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Complex Oscillation of Solutions and Their Arbitrary-Order Derivatives of Linear Differential Equations With Analytic Coefficients of $[p, q]$-Order in the Unit Disc

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Abstract. Throughout this article, we investigate the growth and fixed points of solutions of complex higher order linear differential equations in which the coefficients are analytic functions of $[p, q]-$order in the unit disc. This work improves some results of Belaïdi [3–5], which is a generalization of recent results from Chen *et al.* [9].

1. Introduction and main results

Consider for $k \geq 2$ the following complex linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.2}$$

where $A_i \not\equiv 0$ $(i = 0, 1, ..., k)$ are analytic functions in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. It is well-known that the solutions of (1.1) are analytic in D too and that there are exactly k linearly independent solutions of (1.1), see [13]. Bernal [6] was the first to use the concept of iterated order to study the growth of fast growing solutions of equation (1.1). After that, the iterated order of solutions of higher order equations was investigated by Cao in [8], he extended the results of Chen and Yang [10], Belaïdi [2] on $\mathbb{C}$. In addition, Cao [8] obtained some results concerning the fixed points of homogeneous linear differential equations (1.1) and (1.2). In [15,16], Juneja and his co-authors have investigated some properties of entire functions of $[p, q]$-order, and obtained some results of their growth. In [20], by using the concept of $[p, q]$-order Liu, Tu and Shi have considered the equation (1.1) with entire coefficients and obtained different results concerning the growth of its solutions in the complex plane. In [3], the $[p, q]-$order was introduced in the unit disc D, and many results on $[p, q]-$order of solutions of (1.1) have been found by different researchers [3–5,14,18,22] in D. Recently, Chen *et al.* in [9] gave some results about the growth and fixed points of solutions

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of higher-order linear differential equations in the unit disc, they studied and estimated the fixed points of solutions of (1.1) and (1.2), and also extended the coefficient conditions to a type of one-constant-control coefficient comparison and obtained the same estimates of iterated order of solutions. The aim of this paper is to contrast coefficients by producing better estimates of the growth of solutions by using the concept of $[p, q]-$order, and optimizing the coefficients's conditions with less control constants of the coefficients's modulus or characteristic functions and we will obtain results which improve and generalize those of Chen *et al.*, Belaïdi, Cao, Tu and Xuan.

Throughout this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see, [12, 13, 17, 21]).

Now, we give the definitions of iterated order and growth index to classify generally the functions of fast growth in D as those in $\mathbb{C}$ (see, [6]). Let us define inductively, for $r \in \mathbb{R}$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0, +\infty)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1 (*see* [7]) *Let f be a meromorphic function in D. Then the iterated $n-$order of f is defined by*

$$\sigma_n(f) = \limsup_{r \to 1^-} \frac{\log_n^+ T(r, f)}{\log\left(\frac{1}{1-r}\right)} \quad (n \geq 1 \text{ is an integer}),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{n+1}^+ x = \log^+\left(\log_n^+ x\right)$. For $n = 1$, this notation is called order $(\sigma_1(f) = \sigma(f))$ and for $n = 2$ hyper-order ([19]). *If f is an analytic in D, then the iterated $n-$order of f is defined by*

$$\sigma_{M,n}(f) = \limsup_{r \to 1^-} \frac{\log_{n+1}^+ M(r, f)}{\log\left(\frac{1}{1-r}\right)} \quad (n \geq 1 \text{ is an integer}).$$

For $n = 1$, $\sigma_{M,1}(f) = \sigma_M(f)$.

Now, we introduce the concept of $[p, q]$-order of meromorphic and analytic functions in the unit disc.

Definition 1.2 ([3]) *Let $p \geq q \geq 1$ be integers and f be a meromorphic function in D. Then, the $[p, q]$-order of f is defined by*

$$\sigma_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r, f)}{\log_q\left(\frac{1}{1-r}\right)}.$$

For an analytic function f in D, we also define

$$\sigma_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p+1}^{+} M(r,f)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Remark 1.1 It is easy to see that $0 \le \sigma_{[p,q]}(f) \le \infty$ $(0 \le \sigma_{M,[p,q]}(f) \le \infty)$, for any $p \ge q \ge 1$. By Definition 1.2, we have that $\sigma_{[1,1]} = \sigma(f)$ $(\sigma_{M,[1,1]} = \sigma_M(f))$ and $\sigma_{[2,1]} = \sigma_2(f)$ $\left(\sigma_{M,[2,1]} = \sigma_{M,2}(f)\right)$.

Proposition 1.1 ([3]) *Let $p \ge q \ge 1$ be integers, and let f be an analytic function in D of $[p,q]$-order. The following two statements hold:*
(i) *If $p = q$, then*

$$\sigma_{[p,q]}(f) \le \sigma_{M,[p,q]}(f) \le \sigma_{[p,q]}(f) + 1.$$

(ii) *If $p > q$, then*

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f).$$

Definition 1.3 ([4]) *Let $p \ge q \ge 1$ be integers and f be a meromorphic function in D. Then, the $[p,q]$-exponent of convergence of the sequence of zeros of f is defined by*

$$\lambda_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_p^{+} N\left(r,\frac{1}{f}\right)}{\log_q\left(\frac{1}{1-r}\right)},$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of f in $\{z : |z| \le r\}$. Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_p^{+} \overline{N}\left(r,\frac{1}{f}\right)}{\log_q \frac{1}{1-r}},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of f in $\{z : |z| \le r\}$.

Definition 1.4 *Let $p \ge q \ge 1$ be integers and f be a meromorphic function in D. Then, the $[p,q]$-exponent of convergence of the sequence of fixed points of f is defined by*

$$\lambda_{[p,q]}(f-z) = \limsup_{r\to 1^-} \frac{\log_p^{+} N\left(r,\frac{1}{f-z}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct fixed points of f is defined by

$$\bar{\lambda}_{[p,q]}(f-z) = \limsup_{r\to 1^-} \frac{\log_p^{+} \bar{N}\left(r,\frac{1}{f-z}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Recall that for a measurable set $E \subset [0,1)$, the upper and lower densities of E are defined by

$$\overline{dens}_D E = \limsup_{r\to 1^-} \frac{m(E\cap[0,r))}{m([0,r))} \quad \text{and} \quad \underline{dens}_D E = \liminf_{r\to 1^-} \frac{m(E\cap[0,r))}{m([0,r))},$$

respectively, where $m(F) = \int_F \frac{dt}{1-t}$ for $F \subset [0,1)$. It is clear that $0 \le \underline{dens}_D E \le \overline{dens}_D E \le 1$ for any measurable set $E \subset [0,1)$.

Proposition 1.2 *If a set E satisfies $\overline{dens}_D E > 0$, then $m(E) = \int_E \frac{dt}{1-t} = +\infty$.*

Proof. Suppose that $m(E) = \int_E \frac{dt}{1-t} = \delta < \infty$. We have

$$m([0,r)) = -\log(1-r).$$

Since $m(E \cap [0,r)) \leq m(E)$, then

$$\overline{dens}_D E = \limsup_{r \to 1^-} \frac{m(E \cap [0,r))}{m([0,r))} \leq \limsup_{r \to 1^-} \frac{\delta}{-\log(1-r)} = 0.$$

So $\overline{dens}_D E = 0$. Hence

$$\overline{dens}_D E > 0 \Longrightarrow m(E) = \int_E \frac{dt}{1-t} = +\infty.$$

In 2012, Belaïdi in [4] and [5] treated the growth of solutions of homogeneous linear differential equations in which the coefficients are analytic functions of $[p,q]-$order in D. As for the equation (1.1), he got the following results.

Theorem A (*see* [4]) *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in the unit disc D such that for real constants α, β, where $0 \leq \beta < \alpha$, we have*

$$|A_0(z)| \geq \exp_{p+1}\left\{\alpha \log_q\left(\frac{1}{1-|z|}\right)\right\}$$

and

$$|A_i(z)| \leq \exp_{p+1}\left\{\beta \log_q\left(\frac{1}{1-|z|}\right)\right\} \quad (i = 1, ..., k-1)$$

as $|z| \to 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geq \alpha$.

