $g-$fusion frame operator. It can be easily verify that

$$\langle S_\Lambda f, f\rangle = \sum_{j\in\mathbb{J}} v_j^2 \langle \Lambda_j P_{W_j}(f), \Lambda_j P_{W_j}(f)\rangle, \qquad \forall f \in H. \tag{1.2}$$

Furthermore, if Λ is a $g-$fusion frame with bounds A and B, then

$$A\langle f, f\rangle \leq \langle S_\Lambda f, f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in H.$$

It easy to see that the operator S_Λ is bounded, self-adjoint, positive, now we proof the inversibility of S_Λ. Let $f \in H$ we have

$$||U_\Lambda(f)|| = ||\{v_j \Lambda_j P_{W_j}(f)\}_{j\in I}|| = ||\sum_{j\in\mathbb{J}} v_j^2 \langle \Lambda_j P_{W_j}(f), \Lambda_j P_{W_j}(f)\rangle||^{\frac{1}{2}}.$$

Since Λ is $g-$fusion frame then

$$\sqrt{A}||\langle f, f\rangle||^{\frac{1}{2}} \leq ||U_\Lambda f||.$$

Then

$$\sqrt{A}||f|| \leq ||U_\Lambda f||.$$

Frome lemma 1.3, T_Λ is surjective and by lemma 1.4, $T_\Lambda U_\Lambda = S_\Lambda$ is invertible. We now, $AI_H \leq S_\Lambda \leq BI_H$ and this gives $B^{-1}I_H \leq S_\Lambda^{-1} \leq A^{-1}I_H$.

2. Woven $K-g-$fusion frames in Hilbert C^*-modules

Throughout this paper, $[m] = \{1, 2, ..., m\}$ for each $m > 1$, $\{W_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a collection of closed orthogonally complemented submodules of H, $\{v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a family of weights, $K \in End^*_{\mathcal{A}}(H)$ and $\{\Lambda_{ij}\}_{j\in\mathbb{J}, i\in[m]} \in End^*_{\mathcal{A}}(H, H_{ij})$ where H_{ij} are Hilbert $\mathcal{A}-$modules.

Definition 2.1. A family of $g-$fusion frames $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ for H is said to be $K-g-$fusion woven if there exist universal positive constants $0 < A \leq B$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$, the family $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\sigma_i, i\in[m]}$ is a $K-g-$fusion frame for H with bounds A and B.

In next theorem, we provide a necessary and sufficient condition for weaving $K-g-$fusion frames.

Theorem 2.2. *Assume that $\{W_j, \Lambda_j, v_j\}_{j\in\mathbb{J}}$ and $\{V_j, \theta_j, \mu_j\}_{j\in\mathbb{J}}$ are two $K-g-$fusion frames for H where $\Lambda_j \in End^*_{\mathcal{A}}(H, H_j)$ and $\theta_j \in End^*_{\mathcal{A}}(H, H_j)$ for any $j \in \mathbb{J}$, the following assertions are equivalent.*

(1) *$\{W_j, \Lambda_j, v_j\}_{j\in\mathbb{J}}$ and $\{V_j, \theta_j, \mu_j\}_{j\in\mathbb{J}}$ are $K-g-$fusion woven.*
(2) *there exists $\alpha > 0$ such that for each $\sigma \subset \mathbb{J}$ there exists a bounded linear operator*

$$\psi_\sigma : l_2^\sigma(\{H_j\}_{j\in\mathbb{J}}) \to H,$$
$$\psi_\sigma\{x_j\}_{j\in\mathbb{J}} = \sum_{j\in\sigma} v_j P_{W_j}\Lambda_j^* x_j + \sum_{j\in\sigma^c} \mu_j P_{V_j}\theta_j^* x_j,$$

such that $\alpha KK^* \leq \psi_\sigma\psi_\sigma^*$, where

$$l_2^\sigma(\{H_j\}_{j\in\mathbb{J}}) = \{\{x_j\}_{j\in\mathbb{J}} = \{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c} : f_j \in H_j, g_j \in H_j, \|\sum_{j\in\mathbb{J}}\langle x_j, x_j\rangle\| < \infty\}.$$

