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A Note on the Stability of Functional Equations via a Celebrated Direct Method

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Abstract. More than ten years after Justyna Sikorska [8] attempted to solve the Heyers-Ulam stability of a single variable equation by using direct method. In this paper, we will improve the results of Justyna Sikorska by using a more efficient approach. Relations between the generalized functional equation, the dependence of their different parameters and several properties are also further explored. To achieve the problem, we try to develop some new techniques to overcome the fundamental difficulties caused by the different properties of the function and the presence of several variables in the equation. Furthermore, we continue to construct and study a couple of functional equations by making a new direct method.

1. Introduction

The core idea of the Hyers-Ulam stability for functional equations has been dated back to a well-known problem concerning about group homomorphisms solved by S.M. Ulam and D.H. Hyers (see [1–3]). In the last decades, a great number of papers treating the stability problem about functional equations has already been achieved and a great deal of important problems about this field has been studied ([4–7]). It follows that the most efficient methods have been stated in many papers ([10, 18–24, 27]) such as the direct approach, the shadowing approach, and invariant mean approach and so on. In particular, the direct method is always the main studying tool on the investigation of functional equations of different types.

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The stability problems for an appropriate simple variable functional equations have earlier been investigated by direct method. The direct method is familiar with many readers to derive the solutions of the equations. The author in [8] have made full use of quite a general way to solve the Hyers-Ulam stability problems on the functional equations under which many excellent outcomes have been achieved without reduplicating the similar procedure in the whole process of computation. However, her results can only be used to derive the solutions of the equation where the mediate function is odd. This is exactly our contribution to the paper. In fact, a straightforward observation is that the inequality

$$\|f(x) - uf(e(x)) - vf(-e(x))\| \leqslant \delta(x)$$

can be solved if the function h is even. Next, the present studying approach calls us to investigate the following functional inequality, by using a direct method, under which the result can not be covered by earlier works

$$\begin{aligned}&\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(z+y) - f(x+z)\| \\ &\leqslant K\left(\|x\|^r + \|y\|^r + \|z\|^r\right).\end{aligned} \tag{1.1}$$

In fact, the functional inequality (1.1) comes from some equivalent characterizations of Hilbert space in [15]. The investigator described several properties of an inner product space and applies these results to solve many interesting functional inequalities such as: Zarantone's inequality, Hayashi's inequality and so on. However, the more far reaching work can be done M. Fréchet in [16] under which he ascertained that the corresponding equation is a necessary prerequisite condition when complex or real normed completed spaces become Hilbert spaces. Investigator in [17] studied the stability of Fréchet functional equation from which a characterization of inner product spaces had been achieved by using a stationary point theorem in Banach spaces. Compared with the before studying approaches, we further explored solutions of the equation (1.1) in this literature. Of course, to the best of our knowledge, it has also already been solved by [8] under which a direct method was to derive solutions of the equation (1.1) and to look for some improvement approximations. However, in this literature we make a new direct method to achieve the solution of equation (1.1) must be close to the approximate solution, approximately satisfying the corresponding equation.

Besides this, we will consider that the functions on the functional equation of different types have been defined in a more general domain. For instance, the papers [11, 12] have defined an additive ρ-functional inequalities in nonArchimedean normed spaces and Banach spaces. However, this phenomenon can not attract enough attention to the study of functional equations in the more general and complex nonlinear structure of F-spaces (see the definition in [13, 14]). But, the nonlinear structure of space has always stood in a very important position of leadership in functional analysis. Based on the above analysis, it is of great significance that the functional inequality is considered in β-homogeneous F-space.

In section 2, the counterpart of Theorem 2.1 from [8] where the mediate function is odd will be considered. In the subsequent part, a new direct method for solving (1.1) in F-space will be described and some new extended results of Theorem 2.1 from [8] will be presented. With it, two new different applications of the results will be described in the final part.

2. A simple variable of abstract equation

In Theorem 2.1 from [8], Sikorska solved the equation (2.1) where the mediate function e is odd and the related parameters u, v are restricted on the real field. For simplicity in notation, we provide traditionally our first result with the studying mapping defined in Banach space. By making use of small conjectures the more general form of the results will be provided in β-homogeneous F-space in section 3. Therefore, our first result is simply considered in Banach space.

Theorem 2.1 Let $(X, +)$ be a group, and $(Y, \|\cdot\|)$ be a Banach space, and assume the mapping $f : X \to Y$ satisfying the inequality

$$\|f(x) - uf(e(x)) - vf(-e(x))\| \leqslant \delta(x), \quad x \in X, \tag{2.1}$$

where $u, v \in (-\infty, +\infty)$, and the mappings $e : X \to X$, $\delta : X \to [0, \infty)$ satisfy that e is even (i.e., $e(-x) = e(x)$ for every $x \in X$). Let the infinite progression $\sum_{n=0}^{\infty} [|u_n|\, \delta\left(e^n(x)\right) + |v_n| \delta\left(-e^n(x)\right)]$ with

$$u_0 := 1, \qquad u_n := \left[u(u+v)^{n-1}\right],$$
$$v_0 := 0, \qquad v_n := \left[v(u+v)^{n-1}\right], \quad n \in \mathbb{N}$$

(e^m states the m-th composition of function e), be assumed convergence for every $x \in X$. Then there has a unique even mapping $g : X \to Y$ satisfying

$$g(x) = u_n g(e^n(x)) + v_n g(-e^n(x)),$$

and

$$\|f(x) - g(x)\| \leqslant \sum_{i=0}^{\infty} \left[|u_i|\, \delta\left(e^i(x)\right) + |v_i|\, \delta\left(-e^i(x)\right)\right], \quad x \in X \text{ and } n \in \mathbb{N}. \tag{2.2}$$

Proof. We will prove that

$$\|f(x) - u_n f\left(e^n(x)\right) - v_n f\left(-e^n(x)\right)\| \leqslant \gamma_n(x), \quad x \in X, \tag{2.3}$$

where

$$\gamma_n(x) := \sum_{i=0}^{n-1} \left[|u_i|\, \delta\left(e^i(x)\right) + |v_i|\, \delta\left(-e^i(x)\right)\right], \quad x \in X, n \in \mathbb{N}.$$