Theorem B (*see* [5]) *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in the unit disc D such that for real constants α, β, where $0 \leq \beta < \alpha$, we have*

$$T(r, A_0) \geq \exp_p\left\{\alpha \log_q\left(\frac{1}{1-|z|}\right)\right\}$$

and

$$T(r, A_i) \leq \exp_p\left\{\beta \log_q\left(\frac{1}{1-|z|}\right)\right\} \quad (i = 1, ..., k-1)$$

as $|z| = r \to 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geq \alpha$.

After that in 2021, Chen *et al.* [9] investigated the growth of solutions of equations (1.1) and (1.2) in D by using the iterated order, and they got the following results.

Theorem C (*see* [9]) *Let $n \geq 1$ be an integer. Let H be a set of complex numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0, A_1, ..., A_{k-1}$ be analytic functions in the unit disc D such that*

$$\max \{\sigma_{M,n}(A_i) : i = 1, 2, ..., k-1\} \leq \sigma_{M,n}(A_0) = \mu \ \ (0 < \mu < \infty),$$

and for a constant $\alpha \geq 0$, we have

$$\liminf_{|z| \to 1^-, z \in H} \left((1 - |z|)^{\mu} \log_n |A_0(z)|\right) > \alpha$$

and

$$|A_i(z)| \leq \exp_n \left\{\alpha \left(\frac{1}{1 - |z|}\right)^{\mu}\right\}, \quad (i = 1, 2, ..., k-1)$$

as $|z| \to 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_n(f) = \sigma_{M,n}(f) = \infty$ and $\sigma_{n+1}(f) = \sigma_{M,n}(A_0) = \mu$.

Theorem D (*see* [9]) *Let $n \geq 1$ be an integer. Let H be a set of complex numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0, A_1, ..., A_k$ be analytic functions in the unit disc D, and for some constants $\alpha \geq 0$ and $\mu > 0$, we have*

$$\liminf_{|z| \to 1^-, z \in H} \left((1 - |z|)^{\mu} \log_{n-1} T(r, A_0)\right) > \alpha$$

and

$$T(r, A_i) \leq \exp_{n-1} \left\{\alpha \left(\frac{1}{1 - |z|}\right)^{\mu}\right\}, \quad (i = 1, 2, ..., k)$$

as $|z| = r \to 1^-$ for $z \in H$. Then every meromorphic (or analytic) solution $f \not\equiv 0$ of equation (1.2) satisfies $\sigma_n(f) = \infty$ and $\sigma_{n+1}(f) \geq \mu$.

Theorem E (*see* [9]) *Assume that the assumptions of Theorem C hold. Then every solution $f \not\equiv 0$ of equation (1.1) satisfies*

$$\bar{\lambda}_n\left(f^{(j)} - z\right) = \bar{\lambda}_n(f - z) = \sigma_n(f) = \infty,$$

$$\bar{\lambda}_{n+1}\left(f^{(j)} - z\right) = \bar{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f) = \mu, \quad (j = 1, 2, ...).$$

In this paper, we improve and generalize the recent results of Chen *et al.* [9] by using the concept of $[p, q]-$order instead of the iterated order with less control constant. At the same time, our work improve some results of Belaïdi in [4] and [5]. To be specific, we will decrease the control constants of the coefficients' modulus or characteristic functions and obtain the same results of Belaïdi, Tu and Xuan. Here, we study the problem and get the following results.

Theorem 1.1 *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying*

$$\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0,$$

and let $A_0, ..., A_{k-1}$ be analytic functions in the unit disc D such that

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leq \sigma_{M,[p,q]}(A_0) = \mu \ \ (0 < \mu < +\infty)$$

and for a constant $\alpha \geq 0$, we have

$$\liminf_{|z|\to 1^-, z\in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} > \alpha \tag{1.3}$$

and

$$|A_i(z)| \leq \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \quad (i = 1, ..., k-1) \tag{1.4}$$

as $|z| \to 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

By Theorem 1.1, we easily obtain the following corollary.

Corollary 1.1 ([22]) *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$, and let $A_0, ..., A_{k-1}$ be analytic functions in the unit disc D such that*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leq \sigma_{M,[p,q]}(A_0) = \mu \ \ (0 < \mu < +\infty)$$

and for some real constants α, β where $0 \leq \beta < \alpha$, we have

$$|A_0(z)| \geq \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$|A_i(z)| \leq \exp_p\left\{\beta\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}, \ \ i = 1, ..., k-1$$

as $|z| \to 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Theorem 1.2 *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying*

$$\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0,$$

and let $A_0, ..., A_{k-1}$ be analytic functions in the unit disc D such that

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leq \sigma_{M,[p,q]}(A_0) = \mu \ \ (0 < \mu < +\infty)$$

and

$$\limsup_{|z|\to 1^-, z\in H} \frac{\log_p |A_i(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} < \liminf_{|z|\to 1^-, z\in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} \ \ (i = 1, ..., k-1) \tag{1.5}$$

as $|z| \to 1^-$ *for* $z \in H$*.Then every solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$*.*

Theorem 1.3 *Let* $p \geq q \geq 1$ *be integers. Let* H *be a set of complex numbers satisfying*

$$\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0,$$

and let $A_0, ..., A_{k-1}$ *be analytic functions in the unit disc* D *such that*

$$\max \left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leq \sigma_{M,[p,q]}(A_0) = \mu \ \ (0 < \mu < +\infty)$$

and for a constant $\alpha \geq 0$*, if* $p \geq q \geq 2$ *we have*

$$\liminf_{|z| \to 1^-, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} > \alpha \tag{1.6}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\alpha \left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}, \quad (i = 1, ..., k-1) \tag{1.7}$$

as $|z| = r \to 1^-$ *for* $z \in H$*, then every solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$*. If* $p = q = 1$*, we have*

$$\liminf_{|z| \to 1^-, z \in H} \frac{T(r, A_0)}{\left(\frac{1}{1-|z|}\right)^{\mu}} > (k-1)\alpha \tag{1.8}$$

and

$$T(r, A_i) \leq \alpha \left(\frac{1}{1-|z|}\right)^{\mu}, \quad (i = 1, ..., k-1) \tag{1.9}$$

as $|z| = r \to 1^-$ *for* $z \in H$*, then every nontrivial solution* f *of equation* (1.1) *satisfies* $\sigma(f) = \sigma_M(f) = \infty$ *and* $\sigma_2(f) = \sigma_{M,2}(f) = \mu$*.*

By Theorem 1.3, we easily obtain the following corollary.

Corollary 1.2 ([22]) *Let* $p \geq q \geq 1$ *be integers. Let* H *be a set of complex numbers satisfying* $\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0$*, and let* $A_0, ..., A_{k-1}$ *be analytic functions in the unit disc* D *such that*

$$\max \left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leq \sigma_{M,[p,q]}(A_0) = \mu \, (0 < \mu < +\infty)$$

and for some real constants α, β*, where* $0 \leq \beta < \alpha$*, we have*

$$T(r, A_0) \geq \exp_{p-1}\left\{\alpha \left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\beta \left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \quad (i = 1, ..., k-1)$$

as $|z| = r \to 1^-$ *for* $z \in H$*. Then the following statements hold:*

(i) *If* $p = q = 1$ *and* $0 \leq (k-1)\beta < \alpha$, *then every nontrivial solution* f *of equation* (1.1) *satisfies* $\sigma(f) = \sigma_M(f) = \infty$ *and* $\sigma_2(f) = \sigma_{M,2}(f) = \mu$.

(ii) *If* $p \geq q \geq 2$ *and* $0 \leq \beta < \alpha$, *then every nontrivial solution* f *of equation* (1.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$.