Proof. (1) $\implies$ (2): Suppose that A is an universal lower frame bound for $\{W_j, \Lambda_j, v_j\}_{j\in\mathbb{J}}$ and $\{V_j, \theta_j, \mu_j\}_{j\in\mathbb{J}}$. Choose $\alpha = A$ and $\psi_\sigma = T_\sigma$ for every $\sigma \subset \mathbb{J}$, where T_σ is the synthesis operator of $\{W_j, \Lambda_j, v_j\}_{j\in\sigma} \cup \{V_j, \theta_j, \mu_j\}_{j\in\sigma^c}$. Then, for any $\{x_j\}_{j\in\mathbb{J}} \in l_2^\sigma(\{H_j\}_{j\in\mathbb{J}})$ we have

$$\begin{aligned}\psi_\sigma\{x_j\}_{j\in\mathbb{J}} &= T_\sigma\{x_j\}_{j\in\mathbb{J}} \\ &= \sum_{j\in\sigma} v_j P_{W_j}\Lambda_j^* x_j + \sum_{j\in\sigma^c} \mu_j P_{V_j}\theta_j^* x_j,\end{aligned}$$

and also, for each $f \in H$,

$$A\langle K^*f, K^*f\rangle \leq \langle T_\sigma^* f, T_\sigma^* f\rangle = \langle \psi_\sigma^* f, \psi_\sigma^* f\rangle.$$

Thus, $\alpha KK^* \leq \psi_\sigma\psi_\sigma^*$.

(2) $\implies$ (1): Let $\sigma \subset \mathbb{J}$ and $f \in H$, so it is easy to check that

$$\psi_\sigma^* f = \{v_j\Lambda_j P_{W_j} f\}_{j\in\sigma} \cup \{\mu_j\theta_j P_{V_j} f\}_{j\in\sigma^c}.$$

Therefore,

$$\begin{aligned}
\alpha\langle K^*f, K^*f\rangle &= \langle \alpha KK^*f, f\rangle \\
&\leq \langle \psi_\sigma \psi_\sigma^* f, f\rangle \\
&= \langle \psi_\sigma^* f, \psi_\sigma^* f\rangle \\
&= \sum_{j\in\sigma} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle + \sum_{j\in\sigma^c} \mu_j^2 \langle \theta_j P_{V_j} f, \theta_j P_{V_j} f\rangle.
\end{aligned}$$

This gives that α is an universal lower frame bound of $\{W_j, \Lambda_j, v_j\}_{j\in\mathbb{J}}$ and $\{V_j, \theta_j, \mu_j\}_{j\in\mathbb{J}}$. □

In next results, we construct a $K-g-$fusion woven by using a bounded linear operator.

Theorem 2.3. *Let $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ be a $K-g-$fusion woven for H with common frame bounds A, B and assume that $U \in End_{\mathcal{A}}^*(H)$ has closed range so that $\mathcal{R}(K^*) \subset \mathcal{R}(U)$ and $KU = UK$. Then $\{UW_{ij}, \Lambda_{ij} P_{W_{ij}} U^*, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is also $K-g-$fusion woven for $\mathcal{R}(U)$.*

Proof. By the open mapping theorem, UW_{ij} is closed for any $j \in \mathbb{J}$ and $i \in [m]$. Using Lemme(ref k-g-fusion), we can write for each $f \in \mathcal{R}(U)$,

$$\begin{aligned}
A\langle K^*f, K^*f\rangle &= A\langle (U^+)^* U^* K^* f, (U^+)^* U^* K^* f\rangle \\
&\leq A\|U^+\|^2 \langle K^* U^* f, K^* U^* f\rangle \\
&\leq \|U^+\|^2 \sum_{i\in[m]} \sum_{j\in\mathbb{J}} v_{ij}^2 \langle \Lambda_{ij} P_{W_{ij}} U^* f, \Lambda_{ij} P_{W_{ij}} U^* f\rangle \\
&= \|U^+\|^2 \sum_{i\in[m]} \sum_{j\in\mathbb{J}} v_{ij}^2 \langle \Lambda_{ij} P_{W_{ij}} U^* P_{UW_{ij}} f, \Lambda_{ij} P_{W_{ij}} U^* P_{UW_{ij}} f\rangle.
\end{aligned}$$