First of all, consider with every $m, n \in \mathbb{N}$ and it is easy to observe that

$$u_{n+1} = uu_n + uv_n, \quad v_{n+1} = vv_n + vu_n,$$

and

$$u_{n+m} = u_m u_n + v_m u_n, \quad v_{n+m} = u_m v_n + v_m v_n. \tag{2.4}$$

From the definition of sequences (u_n) and (v_n) we also have

$$uv_n = vu_n, \;\; v_m u_n = u_m v_n.$$

First, (2.1) gives (2.3) with setting $n = 1$, and by mathematical induction, later we suppose that (2.3) establishes for some $n \in N$. We prove that in the case for $n+1$ by virtue of (2.1)

$$\begin{aligned}
&\left\|f(x) - u_{n+1} f\left(e^{n+1}(x)\right) - v_{n+1} f\left(-e^{n+1}(x)\right)\right\| \\
&\leqslant \|f(x) - u_n f\left(e^n(x)\right) - v_n f\left(-e^n(x)\right)\| \\
&\quad + |u_n| \left\|f\left(e^n(x)\right) - uf\left(e^{n+1}(x)\right) - vf\left(-e^{n+1}(x)\right)\right\| \\
&\quad + |v_n| \left\|f\left(-e^n(x)\right) - uf\left(e^{n+1}(x)\right) - vf\left(-e^{n+1}(x)\right)\right\| \\
&\leqslant \sum_{i=0}^{n-1} \left[|u_i|\,\delta\left(e^i(x)\right) + |v_i|\,\delta\left(-e^i(x)\right)\right] + |u_n|\,\delta\left(e^n(x)\right) + |v_n|\,\delta\left(-e^n(x)\right) \\
&= \sum_{i=0}^{n} \left[|u_i|\,\delta\left(e^i(x)\right) + |v_i|\,\delta\left(-e^i(x)\right)\right].
\end{aligned}$$

Since the series $\sum_{i=0}^{\infty} \left[|u_i|\,\delta\left(e^i(x)\right) + |v_i|\,\delta\left(-e^i(x)\right)\right]$ is convergent for every $x \in X$, combined with (2.3) and by virtue of the completeness of Y, the mapping can be well defined as in the following:

$$g(x) := \lim_{n\to\infty} \left[u_n f\left(e^n(x)\right) + v_n f\left(-e^n(x)\right)\right], \quad x \in X, \tag{2.5}$$

and we prove the following properties of the function g.

An easy computation is to prove that

$$\begin{aligned}
&ug(e(x)) + vg(-e(x)) \\
&\quad = u \lim_{n\to\infty} \left[u_n f\left(e^{n+1}(x)\right) + v_n f\left(-e^{n+1}(x)\right)\right] + v \lim_{n\to\infty} \left[u_n f\left(e^{n+1}(x)\right) + v_n f\left(-e^{n+1}(x)\right)\right] \\
&\quad = \lim_{n\to\infty} \left[(uu_n + vu_n) f\left(e^{n+1}(x)\right) + (uv_n + vv_n) f\left(-e^{n+1}(x)\right)\right] = g(x).
\end{aligned}$$

Furthermore, we will prove the more general property of g

$$g(x) = u_n g\left(e^n(x)\right) + v_n g\left(-e^n(x)\right), \quad for\ all\ x \in X\ and\ n \in N. \tag{2.6}$$

By induction, we assume that the equation is true for all natural number k with $k \le n$ for some $n \in N$. Let us calculate with $k = n+1$

$$\begin{aligned}
g(x) &= u_n g\left(e^n(x)\right) + v_n g\left(-e^n(x)\right) \\
&= u_n(ug\left(e^{n+1}(x)\right) + vg\left(-e^{n+1}(x)\right)) + v_n(ug\left(e^{n+1}(x)\right) + vg\left(-e^{n+1}(x)\right)) \\
&= u_{n+1} g\left(e^{n+1}(x)\right) + v_{n+1} g\left(-e^{n+1}(x)\right)).
\end{aligned}$$

In particular, we also have that the function g is even. An easy computation is to state that

$$g(-x) = u_n g\left(e^n(x)\right) + v_n g\left(-e^n(x)\right) = g(x), \quad for\ every\ x \in X\ and\ n \in \mathbb{N}.$$

In order to achieve the uniqueness of g, suppose further that $\bar{g}: X \to Y$ is the another mapping such that (2.2) and (2.6) hold. Then

$$\|g(x)-\bar{g}(x)\| \leqslant 2\sum_{i=0}^{\infty}\left[|u_i|\,\delta\left(e^i(x)\right)+|v_i|\,\delta\left(-e^i(x)\right)\right], \quad x\in X.$$

Moreover, we have

$$g(x)-\bar{g}(x)=u_n\left[g\left(e^n(x)\right)-\bar{g}\left(e^n(x)\right)\right]+v_n\left[g\left(-e^n(x)\right)-\bar{g}\left(-e^n(x)\right)\right], \quad x\in X,$$

and on account of (2.4) and (2.6) we can rewrite

$$\begin{aligned}
\|g(x)-\bar{g}(x)\| &\leqslant |u_n+v_n|\,\|g\left(e^n(x)\right)-\bar{g}\left(e^n(x)\right)\| \\
&\leqslant 2|u_n+v_n|\sum_{i=0}^{\infty}\left[|u_i|\,\delta\left(e^{i+n}(x)\right)+|v_i|\,\delta\left(-e^{i+n}(x)\right)\right] \\
&=2\sum_{i=0}^{\infty}\left[(|u_i(u_n+v_n)|)\,\delta\left(e^{i+n}(x)\right)+|v_i(u_n+v_n)|\delta\left(-e^{i+n}(x)\right)\right] \\
&=2\sum_{i=0}^{\infty}\left[|u_{i+n}|\,\delta\left(e^{i+n}(x)\right)+|v_{i+n}|\,\delta\left(-e^{i+n}(x)\right)\right] \\
&=2\sum_{j=n}^{\infty}\left[\left|u_j\right|\delta\left(e^j(x)\right)+\left|v_j\right|\delta\left(-e^j(x)\right)\right]
\end{aligned}$$

for every $x\in X$ and $n\in\mathbb{N}$, where it states that $g=\bar{g}$ as $n\to\infty$. This proves the theorem. □

The purpose of stating and proving this results is of particular interest and give out a solution of a simple variable functional equation (2.1) at least. In section 3, we will extend the results of Theorem 2.1 form [8] to a more general setting. In particular, the related parameters $u,\ v$ can be extended to complex numbers.

According to the above analysis, we give out a corollary of Theorem 2.1 (still quite general). First of all, we must state that the absolute of an element $x\in X$ can be given out in the real field considering that the function h is even for the meaningful of the results, for example $h(x)=a|x|$. As a matter of fact, we can also present the absolute of $x=(x_1,x_2,\cdots,x_n)\in R^n$ by $|x|=(|x_1|,|x_2|,\cdots,|x_n|)$. Thus, it worth stating the results. In particular, we can present the following results in the Euclidean space if the more general setting can not be judged.