Theorem 1.4 *Let* $p \geq q \geq 1$ *be integers. Let* H *be a set of complex numbers satisfying*

$$\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0,$$

and let $A_0, ..., A_{k-1}$ *be analytic functions in the unit disc* D *such that*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, 2, ..., k-1\right\} \leq \sigma_{M,[p,q]}(A_0) = \mu \ \ (0 < \mu < +\infty)$$

and if $p \geq q \geq 2$, *we have*

$$\limsup_{|z|\to 1^-, z\in H} \frac{\log_{p-1} T(r, A_i)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} < \liminf_{|z|\to 1^-, z\in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}}, \quad (i = 1, ..., k-1) \tag{1.10}$$

as $|z| = r \to 1^-$ *for* $z \in H$, *then every nontrivial solution* f *of equation* (1.1) *satisfies* $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu$. *If* $p = q = 1$, *we have*

$$\limsup_{|z|\to 1^-, z\in H} \frac{(k-1)T(r, A_i)}{\left(\frac{1}{1-|z|}\right)^{\mu}} < \liminf_{|z|\to 1^-, z\in H} \frac{T(r, A_0)}{\left(\frac{1}{1-|z|}\right)^{\mu}}, \ \ (i = 1, ..., k-1) \tag{1.11}$$

as $|z| = r \to 1^-$ *for* $z \in H$, *then every nontrivial solution* f *of equation* (1.1) *satisfies* $\sigma(f) = \sigma_M(f) = \infty$ *and* $\sigma_2(f) = \sigma_{M,2}(f) = \mu$.

Theorem 1.5 *Let* $p \geq q \geq 1$ *be integers. Let* H *be a set of complex numbers satisfying*

$$\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0,$$

and let $A_0, ..., A_k$ *be analytic functions in the unit disc* D *such that for some constants* $\alpha \geq 0$ *and* $\mu > 0$, *we have* (1.3) *and*

$$|A_i(z)| \leq \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}, \quad (i = 1, ..., k)$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma_{[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) \geq \mu$.

Theorem 1.6 *Let* $p \geq q \geq 1$ *be integers. Let* H *be a set of complex numbers satisfying*

$$\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0,$$

and let $A_0, ..., A_k$ *be analytic functions in the unit disc* D *such that for a constant* $\mu > 0$, *we have*

$$\limsup_{|z|\to 1^-, z\in H} \frac{\log_p |A_i(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} < \liminf_{|z|\to 1^-, z\in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}}, \quad (i = 1, ..., k)$$

as $|z| \to 1^-$ *for* $z \in H$.*Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma_{[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) \geq \mu$.

Theorem 1.7 *Let* $p \geq q \geq 1$ *be integers. Let* H *be a set of complex numbers satisfying*

$$\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0,$$

and let $A_0, ..., A_k$ *be analytic functions in the unit disc* D *such that for some constants* $\alpha \geq 0$ *and* $\mu > 0$, *if* $p \geq q \geq 2$ *we have*

$$\liminf_{|z|\to 1^-, z\in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} > \alpha \tag{1.12}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}, \quad (i = 1, ..., k) \tag{1.13}$$

as $|z| = r \to 1^-$ *for* $z \in H$, *then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma_{[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) \geq \mu$. *If* $p = q = 1$, *we have*

$$\liminf_{|z|\to 1^-, z\in H} \frac{T(r, A_0)}{\left(\frac{1}{1-|z|}\right)^{\mu}} > k\alpha \tag{1.14}$$

and

$$T(r, A_i) \leq \alpha\left(\frac{1}{1-|z|}\right)^{\mu}, \quad (i = 1, ..., k) \tag{1.15}$$

as $|z| = r \to 1^-$ *for* $z \in H$, *then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma(f) = \infty$ *and* $\sigma_2(f) \geq \mu$.

Theorem 1.8 *Let* $p \geq q \geq 1$ *be integers. Let* H *be a set of complex numbers satisfying*

$$\overline{dens}_D \{|z| : z \in H \subseteq D\} > 0,$$

and let $A_0(z), ..., A_k(z)$ *be analytic functions in the unit disc* D *such that for a constant* $\mu > 0$, *if* $p \geq q \geq 2$ *we have*

$$\limsup_{|z|\to 1^-, z\in H} \frac{\log_{p-1} T(r, A_i)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} < \liminf_{|z|\to 1^-, z\in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}}, \quad (i = 1, ..., k) \tag{1.16}$$

as $|z| = r \to 1^-$ *for* $z \in H$*, then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma_{[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) \geq \mu$*. If* $p = q = 1$*, we have*

$$\limsup_{|z|\to 1^-, z\in H} \frac{kT(|z|, A_i)}{\left(\frac{1}{1-|z|}\right)^{\mu}} < \liminf_{|z|\to 1^-, z\in H} \frac{T(|z|, A_0)}{\left(\frac{1}{1-|z|}\right)^{\mu}}, \quad (i = 1, ..., k) \tag{1.17}$$

as $|z| = r \to 1^-$ *for* $z \in H$*, then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies* $\sigma(f) = \infty$ *and* $\sigma_2(f) \geq \mu$*.*

Remark 1.2 For equation (1.1), we can easily conclude that Theorems A-C are generalized to Theorems 1.1-1.4.

In the same paper, Chen *et al.* [9] obtained some results of the fixed points of solutions and their arbitrary order derivatives of equations (1.1) and (1.2). Here, we generalize these results, and we obtain our theorems as following.

Theorem 1.9 *Assume that the assumptions of Theorem 1.1 or Theorem 1.2 hold. Then every solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies*

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \lambda_{[p,q]}(f - z) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p+1,q]}(f - z) = \sigma_{[p+1,q]}(f) = \mu, \ (j = 1, 2, ...).$$

Theorem 1.10 *Assume that the assumptions of Theorem 1.3 or Theorem 1.4 hold. Then every solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies*

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \lambda_{[p,q]}(f - z) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p+1,q]}(f - z) = \sigma_{[p+1,q]}(f) = \mu, \ (j = 1, 2, ...).$$

Theorem 1.11 *Assume that the assumptions of one of Theorem 1.5 to Theorem 1.8 hold. Then every meromorphic (or analytic) solution* $f \not\equiv 0$ *of equation* (1.2) *satisfies*

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \lambda_{[p,q]}(f - z) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p+1,q]}(f - z) = \sigma_{[p+1,q]}(f) \geq \mu, \ (j = 1, 2, ...).$$

2. Some lemmas

In this section we give some lemmas which are used in the proofs of our theorems.

Lemma 2.1 ([11], Theorem 3.1) *Let k and j be integers satisfying $k > j \geq 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If f is a meromorphic function in D such that $f^{(j)}$ does not vanish identically, then for $|z| \notin E_1$*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\left(\frac{1}{1-|z|}\right); T\left(s\left(|z|\right), f\right)\right\}\right]^{k-j},$$

where $E_1 \subset [0,1)$ is a set with $\int_{E_1} \frac{dr}{1-r} < \infty$ and $s(|z|) = 1 - d(1-|z|)$.

Lemma 2.2 ([13]) *Let f be a meromorphic function in the unit disc D, and let $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$.

Lemma 2.3 ([1]) *Let $g : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ holds outside of an exceptional set $E_3 \subset [0,1)$ for which $\int_{E_3} \frac{dr}{1-r} < \infty$. Then there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

Lemma 2.4 ([3]) *Let $p \geq q \geq 1$ be integers. If $A_0(z), ..., A_{k-1}(z)$ are analytic functions of $[p,q]-$order in the unit disc D, then every solution $f \not\equiv 0$ of (1.1) satisfies*

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \leq \max\left\{\sigma_{M,[p,q]}\left(A_j\right) : j = 0, 1, ..., k-1\right\}.$$

Lemma 2.5 ([4, 18]) *Let $p \geq q \geq 1$ be integers. If f and g are non-constant meromorphic functions of $[p,q]-$order in D, then we have*
(i) $\sigma_{[p,q]}(f) = \sigma_{[p,q]}\left(\frac{1}{f}\right)$, $\sigma_{[p,q]}(af) = \sigma_{[p,q]}(f)$ *and* $\sigma_{[p,q]}(f+a) = \sigma_{[p,q]}(f)$ $(a \in \mathbb{C}^*)$,
(ii) $\sigma_{[p,q]}(f') = \sigma_{[p,q]}(f)$,
(iii) $\sigma_{[p,q]}(f+g) \leq \max\left\{\sigma_{[p,q]}(f), \sigma_{[p,q]}(g)\right\}$,
(iv) $\sigma_{[p,q]}(fg) \leq \max\left\{\sigma_{[p,q]}(f), \sigma_{[p,q]}(g)\right\}$, *if* $\sigma_{[p,q]}(f) > \sigma_{[p,q]}(g)$, *then we obtain* $\sigma_{[p,q]}(f+g) = \sigma_{[p,q]}(fg) = \sigma_{[p,q]}(f)$.