The upper bound is obvious. □

Theorem 2.4. *Let K have closed range, $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ be a $K-g-$fusion woven for H with the universal bounds A, B and $U \in End_{\mathcal{A}}^*(H)$ has closed range so that $\mathcal{R}(U^*) \subset \mathcal{R}(K)$. Then $\{UW_{ij}, \Lambda_{ij} P_{W_{ij}} U^*, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a $K-g-$fusion woven for H if and only if there exists a $\delta > 0$ such that for every $f \in H$,*

$$\langle U^*f, U^*f\rangle \geq \delta \langle K^*f, K^*f\rangle.$$

Proof. Let $f \in H$ and $\{UW_{ij}, \Lambda_{ij} P_{W_{ij}} U^*, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a $K-g-$fusion woven for H with lower bound C, we get

$$\begin{aligned}
C\langle K^*f, K^*f\rangle &\leq \sum_{i\in[m]} \sum_{j\in\mathbb{J}} v_{ij}^2 \langle \Lambda_{ij} P_{W_{ij}} U^* P_{UW_{ij}} f, \Lambda_{ij} P_{W_{ij}} U^* P_{UW_{ij}} f\rangle \\
&= \sum_{i\in[m]} \sum_{j\in\mathbb{J}} v_{ij}^2 \langle \Lambda_{ij} P_{W_{ij}} U^* f, \Lambda_{ij} P_{W_{ij}} U^* f\rangle \\
&\leq B\langle U^*f, U^*f\rangle.
\end{aligned}$$

Therefore, $\langle U^*f, U^*f\rangle \geq \sqrt{\frac{C}{B}}\langle K^*f, K^*f\rangle$. For the opposite implication, we can write for all $f \in H$,

$$\langle U^*f, U^*f\rangle = \langle (K^+)^*K^*U^*f, (K^+)^*K^*U^*f\rangle \leq \|K^+\|^2\langle K^*U^*f, K^*U^*f\rangle.$$

Hence, we have

$$\begin{aligned} A\delta\|K^+\|^{-2}\langle K^*f, K^*f\rangle &\leq A\|K^+\|^{-2}\langle U^*f, U^*f\rangle \\ &\leq A\langle K^*U^*f, K^*U^*f\rangle \\ &\leq \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\langle \Lambda_{ij}P_{W_{ij}}U^*f, \Lambda_{ij}P_{W_{ij}}U^*f\rangle \\ &= \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\langle \Lambda_{ij}P_{W_{ij}}U^*P_{UW_{ij}}f, \Lambda_{ij}P_{W_{ij}}U^*P_{UW_{ij}}f\rangle \\ &\leq B\|U\|^2\langle f, f\rangle. \end{aligned}$$

So, $\{UW_{ij}, \Lambda_{ij}P_{W_{ij}}U^*, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a $K-g-$fusion woven for H with frame bounds $A\delta\|K^+\|^{-2}$ and $B\|U\|^2$. □

Theorem 2.5. *Let $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ be a $K-g-$fusion woven for H with common frame bounds A and B. Suppose that $0 \leq C \leq |w_j^{(i)}|^2 \leq D < \infty$ for any $i \in [m]$ and $j \in \mathbb{J}$, then $\{W_{ij}, w_j^{(i)}\Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a $K-g-$fusion woven for H with frame bounds AC and BD.*

Proof. For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$ and $f \in H$, we get

$$\begin{aligned} AC\langle K^*f, K^*f\rangle = \min_{i\in[m]}|w_j^{(i)}|^2 A\langle K^*f, K^*f\rangle &\leq \sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\langle w_j^{(i)}\Lambda_{ij}P_{W_{ij}}f, w_j^{(i)}\Lambda_{ij}P_{W_{ij}}f\rangle \\ &\leq \max_{i\in[m]}|w_j^{(i)}|^2 B\langle f, f\rangle \\ &= BD\langle f, f\rangle. \end{aligned}$$

□

Theorem 2.6. *Let $\mathbb{I} \subset \mathbb{J}$ be arbitrary and $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{I}, i\in[m]}$ be a $K-g-$fusion woven for H. Then $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a $K-g-$fusion woven.*

Proof. Assume that $\sigma_i \subset \mathbb{J}$, so $\sigma_i \cap \mathbb{I} \subset \mathbb{I}$ and A is the lower bound of $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\sigma_i\cap\mathbb{I}, i\in[m]}$, then for every $f \in H$ we have

$$\begin{aligned} A\langle K^*f, K^*f\rangle &\leq \sum_{i\in[m]}\sum_{j\in\sigma_i\cap\mathbb{I}} v_{ij}^2\langle \Lambda_{ij}P_{W_{ij}}f, \Lambda_{ij}P_{W_{ij}}f\rangle \\ &\leq \sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\langle \Lambda_{ij}P_{W_{ij}}f, \Lambda_{ij}P_{W_{ij}}f\rangle. \end{aligned}$$

This implies the statement. □

Next theorem is shows that even if one subspace is deleted, it dose not still remain a $K-g-$fusion woven.