Corollary 2.1 Assume that $(X,+)$ is a real or complex normed linear space and set $(Y,\|\cdot\|)$ is a Banach space. Suppose further that the mapping $f: X\to Y$ fulfils the inequality

$$\left\|f(x)-\frac{a+1}{2a^2}f(a|x|)+\frac{a-1}{2a^2}f(-a|x|)\right\|\leqslant\delta(x), \quad x\in X, \tag{2.7}$$

where $a\in R$ with $a>1$ and mappings $e: X\to X,\ \delta: X\to[0,\infty)$ make that e is an even function (i.e., $e(-x)=e(x)$ for every $x\in X$). The infinite progression $\sum_{i=0}^{\infty}\frac{1}{a^i}\delta\left(a^i|x|\right)$ is convergence for every $x \in X$. Then there has a unique mapping $g : X \to Y$ fulfilling the following equations for all $x \in X$

$$g(x) = \frac{a+1}{2a^2} g(a|x|) - \frac{a-1}{2a^2} g(-a|x|),$$

and

$$\|f(x) - g(x)\| \leqslant \Delta(x) + \Lambda(x),$$

where

$$\Delta(x) := \tfrac{1}{2} \textstyle\sum_{i=0}^{\infty} \tfrac{1}{a^i} \left[\delta\left(a^i|x|\right) + \delta\left(-a^i|x|\right)\right],$$
$$\Lambda(x) := \tfrac{1}{2} \textstyle\sum_{i=0}^{\infty} \tfrac{1}{a^{2i}} \left[\delta\left(a^i|x|\right) - \delta\left(-a^i|x|\right)\right], \quad x \in X.$$

Furthermore, g can be obtained in the following limiting equality

$$g(x) := \lim_{n\to\infty} \left(\frac{a^n+1}{2a^{2n}} f\left(a^n|x|\right) - \frac{a^n-1}{2a^{2n}} f\left(-a^n|x|\right) \right), \quad x \in X.$$

Proof. By using the results of Theorem 2.1, $u := \frac{1+a}{2a^2}$, $v := \frac{1-a}{2a^2}$ and together $e(x) := a|x|$, for all $x \in X$, a computation is to prove that

$$u_n := \frac{1+a}{2a^{2n}}, \quad v_n := \frac{1-a}{2a^{2n}}, \quad n \in \mathbb{N}.$$

For the convergent series $\sum_{i=0}^{\infty} \frac{1}{a^i} \delta\left(a^i|x|\right)$ with $x \in X$, therefore $\sum_{i=0}^{\infty} \frac{1}{a^{2i}} \delta\left(a^i|x|\right)$ is convergence. Applying Theorem 2.1, there has a unique limiting function $g : X \to Y$ fulfilling

$$\begin{aligned}\|f(x) - g(x)\| &\leqslant \sum_{i=0}^{\infty} \left[\left|\frac{1+a^i}{2a^{2i}}\right| \delta\left(a^i|x|\right) + \left|\frac{1-a^i}{2a^{2i}}\right| \delta\left(-a^i|x|\right) \right] \\ &= \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \left[\delta\left(a^i|x|\right) - \delta\left(-a^i|x|\right)\right] + \sum_{i=0}^{\infty} \frac{1}{a^i} \left[\delta\left(a^i|x|\right) + \delta\left(-a^i|x|\right)\right] \\ &= \Lambda(x) + \Delta(x).\end{aligned}$$

Function g has been dated back to derived from (2.5). We complete the proof. □

Remark 2.2 If $u = 1$, $v = 0$ and $e(x) = |x|$, the above results may be trivial and meaningless. In the above results, suppose that $a \in (-\infty, \infty)$ which is not equal to $-1, 0, 1$. Exchanging a with $-a$, this transformation may not be different from primary inequality (2.7). This is a basic fact leaving to the reader to check it. Assume that the convergent series $\sum_{i=0}^{\infty} \frac{1}{|a|^i} \delta\left(|a|^i|x|\right)$ establishes for every $x \in X$. In fact, the assertions with $|a|$ exchanging for a has also been achieved by a similar way.

Corollary 2.2 Assume that $(X, +)$ is a group and set $(Y, \|\cdot\|)$ is a Banach space. Suppose further that the mapping $f : X \to Y$ fulfils the inequality

$$\left\| f(x) - \frac{a+1}{2a^2} f(a|x|) + \frac{a-1}{2a^2} f(-a|x|) \right\| \leqslant \delta, \quad x \in X,$$

where $a \in (-\infty, \infty)$ with $|a| > 1$ and $\delta > 0$ is constant. Then there is a unique limiting even function $g : X \to Y$ fulfilling

$$\|f(x) - g(x)\| \leqslant \frac{|a|\delta}{|a| - 1}.$$

Proof. Since the function δ is a positive constant, thus $\Delta(x) = \frac{|a|}{|a|-1}\delta$ and $\Lambda(x) = 0$ for every $x \in X$. The mapping g has been stated in the following shape:

$$g(x) := \lim_{n\to\infty} \left(\frac{|a|^n + 1}{2a^{2n}} f\left(|a|^n|x|\right) - \frac{|a|^n - 1}{2a^{2n}} f\left(-|a|^n|x|\right) \right), \quad x \in X.$$

This proves the proof. □

Remark 2.3 The above corollaries 2.1 and 2.2 will still establish in β-homogeneous F-space with $a \in (-\infty, \infty)$ and $|a| > 1$. If we exchange a for $\frac{1}{a}$ in the equation

$$f(x) - \frac{a+1}{2a^2} f(a|x|) + \frac{a-1}{2a^2} f(-a|x|)$$

from (2.7), the second group of results will also be obtained with a is a positive constant stated in the following results.