Lemma 2.6 ([4]) *Let $p \geq q \geq 1$ be integers. Let $A_0, ..., A_{k-1}$ and $F \not\equiv 0$ be finite $[p,q]-$order analytic functions in the unit disc D. If f is a solution with $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma < \infty$ of equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F, \tag{2.1}$$

then

$$\bar{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma.$$

By using the same arguments of the proof of Lemma 3.5 in the paper [14, p. 4], we obtain the following lemma in the case when $\sigma_{[p,q]}(f) = \sigma = \infty$.

Lemma 2.7 *Let $p \geq q \geq 1$ be integers. Let A_j $(j = 0, ..., k-1)$, $F \not\equiv 0$ be meromorphic functions in D, and let f be a solution of the differential equation* (2.1) *satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) \quad (j = 0, ..., k-1), \sigma_{[p,q]}(F)\right\} < \sigma_{[p,q]}(f) = \sigma \leq \infty.$$

Then we have

$$\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$$

and

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f).$$

3. Proofs of Theorems 1.1 to 1.8

Proof of Theorem 1.1. Suppose that every solution f of equation (1.1) not being identically equal to 0. From the conditions of Theorem 1.1, there exists a set H of complex numbers satisfying $\overline{dens}_D H_1 > 0$, where $H_1 = \{r = |z| : z \in H \subseteq D\}$. Then H_1 is a set with $\int_{H_1} \frac{dr}{1-r} = +\infty$, such that for $z \in H$ we have (1.3) and (1.4) as $|z| \to 1^-$. By Lemma 2.1, there exists a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$ such that for $|z| \notin E_1$, we have for $j = 1, ..., k$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\left(\frac{1}{1-|z|}\right), T\left(s\left(|z|\right), f\right)\right\}\right]^j, \tag{3.1}$$

where $s(|z|) = 1 - d(1-|z|)$, $d \in (0,1)$. From (1.1), we get

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{3.2}$$

By (1.3), we know that

$$\exists \gamma \in \mathbb{R} : \liminf_{|z| \to 1^-, z \in H} \frac{\log_p |A_0(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} > \gamma > \alpha.$$

Obviously

$$\frac{\log_p |A_0(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} > \gamma > \alpha \geq 0 \tag{3.3}$$

as $|z| \to 1^-$ for $z \in H$. By (1.4) and (3.3), we obtain

$$|A_0(z)| > \exp_p\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$> \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \geq |A_i(z)| \quad (i = 1, 2, ..., k-1) \tag{3.4}$$

as $|z| \to 1^-$ for $z \in H$. Applying (3.1) and (3.4) into (3.2), we have

$$\exp_p\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \le |A_0(z)|$$

$$\le k\left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon}\max\left\{\log\left(\frac{1}{1-|z|}\right), T(s(|z|), f)\right\}\right]^k$$

$$\times \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

holds for all z satisfying $|z| \in H_1 \backslash E_1$ as $|z| \to 1^-$. Noting that $\gamma > \alpha$, by the last inequality, we obtain

$$\exp\left((1-o(1))\exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}\right)$$

$$\le\ k\left(\frac{1}{1-|z|}\right)^{k(2+\varepsilon)} T^k(s(|z|), f) \tag{3.5}$$

for all z satisfying $|z| \in H_1 \backslash E_1$ as $|z| \to 1^-$. Then, by (3.5) and combining with Lemma 2.3, we get for all $r = |z| \in H_1$

$$\exp\left((1-o(1))\exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\right)$$

$$\le k\left(\frac{1}{1-s(r)}\right)^{k(2+\varepsilon)} T^k(s_1(r), f), \tag{3.6}$$

where $s_1(r) = 1 - d^2(1-r)$ with $d \in (0,1)$. Therefore, from (3.6) we obtain $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \limsup_{s_1(r)\to 1^-}\frac{\log^+_{p+1} T(s_1(r),\ f)}{\log_q\left(\frac{1}{1-s_1(r)}\right)} \ge \mu. \tag{3.7}$$

By Lemma 2.4, we get

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f)$$

$$\le \max\left\{\sigma_{M,[p,q]}(A_i) : i = 0, 1, ..., k-1\right\} = \sigma_{M,[p,q]}(A_0) = \mu. \tag{3.8}$$

Therefore, by (3.7) and (3.8), we obtain $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0) = \mu.$$

Proof of Theorem 1.2. Set

$$\alpha_0 = \liminf_{|z|\to 1^-, z\in H}\frac{\log_p|A_0(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}},$$

$$\alpha_i = \limsup_{|z|\to 1^-, z\in H}\frac{\log_p|A_i(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}},\ \ (i = 1, 2, ..., k-1).$$

By (1.5), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, ..., k-1$. It yields

$$\frac{\log_p |A_i(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} < \alpha < \gamma < \frac{\log_p |A_0(z)|}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}}$$

as $|z| \to 1^-$ for $z \in H$. Hence, we have (3.4) as $|z| \to 1^-$ for $z \in H$. Then, by using the same proof of Theorem 1.1, we get

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \quad \text{and} \quad \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geq \mu$$

and by Lemma 2.4 we obtain the conclusion of Theorem 1.2.

Proof of Theorem 1.3. Suppose that every solution f of equation (1.1) not being identically equal to 0. By (1.1), we can write

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)}. \tag{3.9}$$

From (3.9), we obtain

$$T(r, A_0) = m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_i) + \sum_{i=1}^{k} m\left(r, \frac{f^{(i)}}{f}\right) + O(1)$$

$$= \sum_{i=1}^{k-1} T(r, A_i) + \sum_{i=1}^{k} m\left(r, \frac{f^{(i)}}{f}\right) + O(1). \tag{3.10}$$

If $p \geq q \geq 2$, then by (1.6), we know that

$$\exists \gamma \in \mathbb{R} : \liminf_{|z| \to 1^-, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} > \gamma > \alpha.$$

Obviously

$$\frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} > \gamma > \alpha \geq 0 \tag{3.11}$$

as $|z| \to 1^-$ for $z \in H$. By (1.7) and (3.11), we obtain

$$T(r, A_0) > \exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$> \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \geq T(r, A_i), \quad (i = 1, 2, ..., k-1) \tag{3.12}$$

as $|z| \to 1^-$ for $z \in H$. By applying Lemma 2.2 and substituting (3.12) into (3.10), we get

$$\exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \leq (k-1)\exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}$$

$$+O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right)$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Noting that $\gamma > \alpha$, by the last inequality, we have

$$\exp\left\{(1 - o(1))\exp_{p-2}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\right\}$$
$$\leq O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right) \tag{3.13}$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Therefore, from (3.13) we obtain

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \quad \text{and} \quad \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geq \mu. \tag{3.14}$$

By Lemma 2.4, we get

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f)$$
$$\leq \max\left\{\sigma_{M,[p,q]}(A_i) : i = 0, 1, \ldots, k-1\right\} = \sigma_{M,[p,q]}(A_0) = \mu. \tag{3.15}$$

Therefore, by (3.14) and (3.15), we obtain

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \text{ and } \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \mu.$$

If $p = q = 1$, then by (1.8), we know that

$$\exists\gamma \in \mathbb{R} : \liminf_{|z|\to 1^-, z\in H} \frac{T(r,A_0)}{\left(\frac{1}{1-|z|}\right)^{\mu}} > \gamma > (k-1)\alpha.$$