Theorem 2.7. *Let K has closed range, $\mathbb{I} \subset \mathbb{J}$ and $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ be a $K-g-$fusion woven for H with the bounds A, B. If*

$$C = \sum_{i\in[m]} \sum_{j\in\mathbb{I}} v_{ij}^2 \|\Lambda_{ij} P_{W_{ij}}\|^2 < A\|K^+\|^2,$$

then $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}-\mathbb{I}, i\in[m]}$ is a $K-g-$fusion woven for $\mathcal{R}(K)$.

Proof. The upper bound is obvious. Suppose that $\sigma_{i\, i\in[m]} \subset \mathbb{J}-\mathbb{I}$ and $f \in \mathcal{R}(K)$, so we get

$$\begin{aligned}\sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \langle \Lambda_{ij} P_{W_{ij}} f, \Lambda_{ij} P_{W_{ij}} f\rangle &= \sum_{i\in[m]} \sum_{j\in\sigma_i\cup\mathbb{I}} v_{ij}^2 \langle \Lambda_{ij} P_{W_{ij}} f, \Lambda_{ij} P_{W_{ij}} f\rangle - \sum_{i\in[m]} \sum_{j\in\mathbb{I}} v_{ij}^2 \langle \Lambda_{ij} P_{W_{ij}} f, \Lambda_{ij} P_{W_{ij}} f\rangle \\ &\geq A\langle K^* f, K^* f\rangle - \sum_{i\in[m]} \sum_{j\in\mathbb{I}} v_{ij}^2 \|\Lambda_{ij} P_{W_{ij}}\|^2 \langle f, f\rangle \\ &\geq (A - C\|K^+\|^2)\langle K^* f, K^* f\rangle.\end{aligned}$$

□

Theorem 2.8. *Let $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ be a $K-g-$fusion woven for H with bounds A, B. For each $i \in [m]$,$j \in \mathbb{J}$ and a index set $\mathbb{I}_{ij}$, suppose that $\{f_{ij}^{(k)}\}_{k\in\mathbb{I}_{ij}} \in \Lambda_{ij}(W_{ij})$ is a Parseval frame for H_{ij} such that for every finite subset $\mathbb{K}_{ij} \subset \mathbb{I}_{ij}$, the set $\{f_{ij}^{k}\}_{k\in\mathbb{I}_{ij}-\mathbb{K}_{ij}}$ is a frame with the lower bound C_{ij}. Let $\tilde{W}_{ij} = \overline{span}\{\Lambda_{ij}^* f_{ij}^{(k)}\}_{k\in\mathbb{I}_{ij}-\mathbb{K}_{ij}}$ for any $i \in [m]$ and $j \in \mathbb{J}$, then $\{\tilde{W}_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is a $K-g-$fusion woven for H with the bounds $(\min_{i\in[m], j\in\mathbb{J}} C_{ij})A$ and B.*

Proof. Obviously, B is the upper bound of $\{\tilde{W}_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$. Assume that $f \in H$ and $\{\sigma_i\}_{i\in[m]} \in \mathbb{J}$, so

$$\begin{aligned}\sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \langle \Lambda_{ij} P_{\tilde{W}_{ij}} f, \Lambda_{ij} P_{\tilde{W}_{ij}} f\rangle &= \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \sum_{k\in\mathbb{I}_{ij}} \langle \Lambda_{ij} P_{\tilde{W}_{ij}} f, f_{ij}^{(k)}\rangle \langle f_{ij}^{(k)}, \Lambda_{ij} P_{\tilde{W}_{ij}} f\rangle \\ &\geq \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \sum_{k\in\mathbb{I}_{ij}-\mathbb{K}_{ij}} \langle \Lambda_{ij} P_{\tilde{W}_{ij}} f, f_{ij}^{(k)}\rangle \langle f_{ij}^{(k)}, \Lambda_{ij} P_{\tilde{W}_{ij}} f\rangle \\ &\geq \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 C_{ij} \langle \Lambda_{ij} P_{W_{ij}} f, \Lambda_{ij} P_{W_{ij}} f\rangle \\ &\geq (\min_{i\in[m], j\in\mathbb{J}} C_{ij}) \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \langle \Lambda_{ij} P_{W_{ij}} f, \Lambda_{ij} P_{W_{ij}} f\rangle \\ &\geq (\min_{i\in[m], j\in\mathbb{J}} C_{ij}) A\langle K^* f, K^* f\rangle.\end{aligned}$$