Corollary 2.3 Assume that $(X, +)$ is a group divisible by a with $a \in (-\infty, \infty)$ and $|a| > 1$ and set $(Y, \|\cdot\|)$ is a Banach space. Suppose further that the mapping $f : X \to Y$ fulfils the inequality

$$\left\| f(x) - \frac{a^2 + a}{2} f\left(\frac{1}{a}|x|\right) - \frac{a^2 - a}{2} f\left(-\frac{1}{a}|x|\right) \right\| \leqslant \delta(x), \quad x \in X,$$

with $\delta : X \to [0, \infty)$ is such that the convergent series $\sum_{i=0}^{\infty} a^{2i}\delta\left(\frac{1}{a^i}|x|\right)$ holds for every $x \in X$. Then there has a unique even limiting mapping $g : X \to Y$ fulfilling for every $x \in X$,

$$g(x) = \frac{a^2 + a}{2} g\left(\frac{1}{a}|x|\right) + \frac{a^2 - a}{2} g\left(-\frac{1}{a}|x|\right),$$

and

$$\|f(x) - g(x)\| \leqslant \widetilde{\Delta}(x) + \widetilde{\Lambda}(x).$$

Furthermore, the mapping g can be stated in the following shape:

$$g(x) := \lim_{n\to\infty} \left[\frac{a^{2n} + a^n}{2} f\left(\frac{1}{a^n}|x|\right) + \frac{a^{2n} - a^n}{2} f\left(-\frac{1}{a^n}|x|\right) \right], \quad x \in X.$$

Proof. Applying for Theorem 2.1 for $u := \frac{a^2+a}{2}$, $v := \frac{a^2-a}{2}$ and $e(x) := \frac{1}{a}|x|$, for all $x \in X$, an easy computation is to show that

$$u_n := \frac{a^{2n} + a^{2n-1}}{2}, \quad v_n := \frac{a^{2n} - a^{2n-1}}{2}, \quad n \in \mathbb{N}.$$

According to the convergent series $\sum_{i=0}^{\infty} a^{2i}\delta\left(\frac{1}{a^i}|x|\right)$ for all $x \in X$, hence the series $\sum_{i=0}^{\infty} a^{2i-1}\delta\left(\frac{1}{a^i}|x|\right)$ is convergence, and there has a unique even limiting mapping $g : X \to Y$

fulfilling

$$\begin{aligned}\|f(x)-g(x)\| &\leqslant \sum_{i=0}^{\infty}\left[\left|\frac{a^{2i}+a^i}{2}\right|\delta\left(\frac{1}{a^i}|x|\right)+\left|\frac{a^{2i}-a^i}{2}\right|\delta\left(-\frac{1}{a^i}|x|\right)\right]\\ &=\sum_{i=0}^{\infty}\frac{a^{2i}}{2}\left[\delta\left(\frac{1}{a^i}|x|\right)+\delta\left(-\frac{1}{a^i}|x|\right)\right]+\sum_{i=0}^{\infty}\frac{a^i}{2}\left[\delta\left(\frac{1}{a^i}|x|\right)-\delta\left(-\frac{1}{a^i}|x|\right)\right]\\ &=\widetilde{\Delta}(x)+\widetilde{\Lambda}(x).\end{aligned}$$

The definition of g is derived from (2.5). We complete the proof. □

Remark 2.4 Corollary 2.3 can be used to investigate the function from which it could be split into even and odd parts. There is a good point of the approach achieved here where the functions split into two two parts of odd and even functions can give more concise approximations than the before approximations in Theorem 2.1. The above results is the counterpart of the corresponding results of Sikorska's paper. However, it is not copied word by word. It is the counterpart of even function.

3. The stability of functional equations in F-space

An F-space is called β-homogeneous if it satisfies $\|tx\| = |t|^\beta\|x\|$ for every $x \in X$, $t \in C$. In this section of the first two theorems, β_1, β_2 are to be $0 < \beta_1 \leq 1$ and $0 < \beta_2 \leq 1$. Furthermore, we suppose X is β_1-homogeneous F-space and Y is β_2-homogeneous F-space. Before applying Theorem 2.1 we would like to make an answer that all roads lead to Rome. Therefore another approach to prove the following functional inequality has been stated in the following. In fact, there is also a similar solution about functional equation being stated in [8].

Theorem 3.1 Assume the mapping $f : X \to Y$ fulfilling for some $K \geq 0$ and $r < \frac{\beta_2}{\beta_1}$

$$\begin{aligned}&\|f(x+y+z)+f(x)+f(y)+f(z)-f(x+y)-f(z+y)-f(x+z)\|\\ &\leqslant K\left(\|x\|^r+\|y\|^r+\|z\|^r\right)\end{aligned} \tag{3.1}$$

for $x, y, z \in X$. Then there has a unique limiting mapping $\psi_1 : X \to Y$ such that

$$\|f(x)-\psi_1(x)\| \leqslant \frac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{(2^{\beta_1 r}-2^{2\beta_2})(2^{\beta_1 r}-2^{\beta_2})}\|x\|^r$$

for $x \in X$. Moreover, ψ_1 satisfying the above inequality is also satisfying the following equation

$$\psi_1(x+y+z)+\psi_1(x)+\psi_1(z)+\psi_1(y)=\psi_1(x+y)+\psi_1(z+y)+\psi_1(x+z) \tag{3.2}$$

for all $x, y, z \in X$.

Proof. From (x, x, x) in place of (x, y, z) in (3.1) we have

$$\|f(3x)+3f(x)-3f(2x)\| \leqslant 3K\left(\|x\|^r\right).$$

Hence

$$\|2f(3x)+6f(x)-6f(2x)\| \leqslant 3\cdot 2^{\beta_2}K\left(\|x\|^r\right).$$

Substitute $(x,x,2x)$ in place of (x,y,z) in (3.1), yielding that

$$\|f(4x)+2f(x)-2f(3x)\| \leqslant (2+2^{r\beta_1})K\left(\|x\|^r\right).$$

And combining the above two inequalities, we get

$$\|f(4x)+8f(x)-6f(2x)\| \leqslant (2+2^{r\beta_1}+3\cdot 2^{\beta_2})K\left(\|x\|^r\right). \tag{3.3}$$

Let us define $g(x)=f(2x)-4f(x)$ for all $x\in X$. Hence

$$\|g(2x)/2-g(x)\| \leqslant (2+2^{r\beta_1}+3\cdot 2^{\beta_2})K\left(\|x\|^r\right)/2^{\beta_2} \tag{3.4}$$

for all $x\in X$. Therefore

$$\|g(2^n x)/2^n-g(2^m x)/2^m\| \leqslant \sum_{j=m}^{n-1}(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K\frac{2^{j\beta_1 r}}{2^{\beta_2}2^{j\beta_2}}\left(\|x\|^r\right) \tag{3.5}$$

for $m,\ n\in N$ with $n>m$ and all $x\in X$. Since the sequence $\{g(2^n x)/2^n\}$ is a Cauchy sequence in Y for all $x\in X$ and Y is complete, the mapping can be well defined as:

$$\phi(x)=\lim_{n\to\infty} g(2^n x)/2^n$$

for all $x\in X$. In particular, letting $m=0$ and setting $n\to\infty$ in (3.4), we have

$$\|\phi(x)-g(x)\| \leqslant \tfrac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{2^{\beta_1 r}-2^{\beta_2}}\left(\|x\|^r\right). \tag{3.6}$$