Obviously

$$\frac{T(r,A_0)}{\left(\frac{1}{1-|z|}\right)^{\mu}} > \gamma > (k-1)\alpha \geq 0 \tag{3.16}$$

as $|z| \to 1^-$ for $z \in H$. By (1.9) and (3.16), we obtain

$$T(r,A_0) > \gamma\left(\frac{1}{1-|z|}\right)^{\mu} > (k-1)\alpha\left(\frac{1}{1-|z|}\right)^{\mu}$$
$$\geq \alpha\left(\frac{1}{1-|z|}\right)^{\mu} \geq T(r,A_i), \quad (i = 1, 2, \ldots, k-1) \tag{3.17}$$

as $|z| \to 1^-$ for $z \in H$. By applying Lemma 2.2 and substituting (3.17) into (3.10), we get

$$\gamma\left(\frac{1}{1-r}\right)^{\mu} \leq (k-1)\alpha\left(\frac{1}{1-r}\right)^{\mu}$$
$$+O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right)$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Noting that $\gamma > (k-1)\alpha$, by the last inequality, we have

$$(\gamma - (k-1)\alpha)\left(\frac{1}{1-r}\right)^{\mu} \leq O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right) \tag{3.18}$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Therefore, from (3.18) we obtain

$$\sigma(f) = \sigma_M(f) = \infty \quad \text{and} \quad \sigma_2(f) = \sigma_{M,2}(f) \geq \mu. \tag{3.19}$$

By Lemma 2.4, we get

$$\sigma_2(f) = \sigma_{M,2}(f) \leq \max\{\sigma_M(A_i) : i = 0, 1, ..., k-1\} = \sigma_M(A_0) = \mu. \tag{3.20}$$

Therefore, by (3.19) and (3.20), we obtain $\sigma(f) = \sigma_M(f) = \infty$ and $\sigma_2(f) = \sigma_{M,2}(f) = \mu$.

Proof of Theorem 1.4. If $p \geq q \geq 2$, we set

$$\alpha_0 = \liminf_{|z| \to 1^-, z \in H} \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}},$$

$$\alpha_i = \limsup_{|z| \to 1^-, z \in H} \frac{\log_{p-1} T(r, A_i)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}}, \ (i = 1, 2, ..., k-1).$$

By (1.10), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, ..., k-1$. It yields

$$\frac{\log_{p-1} T(r, A_i)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}} < \alpha < \gamma < \frac{\log_{p-1} T(r, A_0)}{\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}}$$

as $|z| \to 1^-$ for $z \in H$. Hence, we have

$$T(r, A_0) > \exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$> \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \geq T(r, A_i), \ (i = 1, 2, ..., k-1)$$

as $|z| \to 1^-$ for $z \in H$. Then, by using the same proof of Theorem 1.3, we get

$$\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty \ \text{ and } \ \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \geq \mu,$$

and by Lemma 2.4 we obtain the conclusion of Theorem 1.4.

If $p = q = 1$, we set

$$\alpha_0 = \liminf_{|z| \to 1^-, z \in H} \frac{T(r, A_0)}{\left(\frac{1}{1-|z|}\right)^{\mu}},$$

$$\alpha_i = \limsup_{|z| \to 1^-, z \in H} \frac{(k-1)T(r, A_i)}{\left(\frac{1}{1-|z|}\right)^{\mu}}, \ (i = 1, 2, ..., k-1).$$

By (1.11), there exist real numbers α, γ such that $\alpha_i < \alpha < \gamma < \alpha_0$, $i = 1, 2, ..., k-1$. It yields

$$\frac{(k-1)T(r, A_i)}{\left(\frac{1}{1-|z|}\right)^{\mu}} < \alpha < \gamma < \frac{T(r, A_0)}{\left(\frac{1}{1-|z|}\right)^{\mu}} \tag{3.21}$$

as $|z| \to 1^-$ for $z \in H$. By (3.21), we obtain

$$T(r, A_0) > \gamma\left(\frac{1}{1-|z|}\right)^{\mu}$$

$$> \alpha\left(\frac{1}{1-|z|}\right)^{\mu} \geq (k-1)T(r, A_i), \ (i = 1, 2, ..., k-1) \tag{3.22}$$

as $|z| \to 1^-$ for $z \in H$. By applying Lemma 2.2 and substituting (3.22) into (3.10), we get

$$\gamma\left(\frac{1}{1-r}\right)^{\mu} \leq \alpha\left(\frac{1}{1-r}\right)^{\mu} + O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right)$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Noting that $\gamma > \alpha$, by the last inequality, we have

$$(\gamma-\alpha)\left(\frac{1}{1-r}\right)^{\mu} \leq O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right) \tag{3.23}$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Therefore, from (3.23) we obtain

$$\sigma(f) = \sigma_M(f) = \infty \text{ and } \sigma_2(f) = \sigma_{M,2}(f) \geq \mu. \tag{3.24}$$

By Lemma 2.4, we get

$$\sigma_2(f) = \sigma_{M,2}(f) \leq \max\{\sigma_M(A_i) : i = 0, 1, ..., k-1\} = \sigma_M(A_0) = \mu. \tag{3.25}$$

Therefore, by (3.24) and (3.25), we obtain $\sigma(f) = \sigma_M(f) = \infty$ and $\sigma_2(f) = \sigma_{M,2}(f) = \mu$.

Proof of Theorems 1.5 and 1.6. Suppose that every meromorphic (or analytic) solution f of equation (1.2) not being identically equal to 0. From (1.2), we get

$$A_0(z) \leq |A_k(z)|\left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right|. \tag{3.26}$$

By using a similar proof as in Theorem 1.1 or Theorem 1.2, we obtain

$$|A_0(z)| > \exp_p\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$> \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \geq |A_i(z)| \quad (i = 1, 2, ..., k) \tag{3.27}$$

for $|z| \in H_1 \backslash E_1$ as $|z| \to 1^-$. Applying (3.1) and (3.27) into (3.26), we get

$$\exp_p\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \leq |A_0(z)|$$

$$\leq k\left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\left(\frac{1}{1-|z|}\right), T(s(|z|), f)\right\}\right]^k$$

$$\times \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

for all z satisfying $|z| \in H_1 \backslash E_1$ as $|z| \to 1^-$. Noting that $\gamma > \alpha$, by the last inequality, we have

$$\exp\left((1-o(1))\exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}\right)$$

$$\leq k\left(\frac{1}{1-|z|}\right)^{k(2+\varepsilon)} T^k(s(|z|), f) \tag{3.28}$$

for all z satisfying $|z| \in H_1 \backslash E_1$ as $|z| \to 1^-$. Then, by (3.28) and combining with Lemma 2.3, we get for all $r = |z| \in H_1$

$$\exp\left((1-o(1))\exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\right)$$
$$\leq k\left(\frac{1}{1-s(r)}\right)^{k(2+\varepsilon)} T^k\left(s_1(r), f\right), \tag{3.29}$$

where $s_1(r) = 1 - d^2(1-r)$ with $d \in (0,1)$. Therefore, from (3.29) we obtain $\sigma_{[p,q]}(f) = \infty$ and

$$\sigma_{[p+1,q]}(f) = \limsup_{s_1(r)\to 1^-} \frac{\log^+_{p+1} T\left(s_1(r),\ f\right)}{\log_q\left(\frac{1}{1-s_1(r)}\right)} \geq \mu.$$

Proof of Theorems 1.7 and 1.8. Suppose that every meromorphic (or analytic) solution f of equation (1.2) not being identically equal to 0. By (1.2), we can write

$$-A_0(z) = A_k(z)\frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)}. \tag{3.30}$$

From (3.30), we have

$$T(r, A_0) = m(r, A_0) \leq \sum_{i=1}^{k} m(r, A_i) + \sum_{i=1}^{k} m\left(r, \frac{f^{(i)}}{f}\right) + O(1)$$
$$= \sum_{i=1}^{k} T(r, A_i) + \sum_{i=1}^{k} m\left(r, \frac{f^{(i)}}{f}\right) + O(1). \tag{3.31}$$