□

Theorem 2.9. *Let $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}}$ is a $K-g-$fusion frame for H for each $i \in [m]$. Suppose that for a partition collection of disjoint finite sets $\{\delta_i\}_{i\in[m]}$ of $\mathbb{J}$ and for any $\epsilon > 0$ there exists a partition $\{\sigma_i\}_{i\in[m]}$ of the set $\mathbb{J} - \cup_{i\in[m]}\delta_i$ such that $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in(\sigma_i\cup\delta_i), i\in[m]}$ has a lower $K-g-$fusion frame bound less than ϵ. Then $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is not a woven.*

Proof. We can write $\mathbb{J} = \cup_{j\in\mathbb{N}}\mathbb{J}_j$, where $\mathbb{J}_j$ are disjoint index sets. Assume that $\delta_{1j} = \emptyset$ for all $i \in [m]$ and $\epsilon = 1$. Then, there exists a partition $\sigma_{i1}{}_{i\in[m]}$ of $\mathbb{J}$ such that $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in(\sigma_{i1}\cup\delta_{i1}),i\in[m]}$ has a lower bound (also, optimal lower bound) less than 1. Thus, there is a $f_1 \in H$ such that

$$\sum_{i\in[m]}\sum_{j\in(\sigma_{i1}\cup\delta_{i1})} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_1, \Lambda_{ij}P_{W_{ij}}f_1\rangle < \langle K^*f_1, K^*f_1\rangle.$$

Since

$$\sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_1, \Lambda_{ij}P_{W_{ij}}f_1\rangle < \infty,$$

so, there is a $k_1 \in \mathbb{N}$ such that

$$\sum_{i\in[m]}\sum_{j\in\mathbb{K}_1} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_1, \Lambda_{ij}P_{W_{ij}}f_1\rangle < \langle K^*f_1, K^*f_1\rangle,$$

where, $\mathbb{K}_1 = \cup_{i\geq k_1+1}\mathbb{J}_j$. Continuing this way, for $\epsilon = \frac{1}{n}$ and a partition $\{\delta_{ni}\}_{i\in[m]}$ of $\mathbb{J}_1 \cup \ldots \cup \mathbb{J}_{k_n-1}$ such that

$$\delta_{ni} = \delta_{(n-1)i} \cup (\sigma_{(n-1)i} \cap (\mathbb{J}_1 \cup \ldots \cup \mathbb{J}_{k_n-1}))$$

for all $i \in [m]$, there exists a partition $\{\sigma_{ni}\}_{i\in[m]}$ of $\mathbb{J} - (\mathbb{J}_1 \cup \ldots \cup \mathbb{J}_{k_n-1})$ such that $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in(\sigma_{ni}\cup\delta_{ni}),i\in[m]}$ has a lower bound less than $\frac{1}{n}$. Therefore, there is a $f_n \in H$ and $k_n \in \mathbb{N}$ such that $k_n > k_{n-1}$ and

$$\sum_{i\in[m]}\sum_{j\in\mathbb{K}_n} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_n, \Lambda_{ij}P_{W_{ij}}f_n\rangle < \frac{1}{n}\langle K^*f_n, K^*f_1\rangle,$$

where, $\mathbb{K}_n = \cup_{i\geq k_n+1}\mathbb{J}_j$. Choose a partition $\{\alpha_i\}_{i\in[m]}$ of $\mathbb{J}$, where $\alpha_i = \cup_{j\in\mathbb{N}}\{\delta_{ji}\} = \delta_{(n+1)i} \cup (\alpha_i \cap \mathbb{J} - (\mathbb{J}_1 \cup \ldots \cup \mathbb{J}_n))$. Assume that $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\alpha_i,i\in[m]}$ is a $K - g-$fusion frame for H with the optimal lower bound A. Then, by the Archimedean Property, there exits a $r \in \mathbb{N}$ such that $r > \frac{2}{A}$. Now, there exists a $f_r \in H$ such that