Now, we prove the mapping ϕ is additive and is unique. From $(x,y,y+x)$ in (3.1) yields that

$$\|f(2x+2y)+f(x)+f(y)-f(x+2y)-f(2x+y)\| \leqslant K\left(\|x\|^r+\|y\|^r+\|x+y\|^r\right).$$

From (x,x,y) in equation (3.1) yields that

$$\|f(2x+y)+2f(x)+f(y)-f(2x)-2f(x+y)\| \leqslant K\left(2\|x\|^r+\|y\|^r\right).$$

From (x,y,y) in (3.1) we have

$$\|f(x+2y)+f(x)+2f(y)-f(2y)-2f(x+y)\| \leqslant K\left(\|x\|^r+2\|y\|^r\right).$$

Combining the above three inequalities, we have that for $x, y, z \in X$

$$\begin{aligned}
&\|\phi(x+y)-\phi(x)-\phi(y)\| \\
=&\lim_{n\to\infty}\frac{1}{2^{\beta_2 n}}\|f(2^{n+1}x+2^{n+1}y)+4f(2^nx)+4f(2^ny)-f(2^{n+1}x)-f(2^{n+1}y)-4f(2^nx+2^ny)\| \\
\leqslant&\lim_{n\to\infty}\frac{1}{2^{\beta_2 n}}\left\|f(2^{n+1}x+2^{n+1}y)+f(2^nx)+f(2^ny)-f(2^{n+1}x+2^ny)-f(2^{n+1}y+2^nx)\right\| \\
&+\lim_{n\to\infty}\frac{1}{2^{\beta_2 n}}\left\|f(2^{n+1}x+2^ny)+2f(2^nx)+f(2^ny)-f(2^{n+1}x)-2f(2^nx+2^ny)\right\| \\
&+\lim_{n\to\infty}\frac{1}{2^{\beta_2 n}}\left\|f(2^nx+2^{n+1}y)+f(2^nx)+2f(2^ny)-f(2^{n+1}y)-2f(2^nx+2^ny)\right\| \\
\leqslant&\lim_{n\to\infty}\frac{2^{\beta_1 rn}}{2^{\beta_2 n}}K\left(4\|x\|^r+4\|y\|^r+\|x+y\|^r\right).
\end{aligned}$$

So we have $\phi(x+y)=\phi(x)+\phi(y)$ for all $x, y \in X$.

Next, the uniqueness of the mapping ϕ will be proved. Let $u(x)$ be another additive mapping such that for some $K_2 \geq 0$ and $r < \frac{\beta_2}{\beta_1}$, $\|g(x)-u(x)\| \leqslant K_2\|x\|^{r_2}$. Hence

$$\begin{aligned}
\|\phi(x)-u(x)\| =&\|\phi(nx)-u(nx)\|/n^{\beta_2} \\
\leqslant&\|\phi(nx)-g(nx)\|/n^{\beta_2}+\|g(nx)-u(nx)\|/n^{\beta_2} \\
\leqslant&\frac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{2^{\beta_1 r}-2^{\beta_2}}\|x\|^r n^{r\beta_1-\beta_2}+K_2\|x\|^{r_2}n^{r_2\beta_1-\beta_2}
\end{aligned}$$

for all $x \in X$. Therefore $\phi(x)=u(x)$ for all $x \in X$. by the condition $r < \frac{\beta_2}{\beta_1}$. So there has a unique additive limiting mapping ϕ fulfilling

$$\|(f(x)-\frac{1}{2}\phi(x))-(f(2x)-\frac{1}{2}\phi(2x))/4\| \leq \frac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{2^{\beta_1 r}-2^{\beta_2}}\|x\|^r/2^{2\beta_2}.$$

Hence

$$\|(f(x)-\frac{1}{2}\phi(x))-(f(2^nx)-\frac{1}{2}\phi(2^nx))/4^n\| \leq \sum_{j=0}^{n-1}\frac{2^{\beta_1 rj}}{2^{2\beta_2 j}}\frac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{2^{2\beta_2}(2^{\beta_1 r}-2^{\beta_2})}\|x\|^r.$$

Then the mapping can be well defined as

$$\psi(x)=\lim_{n\to\infty}(f(2^nx)-\frac{1}{2}\phi(2^nx))/4^n$$

for all $x \in X$, by the completeness of the space Y. Thus

$$\|\psi(x)-f(x)+\phi(x)/2\| \leq \frac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{(2^{\beta_1 r}-2^{\beta_2})(2^{\beta_1 r}-2^{2\beta_2})}\|x\|^r.$$

Let $u(x)$ be another limiting mapping which has the same property to the function $\psi(x)$ such that,

$$\begin{aligned}
\|u(x)-\phi(x)\| \leqslant&\|u(x)-(f(2^nx)-\frac{1}{2}\phi(2^nx))/4^n\|+\|(f(2^nx)-\frac{1}{2}\phi(2^nx))/4^n-\phi(x)\| \\
\leqslant&2\sum_{j=n}^{\infty}\frac{2^{\beta_1 rj}}{2^{2\beta_2 j}}\frac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{2^{2\beta_2}(2^{\beta_1 r}-2^{\beta_2})}\|x\|^r
\end{aligned}$$

which shows that the approximation function $\phi(x)$ is unique. Finally, it remains to prove that $\phi(x)$ satisfies (3.2) and we obtain

$$\begin{aligned}&\tfrac{1}{4^n}\|f(2^nx+2^ny+2^nz)+f(2^nx)+f(2^ny)+f(2^nz)-\\&f(2^nx+2^ny)-f(2^nz+2^ny)-f(2^nx+2^nz)\|\\&\leqslant \tfrac{2^{\beta_1 n}}{4^n}K\left(\|x\|^r+\|y\|^r+\|z\|^r\right).\end{aligned} \tag{3.7}$$

Letting $n\to\infty$, and we get our assertion by using the additivity of $\phi(x)$. □

In another direction, we will describe the similar stability results of the above Theorem 3.1.