If $p \geq q \geq 2$, then by using a similar proof as in Theorem 1.3 or Theorem 1.4, we obtain

$$T(r, A_0) > \exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$
$$> \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \geq T(r, A_i), \quad (i = 1, 2, ..., k) \tag{3.32}$$

as $|z| \to 1^-$ for $z \in H$. By applying Lemma 2.2 and substituting (3.32) into (3.31), we get

$$\exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\} \leq k\exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}$$
$$+O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right)$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Noting that $\gamma > \alpha$, by the last inequality, we have

$$\exp\left\{(1-o(1))\exp_{p-2}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-r}\right)\right)^{\mu}\right\}\right\}$$
$$\leq O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right) \tag{3.33}$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Therefore, from (3.33) we obtain

$$\sigma_{[p,q]}(f) = \infty \quad \text{and} \quad \sigma_{[p+1,q]}(f) \geq \mu.$$

If $p = q = 1$, then by using a similar proof as in Theorem 1.3 or Theorem 1.4, we get

$$T(r, A_0) > \gamma \left(\frac{1}{1-|z|}\right)^{\mu} > k\alpha \left(\frac{1}{1-|z|}\right)^{\mu}$$

$$> \alpha \left(\frac{1}{1-|z|}\right)^{\mu} \geq T(r, A_i), \quad (i = 1, 2, ..., k) \tag{3.34}$$

as $|z| \to 1^-$ for $z \in H$. By applying Lemma 2.2 and substituting (3.34) into (3.31), we obtain

$$\gamma \left(\frac{1}{1-r}\right)^{\mu} \leq k\alpha \left(\frac{1}{1-r}\right)^{\mu} + O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right)$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Noting that $\gamma > k\alpha$, by the last inequality, we have

$$(\gamma - k\alpha) \left(\frac{1}{1-r}\right)^{\mu} \leq O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right) \tag{3.35}$$

for all z satisfying $|z| = r \in H_1 \backslash E_2$ as $|z| = r \to 1^-$. Therefore, from (3.35) we obtain

$$\sigma(f) = \sigma_M(f) = \infty \text{ and } \sigma_2(f) = \sigma_{M,2}(f) \geq \mu.$$

4. Proof of Theorem 1.9

Suppose that every solution f of equation (1.1) not being identically equal to 0.

First step. We consider the fixed points of f. Define the function g by setting

$$g(z) := f(z) - z, \; z \in D.$$

It follows from (1.1) that

$$g^{(k)} + A_{k-1} g^{(k-1)} + \cdots + A_1 g' + A_0 g = -A_1 - z A_0 \tag{4.1}$$

and by Theorem 1.1 or Theorem 1.2, we get

$$\begin{aligned} &\sigma_{[p,q]}(g) = \sigma_{[p,q]}(f) = \infty, \; \sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f) = \mu, \\ &\bar{\lambda}_{[p+1,q]}(g) = \bar{\lambda}_{[p+1,q]}(f - z). \end{aligned} \tag{4.2}$$

Now, we prove that $-A_1 - zA_0 \not\equiv 0$. Assume that $-A_1 - zA_0 \equiv 0$. Clearly $A_0 \not\equiv 0$. Then $\lim\limits_{|z| \to 1^-, z \in H} \left|\frac{A_1}{A_0}\right| = 1$ and by (3.4), we have

$$\left|\frac{A_1(z)}{A_0(z)}\right| < \frac{\exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}}{\exp_p\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}}$$

$$= \frac{1}{\exp\left\{(1 - o(1)) \exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}\right\}} \to 0$$

as $|z| \to 1^-$ for $z \in H$. Then $\lim_{|z|\to 1^-, z\in H} \left|\frac{A_1}{A_0}\right| = 0$. It is easy to see the contradiction. Hence, $-A_1 - zA_0 \not\equiv 0$. Next by Lemma 2.5, we get

$$\max\left\{\sigma_{[p,q]}(A_i) \quad (i = 0, 1, ..., k-1), \sigma_{[p,q]}(-A_1 - zA_0)\right\} < \infty.$$

We deduce, by using (4.1), (4.2) and Lemma 2.6 that

$$\bar{\lambda}_{[p,q]}(g) = \sigma_{[p,q]}(g) = \infty, \ \bar{\lambda}_{[p+1,q]}(g) = \sigma_{[p+1,q]}(g) = \mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}(f - z) = \bar{\lambda}_{[p,q]}(g) = \sigma_{[p,q]}(g) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}(f - z) = \bar{\lambda}_{[p+1,q]}(g) = \sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f) = \mu.$$

Second step. For the following proof, we use the principle of mathematical induction. Set $A_k(z) \equiv 1$, then

$$|A_k(z)| \le \exp_p\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

and equation (1.1) becomes (1.2). We consider the fixed points of $f^{(j)}(z)$ $(j = 1, 2, ...)$. Define the function g_1 by setting

$$g_1(z) := f'(z) - z, \ z \in D.$$

Then, by Lemma 2.5 and (4.2), we have

$$\begin{aligned} &\sigma_{[p,q]}(g_1) = \sigma_{[p,q]}(f') = \infty, \ \sigma_{[p+1,q]}(g_1) = \sigma_{[p+1,q]}(f') = \mu, \\ &\bar{\lambda}_{[p+1,q]}(g_1) = \bar{\lambda}_{[p+1,q]}(f' - z). \end{aligned} \tag{4.3}$$

Dividing both sides of (1.2) by A_0, we obtain

$$\frac{A_k}{A_0} f^{(k)} + \frac{A_{k-1}}{A_0} f^{(k-1)} + \cdots + \frac{A_1}{A_0} f' + f = 0. \tag{4.4}$$

It follows, by differentiating both sides of equation (4.4) that

$$\begin{aligned} &\tfrac{A_k}{A_0} f^{(k+1)} + \left(\left(\tfrac{A_k}{A_0}\right)' + \tfrac{A_{k-1}}{A_0}\right) f^{(k)} + \cdots + \left(\left(\tfrac{A_2}{A_0}\right)' + \tfrac{A_1}{A_0}\right) f'' \\ &+ \left(\left(\tfrac{A_1}{A_0}\right)' + 1\right) f' = 0. \end{aligned} \tag{4.5}$$

Multiplying (4.5) by A_0, we have

$$A_{k,1} f^{(k+1)} + A_{k-1,1} f^{(k)} + \cdots + A_{1,1} f'' + A_{0,1} f' = 0. \tag{4.6}$$

Substituing $f' = g_1 + z$ into (4.6), we obtain

$$A_{k,1} g_1^{(k)} + A_{k-1,1} g_1^{(k-1)} + \cdots + A_{1,1} g_1' + A_{0,1} g_1 = F_1, \tag{4.7}$$

where

$$A_{k,1} = A_k = 1, \quad A_{i,1} = A_0\left(\left(\tfrac{A_{i+1}}{A_0}\right)' + \tfrac{A_i}{A_0}\right) \quad (i = 1, 2, ..., k-1), \tag{4.8}$$

$$A_{0,1}=A_0\left(\left(\frac{A_1}{A_0}\right)'+1\right), \tag{4.9}$$

$$F_1=-\left(A_{1,1}+zA_{0,1}\right). \tag{4.10}$$

Next, we prove that $A_{0,1}\not\equiv 0$ and $F_1\not\equiv 0$. Assume that $A_{0,1}\equiv 0$, then $\frac{A_1}{A_0}=-z+C_0$, where C_0 is an arbitrary constant. Hence, we have $A_1+(z-C_0)A_0=0$. Then, $f_0=z-C_0$ is a solution of (1.1) and $\sigma_{[p,q]}(f_0)<\infty$. This contradicts (4.2). Now, assume that $F_1\equiv 0$. By (4.6) and (4.10), we know that the function f_1 such that $f_1'=z$ is a solution of equation (4.6) and $\sigma_{[p,q]}(f_1)<\infty$. This contradicts (4.2). Therefore, $A_{0,1}\not\equiv 0$ and $F_1\not\equiv 0$. It follows by (4.8) – (4.10) and Lemma 2.5 that

$$\max\left\{\sigma_{[p,q]}(A_{i,1})\quad(i=0,1,\dots,k),\sigma_{[p,q]}(F_1)\right\}<\infty.$$