$$\begin{aligned}
\sum_{i\in[m]}\sum_{j\in\alpha_i} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_r, \Lambda_{ij}P_{W_{ij}}f_r\rangle &= \sum_{i\in[m]}\sum_{j\in\delta_{(r+1)i}} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_r, \Lambda_{ij}P_{W_{ij}}f_r\rangle \\
&+ \sum_{i\in[m]}\sum_{j\in\alpha_i\cap\mathbb{J}-(\mathbb{J}_1\cup\ldots\cup\mathbb{J}_r)} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_r, \Lambda_{ij}P_{W_{ij}}f_r\rangle \\
&\leq \sum_{i\in[m]}\sum_{j\in(\sigma_{ri}\cup\delta_{ri})} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_r, \Lambda_{ij}P_{W_{ij}}f_r\rangle \\
&+ \sum_{i\in[m]}\sum_{j\in\cup_{k\geq r+1}\mathbb{J}_k} v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f_r, \Lambda_{ij}P_{W_{ij}}f_r\rangle \\
&< \frac{1}{r}\langle K^*f_r, K^*f_r\rangle + \frac{1}{r}\langle K^*f_r, K^*f_r\rangle \\
&< A\langle K^*f_r, K^*f_r\rangle
\end{aligned}$$

and this is a contradiction with the lower bound of A. □

Corollary 2.10. *Let $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ be a $K-g-$fusion woven for H. Then there exists a collection of disjoint finite subsets $\{\delta_i\}_{i\in[m]}$ of $\mathbb{J}$ and $A>0$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of the set $\mathbb{J}-\cup_{i\in[m]}\delta_i$, some the family $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in(\sigma_i\cup\delta_i), i\in[m]}$ is a $K-g-$fusion frame for H with the lower frame bound A.*

Theorem 2.11. *Let $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}}$ be a $K-g-$fusion frame for H with bounds A_i and B_i for each $i\in[m]$. Suppose that there exists $N>0$ such that for all $i,k\in[m]$ with $i\neq k$, $\mathbb{I}\subset\mathbb{J}$ and $f\in H$,*

$$\sum_{j\in\mathbb{I}}\langle(v_{ij}\Lambda_{ij}P_{W_{ij}}-v_{kj}\Lambda_{kj}P_{W_{kj}})f,(v_{ij}\Lambda_{ij}P_{W_{ij}}-v_{kj}\Lambda_{kj}P_{W_{kj}})f\rangle\leq N\min\{\sum_{j\in\mathbb{I}}v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f,\Lambda_{ij}P_{W_{ij}}f\rangle,$$
$$\sum_{j\in\mathbb{I}}v_{kj}^2\langle\Lambda_{kj}P_{W_{kj}}f,\Lambda_{kj}P_{W_{kj}}f\rangle\}.$$

Then the family $\{W_{ij}, \Lambda_{ij}, v_{ij}\}_{j\in\mathbb{J}, i\in[m]}$ is woven with universal bounds

$$\frac{A}{(m-1)(N+1)+1}\quad and\quad B,$$

where $A=\sum_{i\in[m]}A_i$ and $B=\sum_{i\in[m]}B_i$.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be a partition of $\mathbb{J}$ and $f\in H$. Therefore,

$$\begin{aligned}\sum_{i\in[m]}A_i\langle K^*f,K^*f\rangle&\sum_{i\in[m]}\sum_{j\in\mathbb{J}}v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f,\Lambda_{ij}P_{W_{ij}}f\rangle\\
&=\sum_{i\in[m]}\sum_{k\in[m]}\sum_{j\in\sigma_k}v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f,\Lambda_{ij}P_{W_{ij}}f\rangle\\
&\leq\sum_{i\in[m]}\Bigg(\sum_{j\in\sigma_i}v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f,\Lambda_{ij}P_{W_{ij}}f\rangle+\sum_{k\in[m],k\neq i}\sum_{j\in\sigma_k}\{v_{kj}^2\langle\Lambda_{kj}P_{W_{kj}}f,\Lambda_{kj}P_{W_{kj}}f\rangle\\
&+\langle(v_{ij}\Lambda_{ij}P_{W_{ij}}-v_{kj}\Lambda_{kj}P_{W_{kj}})f,(v_{ij}\Lambda_{ij}P_{W_{ij}}-v_{kj}\Lambda_{kj}P_{W_{kj}})f\rangle\}\Bigg)\\
&\leq\sum_{i\in[m]}\Bigg(\sum_{j\in\sigma_i}v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f,\Lambda_{ij}P_{W_{ij}}f\rangle\\
&+\sum_{k\in[m],k\neq i}\sum_{j\in\sigma_k}(N+1)v_{kj}^2\langle\Lambda_{kj}P_{W_{kj}}f,\Lambda_{kj}P_{W_{kj}}f\rangle\Bigg)\\
&=\{(m-1)(N+1)+1\}\sum_{i\in[m]}\Bigg(\sum_{j\in\sigma_i}v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f,\Lambda_{ij}P_{W_{ij}}f\rangle\Bigg).\end{aligned}$$