Theorem 3.2 Let $r>\frac{\beta_2}{\beta_1}$ and assume that $f:X\to Y$ is a mapping satisfying the equation (3.1). Then there has a unique limiting mapping $\psi_1:X\to Y$ satisfying

$$\|f(x)-\psi_1(x)\|\leqslant\frac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{(2^{\beta_1 r}-2^{2\beta_2})(2^{\beta_1 r}-2^{\beta_2})}\|x\|^r$$

for all $x\in X$. Moreover, ψ_1 solves also the following equation

$$\psi_1(x+y+z)+\psi_1(x)+\psi_1(z)+\psi_1(y)=\psi_1(x+y)+\psi_1(z+y)+\psi_1(x+z) \tag{3.8}$$

for all $x,y,z\in X$.

Proof. According to the equation (3.3), we obtain

$$\|g(x)-2g(\frac{x}{2})\|\leqslant(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K\left(\|x\|^r\right)/2^{\beta_1 r}.$$

Therefore

$$\|2^ng(\frac{x}{2^n})-2^mg(\frac{x}{2^m})\|\leqslant\sum_{j=m}^{n-1}(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K\frac{2^{j\beta_2}}{2^{j\beta_1 r}2^{\beta_1 r}}\left(\|x\|^r\right)$$

for $m,n\in N$ with $n>m$ and $x\in X$. Since the sequence $\{2^ng(\frac{x}{2^n})\}$ is a Cauchy sequence in Y for all $x\in X$ and Y is complete, the mapping can be well defined as:

$$\phi(x)=\lim_{n\to\infty}2^ng(\frac{x}{2^n})$$

for all $x\in X$. Using a similar manner, we can complete the rest part. □

If $f(x)$ is odd, then $(x,y,-x-y)$ in (3.2) can give a precise condition to ascertain the additive property of the function $f(x)$ (See [9]). Obviously, the additive property is stronger than the property of the equation (3.2), but vice versa is not true. In contrast with the subadditive property, we can not get obvious strong or weak property temporarily. By using another approach to solve the Theorem 3.1, according to (3.6), we have

$$\|f(2^nx)/2^{2n}-f(x)-\sum_{j=0}^{n-1}\phi(2^jx)/2^{2(j+1)}\|\leq\sum_{j=0}^{n-1}\frac{2^{\beta_1 rj}}{2^{2\beta_2 j}}\frac{(2+2^{r\beta_1}+3\cdot 2^{\beta_2})K}{2^{2\beta_2}(2^{\beta_1 r}-2^{\beta_2})}\|x\|^r.$$

Then the mapping can be well defined as

$$\psi(x)=\lim_{n\to\infty}f(2^nx)/2^{2n}$$

for all $x \in X$, by the completeness of the space Y. Thus

$$\|\psi(x) - f(x) - \phi(x)/2\| \leq \frac{(2 + 2^{r\beta_1} + 3 \cdot 2^{\beta_2})K}{(2^{\beta_1 r} - 2^{\beta_2})(2^{\beta_1 r} - 2^{2\beta_2})}\|x\|^r.$$

In a similar way, we can use two steps to prove that the mapping $\psi(x)$ is unique. The first step we show that the mapping satisfies the property: $\psi(kx) = k^2\psi(x)$ for all $k \in N$, $x \in X$. We prove this by mathematical induction, for a fixed element $x \in X$. We will prove that the property is true for $k = 2$. From $(x, -x, x)$ in equation (3.1), we can get that

$$\|3f(x) + f(-x) - f(2x)\| \leqslant 3K\left(\|x\|^r\right)$$

for all $x \in X$.

Thus

$$\begin{aligned}\|f(-x) - f(x) - \phi(x)\| &\leq \|f(2x) - 4f(x) - \phi(x)\| + \|3f(x) + f(-x) - f(2x)\| \\ &\leqslant \left(\frac{(2 + 2^{r\beta_1} + 3 \cdot 2^{\beta_2})K}{2^{\beta_1 r} - 2^{\beta_2}} + 3K\right)\left(\|x\|^r\right)\end{aligned}$$

for all $x \in X$. Using the similar above argumentation together the above inequality and equation (3.1), yields

$$\psi(-x) = \psi(x) + \lim_{n\to\infty}\frac{\phi(x)}{2^n}$$

and

$$\psi(x + y + z) + \psi(x) + \psi(z) + \psi(y) = \psi(x + y) + \psi(z + y) + \psi(x + z) \tag{3.9}$$

for all $x, y, z \in X$. From $(x, -x, x)$ in equation (3.7), we achieve

$$\psi(2x) = 3\psi(x) + \psi(-x) = 4\psi(x).$$

Fixed $x \in X$, we prove this by induction. We have already proved that the property is true for $n = 2$. Supposing that $\psi(nx) = n^2\psi(x)$ for all natural $n \leq 2k$, with $k \geq 1$, let us calculate $\psi((2k + 1)x)$. From (kx, kx, x) in (3.7), we know

$$\begin{aligned}\psi((2k + 1)x) &= \psi(2kx) + 2\psi((k + 1)x) - 2\psi(kx) - \psi(x) \\ &= (4k^2 + 2(k + 1)^2 - 2k^2 - 1)\psi(x) \\ &= (2k + 1)^2\psi(x).\end{aligned}$$

Now, we show the mapping ψ satisfies the property $\psi(kx) = k^2\psi(x)$ for all $k \in N$, $x \in X$. The second step, we claim that the mapping ϕ is unique. Let $u(x)$ be another limiting mapping such that for some $K_2 \geq 0$ and $r < \frac{\beta_2}{\beta_1}$,

$$\|u(x) - f(x) - \phi(x)/2\| \leqslant K_2\|x\|^{r_2}$$

which satisfies the property $u(kx) = k^2 u(x)$ for all $k \in N$ and $x \in X$. Therefore

$$\begin{aligned}
&\|\psi(x) - u(x)\| \\
=&\|\psi_1(kx) - u(kx)\|/k^{2\beta_2} \\
\leqslant&\|u(xk) - f(kx) - \phi(kx)/2\|/k^{2\beta_2} + |\psi(xk) - f(kx) - \phi(kx)/2\|/k^{2\beta_2} \\
\leqslant&\frac{(2 + 2^{r\beta_1} + 3 \cdot 2^{\beta_2})K}{(2^{\beta_1 r} - 2^{2\beta_2})(2^{\beta_1 r} - 2^{\beta_2})}\|x\|^r k^{r\beta_1 - 2\beta_2} + |K_2\|x\|^{r_2} k^{r_2\beta_1 - 2\beta_2}.
\end{aligned}$$

Hence $\phi(x) = u(x)$ for all $x \in X$. This shows that ψ is unique. Let $\psi_1(x) = \psi(x) - \phi(x)/2$. This completes the uniqueness of $\psi_1(x)$. We have

$$\|\psi_1(x) - f(x)\| \leqslant \frac{(2 + 2^{r\beta_1} + 3 \cdot 2^{\beta_2})K}{(2^{\beta_1 r} - 2^{2\beta_2})(2^{\beta_1 r} - 2^{\beta_2})}\|x\|^r$$

for all $x \in X$ and also the equation (3.2) holds by using the additive property of ϕ and equation (3.7). We complete the proof. We may also assume that $\lim_{n\to\infty} \frac{\phi(x)}{2^n} = \lim_{n\to\infty} \frac{g(2^n x)/2^n}{2^n} = 0$. Otherwise, this limit may not be convergence to zero. Conversely, we may add some similar small additional assumptions to guarantee the convergence in Theorem 3.2.