We deduce by using (4.3), (4.7) and Lemma 2.6 that

$$\bar{\lambda}_{[p,q]}(g_1)=\sigma_{[p,q]}(g_1)=\infty,\quad \bar{\lambda}_{[p+1,q]}(g_1)=\sigma_{[p+1,q]}(g_1)=\mu.$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}\left(f'-z\right)=\bar{\lambda}_{[p,q]}(g_1)=\sigma_{[p,q]}(g_1)=\sigma_{[p,q]}(f)=\infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f'-z\right)=\bar{\lambda}_{[p+1,q]}(g_1)=\sigma_{[p+1,q]}(g_1)=\sigma_{[p+1,q]}(f)=\mu.$$

Set $g_2(z)=f''(z)-z$, $z\in D$. Then, by using a similar discussion as in the case of the function g_1, we can get

$$A_{k,2}f^{(k+2)}+A_{k-1,2}f^{(k+1)}+\cdots+A_{1,2}f^{(3)}+A_{0,2}f''=0$$

and

$$A_{k,2}g_2^{(k)}+A_{k-1,2}g_2^{(k-1)}+\cdots+A_{1,2}g_2'+A_{0,2}g_2=F_2,$$

where

$$A_{k,2}=1,\quad A_{i,2}=A_{0,1}\left(\left(\frac{A_{i+1,1}}{A_{0,1}}\right)'+\frac{A_{i,1}}{A_{0,1}}\right)\quad(i=1,2,\dots,k-1),$$

$$A_{0,2}=A_{0,1}\left(\left(\frac{A_{1,1}}{A_{0,1}}\right)'+1\right),$$

$$F_2=-\left(A_{1,2}+zA_{0,2}\right).$$

Therefore, by the same procedure as for g_1, we obtain

$$\bar{\lambda}_{[p,q]}\left(f''-z\right)=\bar{\lambda}_{[p,q]}(g_2)=\sigma_{[p,q]}(g_2)=\sigma_{[p,q]}(f)=\infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f''-z\right)=\bar{\lambda}_{[p+1,q]}(g_2)=\sigma_{[p+1,q]}(g_2)=\sigma_{[p+1,q]}(f)=\mu.$$

Now, assume that

$$\begin{aligned}&A_{0,s}\not\equiv 0,\\ &\bar{\lambda}_{[p,q]}\left(f^{(s)}-z\right)=\sigma_{[p,q]}(f)=\infty,\\ &\bar{\lambda}_{[p+1,q]}\left(f^{(s)}-z\right)=\sigma_{[p+1,q]}(f)=\mu\end{aligned} \tag{4.11}$$

for all $s = 0, 1, ..., j-1$, and we prove that for $s = j$ we have (4.11) holds. Set $g_j(z) = f^{(j)}(z) - z$, $z \in D$. Then, by using (4.2), we obtain

$$\begin{aligned} &\sigma_{[p,q]}(g_j) = \sigma_{[p,q]}\left(f^{(j)}\right) = \infty,\ \sigma_{[p+1,q]}(g_j) = \sigma_{[p+1,q]}\left(f^{(j)}\right) = \mu, \\ &\bar{\lambda}_{[p+1,q]}(g_j) = \bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right). \end{aligned} \tag{4.12}$$

By following the same procedure as before, we have

$$A_{k,j} f^{(k+j)} + A_{k-1,j} f^{(k+j-1)} + \cdots + A_{1,j} f^{(j+1)} + A_{0,j} f^{(j)} = 0 \tag{4.13}$$

and

$$A_{k,j} g_j^{(k)} + A_{k-1,j} g_j^{(k-1)} + \cdots + A_{1,j} g_j' + A_{0,j} g_j = F_j, \tag{4.14}$$

where

$$A_{k,j} = 1, \quad A_{i,j} = A_{0,j-1}\left(\left(\tfrac{A_{i+1,j-1}}{A_{0,j-1}}\right)' + \tfrac{A_{i,j-1}}{A_{0,j-1}}\right) \quad (i = 1, 2, ..., k-1),$$

$$A_{0,j} = A_{0,j-1}\left(\left(\tfrac{A_{1,j-1}}{A_{0,j-1}}\right)' + 1\right) \not\equiv 0 \quad (A_{0,0} = A_0, \quad A_{1,0} = A_1),$$

$$F_j = -\left(A_{1,j} + zA_{0,j}\right) \not\equiv 0.$$

We deduce by applying Lemma 2.6 in (4.14) that

$$\begin{aligned} \bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) &= \bar{\lambda}_{[p,q]}(g_j) = \sigma_{[p,q]}(g_j) = \sigma_{[p,q]}\left(f^{(j)}\right) = \infty, \\ \bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right) &= \bar{\lambda}_{[p+1,q]}(g_j) = \sigma_{[p+1,q]}(g_j) \\ &= \sigma_{[p+1,q]}\left(f^{(j)}\right) = \mu\,(j = 1, 2, ...). \end{aligned}$$

Therefore, we obtain

$$\bar{\lambda}_{[p,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p,q]}(f - z) = \sigma_{[p,q]}(f) = \infty,$$

$$\bar{\lambda}_{[p+1,q]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[p+1,q]}(f - z) = \sigma_{[p+1,q]}(f) = \mu \quad (j = 1, 2, ...).$$

5. Proofs of Theorem 1.10 and 1.11

Proof of Theorem 1.10. Suppose that every solution f of equation (1.1) not being identically equal to 0. By applying Theorem 1.3 or Theorem 1.4, we get

$$\sigma_{[p,q]}(f) = \infty, \quad \sigma_{[p+1,q]}(f) = \mu.$$

Now, we prove that $-A_1 - zA_0 \not\equiv 0$. Assume that $-A_1 - zA_0 \equiv 0$, then we can easily obtain

$$\begin{aligned} &T(r, A_1) = T(r, -zA_0) \le T(r, A_0) + T(r, z), \\ &T(r, A_0) = T\left(r, \tfrac{A_1}{-z}\right) \le T(r, A_1) + T(r, z) + O(1). \end{aligned} \tag{5.1}$$

It follows from (5.1) that

$$1 - \frac{T(r, z) + O(1)}{T(r, A_0)} \le \frac{T(r, A_1)}{T(r, A_0)} \le 1 + \frac{T(r, z)}{T(r, A_0)}. \tag{5.2}$$

By following the same reasoning as in the proof of Theorem 1.3 or Theorem 1.4, we have

$$T(r, A_0) > \exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$$

$$> \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\} \geq T(r, A_1) \tag{5.3}$$

as $r = |z| \to 1^-$ for $z \in H$. By using (5.3), we obtain

$$\frac{T(r,z)}{T(r,A_0)} \leq \frac{T(r,z)}{\exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}} \to 0 \tag{5.4}$$

as $|z| \to 1^-$ for $z \in H$. Then, by (5.2) and (5.4), we get

$$\lim_{|z|\to 1^-, z\in H} \frac{T(r,A_1)}{T(r,A_0)} = 1. \tag{5.5}$$

On the other hand, we have for $p = q = 1$

$$\frac{T(r,A_1)}{T(r,A_0)} < \frac{\alpha}{\gamma} < 1 \tag{5.6}$$

and for $p \geq q \geq 2$

$$\frac{T(r,A_1)}{T(r,A_0)} < \frac{\exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}}{\exp_{p-1}\left\{\gamma\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}} \to 0 \tag{5.7}$$

as $|z| \to 1^-$ for $z \in H$. It follows by (5.6) and (5.7) that

$$\lim_{|z|\to 1^-, z\in H} \frac{T(r,A_1)}{T(r,A_0)} \neq 1. \tag{5.8}$$

Obviously, (5.5) contradicts with (5.8). Hence, $-A_1 - zA_0 \not\equiv 0$. Set $A_k(z) \equiv 1$, then $T(r, A_k) \leq \exp_{p-1}\left\{\alpha\left(\log_{q-1}\left(\frac{1}{1-|z|}\right)\right)^{\mu}\right\}$. Clearly, $A_0 \not\equiv 0$. We can get the conclusion of Theorem 1.10, by reasoning in the same way as we did in the proof of Theorem 1.9.