Thus, we get

$$\frac{A}{(m-1)(N+1)+1}\langle K^*f,K^*f\rangle\leq\sum_{i\in[m]}\Bigg(\sum_{j\in\sigma_i}v_{ij}^2\langle\Lambda_{ij}P_{W_{ij}}f,\Lambda_{ij}P_{W_{ij}}f\rangle\Bigg)\leq B\langle f,f\rangle.$$

□

In next theorem we study a Paley-Wiener type perturbation for weaving $K-g-$fusion frames.

Theorem 2.12. *Let $\{W_j, \Lambda_j, w_j\}_{j\in\mathbb{J}}$ and $\{V_j, \theta_j, v_j\}_{j\in\mathbb{J}}$ be two $K-g-$fusion frames for H with frame bounds A_1, B_1 and A_2, B_2, respectively. Suppose that there exist non-negative scalers μ and $0 \le \lambda < \frac{1}{2}$ such that $(\frac{1}{2}-\lambda)A_1 > \mu$ and for each $f \in H$,*

$$\sum_{j\in\mathbb{J}}\langle (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f, (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f\rangle \le \lambda \sum_{j\in\mathbb{J}}\langle w_j\Lambda_j P_{W_j} f, w_j\Lambda_j P_{W_j} f\rangle + \mu\langle K^*f, K^*f\rangle.$$

Then, $\{W_j, \Lambda_j, w_j\}_{j\in\mathbb{J}}$ and $\{V_j, \theta_j, v_j\}_{j\in\mathbb{J}}$ are $K-g-$fusion woven for H with universal frame bounds $(\frac{1}{2}-\lambda)A_1 - \mu$ and $B_1 + B_2$.

Proof. The upper frame bound is clear. For the lower frame bound, assume that $\sigma \subset \mathbb{J}$ and we get, by the arithmetic-quadratic mean, for any $f \in H$

$$\begin{aligned}
&\sum_{j\in\sigma} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle + \sum_{j\in\sigma^c} v_j^2\langle \theta_j P_{V_j} f, \theta_j P_{V_j} f\rangle \\
&\quad= \sum_{j\in\sigma} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \\
&\quad+ \sum_{j\in\sigma^c}\langle w_j\Lambda_j P_{W_j} f - (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f, w_j\Lambda_j P_{W_j} f - (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f\rangle \\
&\quad\ge \sum_{j\in\sigma} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle + \frac{1}{2}\sum_{j\in\sigma^c} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \\
&\quad- \sum_{j\in\sigma^c}\langle (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f, (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f\rangle \\
&\quad= \frac{1}{2}\sum_{j\in\mathbb{J}} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle + \frac{1}{2}\sum_{j\in\sigma} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \\
&\quad- \sum_{j\in\sigma^c}\langle (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f, (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f\rangle \\
&\quad\ge \frac{1}{2}\sum_{j\in\mathbb{J}} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle - \sum_{j\in\sigma^c}\langle (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f, (w_j\Lambda_j P_{W_j} - v_j\theta_j P_{V_j})f\rangle \\
&\quad\ge \frac{1}{2}\sum_{j\in\mathbb{J}} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle - \lambda\sum_{j\in\mathbb{J}} w_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle - \mu\langle K^*f, K^*f\rangle \\
&\quad\ge \left((\frac{1}{2-\lambda})A_1 - \mu\right)\langle K^*f, K^*f\rangle.
\end{aligned}$$

This completes the proof. □

Declarations

Availablity of data and materials

Not applicable.

Human and animal rights

We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Competing interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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