4. The stability of functional equations in Banach space

In this section, we will prove the counterpart of the results of Theorem 2.1 from [8] to more general case. We generalize the results of Sikorska in 2010. In particular, the related parameters $u,\ v$ can be extended to complex numbers by using a more efficient approach. Beyond that, we state that the first results in section 2 are presented and combined the first results in [8]. Our contribution to the parameters u, v are complex numbers. The results is stated in this section in more detail.

Theorem 4.1 Suppose that $(X, +)$ is a group, and $(Y, \|\cdot\|)$ is a Banach space, and let the mapping $f : X \to Y$ satisfy the inequality

$$\|f(x) - uf(e(x)) - vf(-e(x))\| \leqslant \delta(x), \quad x \in X,$$

where $u, v \in C$ (C denotes the complex field.), and $e : X \to X,\ \delta : X \to [0, \infty)$ are arbitrary given functions.

(1): If e is a even function (i.e., $e(-x) = e(x)$ for $x \in X$) and the convergent series

$$\sum_{n=0}^{\infty} [|u_n|\,\delta\,(e^n(x)) + |v_n|\delta\,(-e^n(x))]$$

with

$$\begin{aligned}
u_0 &:= 1, \quad u_n := \left[u(u+v)^{n-1}\right], \quad n \in \mathbb{N}, \\
v_0 &:= 0, \quad v_n := \left[v(u+v)^{n-1}\right], \quad n \in \mathbb{N}
\end{aligned}$$

(and where e^n states the n-th composition of the function e), establishes for every $x \in X$. Then there has a unique even limiting function $g : X \to Y$ fulfilling

$$g(x) = u_n g(e^n(x)) + v_n g(-e^n(x)), \quad x \in X \text{ and } n \in N, \tag{4.1}$$

and

$$\|f(x) - g(x)\| \leqslant \sum_{i=0}^{\infty} \left[|u_i|\, \delta\left(e^i(x)\right) + |v_i|\, \delta\left(-e^i(x)\right)\right], \quad x \in X. \tag{4.2}$$

(2): If e is odd (i.e., $e(-x) = -e(x)$ for *all* $x \in X$) and the convergent series

$$\sum_{n=0}^{\infty} \left[|u_n|\, \delta\left(e^n(x)\right) + |v_n|\, \delta\left(-e^n(x)\right)\right].$$

with

$$u_0 := 1, \quad u_n := \tfrac{1}{2}\left[(u+v)^n + (u-v)^n\right], \quad n \in \mathbb{N},$$
$$v_0 := 0, \quad v_n := \tfrac{1}{2}\left[(u+v)^n - (u-v)^n\right], \quad n \in \mathbb{N}$$

establishes for all $x \in X$. Then there has a unique limiting mapping $g : X \to Y$ fulfilling (3.10) and (3.11).

Proof. We only need to prove the uniqueness of the approximation function. (1): Let us suppose that $\widetilde{g} : X \to Y$ is another approximating mapping. So let's first prove the inequality together with the equation (2.5) and $g(-x) = g(x)$

$$\begin{aligned}
&\|f(e^m(x)) - u_m(u_n f\left(e^{n+m}(x)\right) + v_n f\left(-e^{m+n}(x)\right)) \\
&- v_m(u_n f\left(e^{n+m}(x)\right) + v_n f\left(-e^{m+n}(x)\right))\| \\
=&\|f(e^m(x)) - u_{n+m} f\left(e^{n+m}(x)\right) - v_{n+m} f\left(-e^{n+m}(x)\right)\| \\
\leqslant& \sum_{j=m}^{n+m-1} \left[\left|u_j\right| \delta\left(e^j(x)\right) + \left|v_j\right| \delta\left(-e^j(x)\right)\right],
\end{aligned}$$

and letting $n \to \infty$ we have for any $m \in \mathbb{N}$

$$\|f(e^m(x)) - u_m g\left(e^m(x)\right) - v_m g\left(-e^m(x)\right)\| \leqslant \sum_{j=m}^{\infty} \left[\left|u_j\right| \delta\left(e^j(x)\right) + \left|v_j\right| \delta\left(-e^j(x)\right)\right],$$

and we can rewrite

$$\begin{aligned}
\|g(x) - \widetilde{g}(x)\| \leqslant& \|f(k^m(x)) - u_m g\left(e^m(x)\right) - v_m g\left(-e^m(x)\right)\| \\
&+ \|f(k^m(x)) - u_m \widetilde{g}\left(e^m(x)\right) - v_m \widetilde{g}\left(e^m(x)\right)\| \\
\leqslant& 2\sum_{j=m}^{\infty} \left[\left|u_j\right| \delta\left(e^j(x)\right) + \left|v_j\right| \delta\left(-e^j(x)\right)\right]
\end{aligned}$$

for any $x \in X$ and $m \in \mathbb{N}$, which yields $g = \widetilde{g}$ in X as $m \to \infty$.

(2): Combined with the results of Theorem 2.1 from [8] where e is odd, we only need to prove the uniqueness of the approximation function. Let us suppose that $\widetilde{g} : X \to Y$ is another approximating mapping. So let's first prove the inequality together with the equation the results in Theorem 2.1 in [8]

$$\begin{aligned}&\|f(e^m(x)) - u_m(u_nf\left(e^{n+m}(x)\right) + v_nf\left(-e^{m+n}(x)\right))\\&- v_m(u_nf\left(-e^{n+m}(x)\right) + v_nf\left(e^{m+n}(x)\right))\|\\=&\|f(e^m(x)) - u_{n+m}f\left(e^{n+m}(x)\right) - v_{n+m}f\left(-e^{n+m}(x)\right)\|\\\leqslant&\sum_{j=m}^{n+m-1}\left[|u_j|\,\delta\left(e^j(x)\right) + |v_j|\,\delta\left(-e^j(x)\right)\right],\end{aligned}$$