Proof of Theorem 1.11. Suppose that every meromorphic (or analytic) solution f of equation (1.2) not being identically equal to 0. By applying one of Theorem 1.5 to Theorem 1.8, we get

$$\sigma_{[p,q]}(f) = \infty,\ \ \sigma_{[p+1,q]}(f) \geq \mu.$$

Then, we can get the conclusion of Theorem 1.11, by reasoning in the same way as we did in the proof of Theorem 1.9 and Theorem 1.10 by using $\sigma_{[p+1,q]}(f) \geq \mu$ instead of $\sigma_{[p+1,q]}(f) = \mu$ and $\sigma_{[p+1,q]}\left(f^{(j)}\right) \geq \mu$ instead of $\sigma_{[p+1,q]}\left(f^{(j)}\right) = \mu$ $(j = 1, 2, ...)$.

6. Examples

Example 6.1 Consider the following differential equation

$$f'' + K_1(z)\exp_4\left\{\left(\log_2\left(\frac{1}{1-z}\right)\right)^5\right\} f'$$
$$+ K_0(z)\exp_4\left\{3\left(\log_2\left(\frac{1}{1-z}\right)\right)^5\right\} f = 0, \tag{6.1}$$

where K_0 and K_1 are analytic functions in the unit disc D such that $|K_0| > 1$, $|K_1| < 1$ and

$$\max\left\{\sigma_{M,[4,3]}(K_0), \sigma_{M,[4,3]}(K_1)\right\} < 5.$$

In the equation (6.1), we have

$$A_0(z) = K_0(z)\exp_4\left\{3\left(\log_2\left(\frac{1}{1-z}\right)\right)^5\right\},$$

$$A_1(z) = K_1(z)\exp_4\left\{\left(\log_2\left(\frac{1}{1-z}\right)\right)^5\right\}.$$

Then

$$\max\left\{\sigma_{M,[4,3]}(A_0), \sigma_{M,[4,3]}(A_1)\right\} = 5.$$

Let $H = \{z \in \mathbb{C} : |z| = r < 1 \text{ and } \arg z = 0\} \subset D$ be a set of complex numbers satisfying

$$\overline{dens_D}\{|z| : z \in H\} = 1 > 0.$$

Then

$$|A_0(z)| = |K_0(z)|\left|\exp_4\left\{3\left(\log_2\left(\frac{1}{1-z}\right)\right)^5\right\}\right|$$
$$> \exp_4\left\{3\left(\log_2\left(\frac{1}{1-r}\right)\right)^5\right\}$$
$$\Rightarrow \frac{\log_4|A_0(z)|}{\left(\log_2\left(\frac{1}{1-r}\right)\right)^5} > 3 \Rightarrow \liminf_{r\to1^-, z\in H}\frac{\log_4|A_0(z)|}{\left(\log_2\left(\frac{1}{1-r}\right)\right)^5} \geq 3 > 1,$$

and

$$\begin{aligned}|A_1(z)| &= |K_1(z)|\left|\exp_4\left\{\left(\log_2\left(\frac{1}{1-z}\right)\right)^5\right\}\right| \\ &\leq \exp_4\left\{\left(\log_2\left(\frac{1}{1-r}\right)\right)^5\right\}\end{aligned}$$

as $r \to 1^-$ for $z \in H$. It is clear that the conditions of Theorem 1.1 hold with $\alpha = 1, \mu = 5,\ p = 4$ and $q = 3$ on the set H. By Theorem 1.1, every solution $f \not\equiv 0$ of equation (6.1) satisfies

$$\sigma_{[4,3]}(f) = \sigma_{M,[4,3]}(f) = \infty \quad \text{and} \quad \sigma_{[5,3]}(f) = \sigma_{M,[5,3]}(f) = \sigma_{M,[5,3]}(A_0) = 5.$$

Example 6.2 Consider the following differential equation

$$K_2(z)\exp_4\left\{\left(\log_2\left(\tfrac{1}{1-z}\right)\right)^7\right\}f''+K_1(z)\exp_4\left\{2\left(\log_2\left(\tfrac{1}{1-z}\right)\right)^7\right\}f'$$
$$+K_0(z)\exp_4\left\{5\left(\log_2\left(\tfrac{1}{1-z}\right)\right)^7\right\}f=0, \tag{6.2}$$

where K_0, K_1 and K_2 are analytic functions in the unit disc D such that $|K_0|>1$, $|K_1|<1$, $|K_2|<1$ and

$$\max\left\{\sigma_{M,[4,3]}(K_0),\sigma_{M,[4,3]}(K_1),\sigma_{M,[4,3]}(K_2)\right\}<7.$$

In the equation (6.2), we have

$$A_0(z)=K_0(z)\exp_4\left\{5\left(\log_2\left(\tfrac{1}{1-z}\right)\right)^7\right\},$$

$$A_1(z)=K_1(z)\exp_4\left\{2\left(\log_2\left(\tfrac{1}{1-z}\right)\right)^7\right\},$$

$$A_2(z)=K_2(z)\exp_4\left\{\left(\log_2\left(\tfrac{1}{1-z}\right)\right)^7\right\}.$$

Then

$$\max\left\{\sigma_{M,[4,3]}(A_0),\sigma_{M,[4,3]}(A_1),\sigma_{M,[4,3]}(A_2)\right\}=7.$$

Let $H=\{z\in\mathbb{C}:|z|=r<1 \text{ and } \arg z=0\}\subset D$ be a set of complex numbers satisfying

$$\overline{dens_D}\{|z|:z\in H\}=1>0.$$

Then

$$|A_0(z)|=|K_0(z)|\left|\exp_4\left\{5\left(\log_2\left(\tfrac{1}{1-z}\right)\right)^7\right\}\right|$$
$$>\exp_4\left\{5\left(\log_2\left(\tfrac{1}{1-r}\right)\right)^7\right\}$$
$$\Rightarrow\frac{\log_4|A_0(z)|}{\left(\log_2\left(\frac{1}{1-r}\right)\right)^7}>5\Rightarrow\liminf_{r\to1^-,z\in H}\frac{\log_4|A_0(z)|}{\left(\log_2\left(\frac{1}{1-r}\right)\right)^7}\geq5>2,$$

and

$$\begin{aligned}|A_1(z)|&=|K_1(z)|\left|\exp_4\left\{2\left(\log_2\left(\frac{1}{1-z}\right)\right)^7\right\}\right|\\&\leq\exp_4\left\{2\left(\log_2\left(\frac{1}{1-r}\right)\right)^7\right\}\end{aligned}$$

$$\begin{aligned}|A_2(z)|&=|K_2(z)|\left|\exp_4\left\{\left(\log_2\left(\frac{1}{1-z}\right)\right)^7\right\}\right|\\&\leq\exp_4\left\{2\left(\log_2\left(\frac{1}{1-r}\right)\right)^7\right\}\end{aligned}$$

as $r \to 1^-$ for $z \in H$. It is clear that the conditions of Theorem 1.5 hold with $\alpha = 2$, $\mu = 7$, $p = 4$ and $q = 3$ on the set H. By Theorem 1.11, every meromorphic (or analytic) solution $f \not\equiv 0$ of equation (6.2) satisfies

$$\bar{\lambda}_{[4,3]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[4,3]}\left(f - z\right) = \sigma_{[4,3]}\left(f\right) = \infty$$

and

$$\bar{\lambda}_{[5,3]}\left(f^{(j)} - z\right) = \bar{\lambda}_{[5,3]}\left(f - z\right) = \sigma_{[5,3]}\left(f\right) \geq 7, \quad (j = 1, 2, \ldots).$$

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