and letting $n\to\infty$ we have for any $m\in\mathbb{N}$

$$\|f(e^m(x)) - u_mg\left(e^m(x)\right) - v_mg\left(-e^m(x)\right)\| \leqslant \sum_{j=m}^{\infty}\left[|u_j|\,\delta\left(e^j(x)\right) + |v_j|\,\delta\left(-e^j(x)\right)\right],$$

and we can rewrite

$$\begin{aligned}\|g(x) - \widetilde{g}(x)\| \leqslant&\|f(k^m(x)) - u_mg\left(e^m(x)\right) - v_mg\left(-e^m(x)\right)\|\\&+ \|f(k^m(x)) - u_m\widetilde{g}\left(e^m(x)\right) - v_m\widetilde{g}\left(e^m(x)\right)\|\\\leqslant&2\sum_{j=m}^{\infty}\left[|u_j|\,\delta\left(e^j(x)\right) + |v_j|\,\delta\left(-e^j(x)\right)\right]\end{aligned}$$

for any $x\in X$ and $m\in\mathbb{N}$, which yields $g=\widetilde{g}$ in X as $m\to\infty$. This completes the proof. □

For the Euler-Lagrange equation, we provide another method to solve it in contrast with [10].

Theorem 4.2 Suppose that $(X,+)$ is a group, and $(Y,\|\cdot\|)$ is a Banach space and let the mapping $f: X\to Y$ satisfy the inequality for all $x,y,z\in X$ and some $\varepsilon>0$

$$\|f(x+y+z)+f(x-y+z)+f(x+y-z)+f(x-y-z)-4f(x)-4f(y)-4f(z)\|\leqslant\varepsilon. \quad (4.3)$$

Then there has *a* unique limiting function $g: X\to Y$ such that

$$g(x)=\frac{2}{9}g(3x)-\frac{1}{9}g(-3x), \quad x\in X$$

and

$$\|f(x)-g(x)\|\leqslant\frac{3\varepsilon}{8} \quad x\in X.$$

In particular, if X is Abelian, then g is a solution of the equation in the following

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(x-y-z)=4f(x)+4f(y)+4f(y), \quad (4.4)$$

for all $x,y\in X$.

Proof. From $(x,x,-x)$ in (4.3), we obtain

$$\|6f(x)+3f(-x)-f(3x)\|\leqslant\varepsilon, \quad x\in X.$$

Replacing x by $-x$ in the above inequality we obtain

$$\|6f(-x)+3f(x)-f(-3x)\|\leqslant\varepsilon, \quad x\in X.$$

Consequently, combining the above two inequalities yield that

$$\|9f(x)+f(-3x)-2f(3x)\| \leqslant 3\varepsilon, \quad x \in X.$$

By using the results second part of Theorem 4.1, a computation is to prove that

$$u_n := \frac{3^n+1}{2\cdot 9^n}, \quad v_n := \frac{1-3^n}{2\cdot 9^n}, \quad n \in \mathbb{N}.$$

and the convergent series can be described as

$$\sum_{n=0}^{\infty}\left[|u_n|\,\delta\left(e^n(x)\right)+|v_n|\delta\left(-e^n(x)\right)\right] = \frac{3\varepsilon}{8}.$$

And we show that if X is commutative, by using $(x,y,z)=(3^n x, 3^n y, 3^n z)$, then

$$\begin{aligned}
&\|u_n[f(3^n(x+y+z))+f(3^n(x-y+z))+f(3^n(x+y-z))+f(3^n(x-y-z))\\
&-4f(3^n x)-4f(3^n y)-4f(3^n y)]+v_n[f(3^n(x+y+z))+f(3^n(x-y+z))\\
&+f(3^n(x+y-z))+f(3^n(x-y-z))-4f(3^n x)-4f(3^n y)-4f(3^n y)]\|\\
\leqslant&\frac{\varepsilon}{9^n}
\end{aligned}$$

which we achieve our result (3.13) by letting $n\to\infty$.

□

Theorem 4.3 Suppose that X is a group, and $(Y,\|\cdot\|)$ is a Banach space and let the mapping $f: X\to Y$ satisfy the inequality for all $x,y\in X$ and some $\varepsilon>0$

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\| \leqslant \varepsilon, \quad x,y \in X. \tag{4.5}$$

Then there has *a* unique limiting function $g: X\to Y$ fulfilling

$$g(x) = \frac{3}{8}g(2x)-\frac{1}{8}g(-2x), \quad x \in X$$

and

$$\|f(x)-g(x)\| \leqslant \frac{2\varepsilon}{3} \quad x \in X.$$

In particular, if X is commutative, then g also fulfils

$$g(x+y)+g(x-y) = 2g(x)+g(y)+g(-y), \quad x,y \in X.$$

Proof. Substituting in the sequel (x,x) in (4.5), we obtain

$$\|f(2x)+f(0)-3f(x)-f(-x)\| \leqslant \varepsilon, \quad x \in X. \tag{4.6}$$

Replacing x by $-x$ in (4.6) we have

$$\|f(-2x)+f(0)-3f(-x)-f(x)\| \leqslant \varepsilon, \quad x \in X. \tag{4.7}$$

Consequently, (4.6) and (4.7) yield that

$$\|8f(x)+f(-2x)-3f(2x)\| \leqslant 4\varepsilon, \quad x \in X.$$

By using the results of Theorem 3.3, a computation is to prove that

$$u_n := \frac{2^n+1}{2\cdot 4^n}, \quad v_n := \frac{1-2^n}{2\cdot 4^n}, \quad n\in\mathbb{N}.$$

and the convergent series

$$\sum_{n=0}^{\infty}\left[|u_n|\,\delta\left(e^n(x)\right)+|v_n|\delta\left(-e^n(x)\right)\right]=\frac{2\varepsilon}{3}.$$

And we show that if X is commutative, by using $(x,y)=(2^nx,2^ny)$, then

$$\begin{aligned} &\|u_n[f(2^nx+2^ny)+f(2^nx-2^ny)-2f(2^nx)-f(2^ny)-f(-2^ny)]\\ &+v_n[f(2^nx+2^ny)+f(2^nx-2^ny)-2f(2^nx)-f(2^ny)-f(-2^ny)]\|\\ \leqslant&\frac{\varepsilon}{4^n}\end{aligned}$$

which we achieve our result $(*)$ by letting $n\to\infty$. □

If we can not set $f(0)=0$, then the approximate constat is $\frac{5}{6}\varepsilon$.

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Conflict of Interest

The author(s) declare(s) that there is no conflict of interest regarding this manuscript.

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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