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Katugampola Fractional Calculus With Generalized $k-$Wright Function

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Abstract. In this article, we present some properties of the Katugampola fractional integrals and derivatives. Also, we study the fractional calculus properties involving Katugampola Fractional integrals and derivatives of generalized $k-$Wright function ${}_n\Phi^k_m(z)$.

1. Introduction and Preliminaries

In recent years, researchers have introduced new fractional integral and differential operators which are generalizations of the famous definitions of Riemann-Liouville, Caputo, Hadamard, Hilfer, etc. They have made a qualitative contribution to fractional differential equations. For more details, see [1,5-7,9-14] and references therein.

Definition 1.1. [9] Let $\Omega = [a, b]$, the Katugampola fractional integrals ${}_\rho I^\gamma_{0+}\varphi$ and ${}_\rho I^\gamma_{-}\varphi$ of order $\gamma \in \mathbb{C}(\Re(\gamma) > 0)$ are defined for $\rho > 0$, $a = 0$ and $b = \infty$ as

$$({}_\rho I^\gamma_{0+}\varphi)(s) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^s \frac{\tau^{\rho-1}\varphi(\tau)}{(s^\rho - \tau^\rho)^{1-\gamma}}d\tau \quad (s > 0), \tag{1.1}$$

and

$$({}_\rho I^\gamma_{-}\varphi)(s) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_s^\infty \frac{\tau^{\rho-1}\varphi(\tau)}{(\tau^\rho - s^\rho)^{1-\gamma}}d\tau \quad (s > 0), \tag{1.2}$$

the corresponding Katugampola fractional derivatives ${}_\rho D^\gamma_{0+}\varphi$ *and* ${}_\rho D^\gamma_{-}\varphi$ are defined with $\big(n = 1 + [\Re(\gamma)]\big)$ as

$$({}_\rho D^\gamma_{0+}\varphi)(s) := \big(s^{1-\rho}\frac{d}{ds}\big)^{1+[\Re(\gamma)]}\big({}_\rho I^{1-\gamma+[\Re(\gamma)]}_{0+}\varphi\big)(s)$$

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$$= \frac{\rho^{\gamma-[\Re(\gamma)]}}{\Gamma(1-\gamma+[\Re(\gamma)])}\Big(s^{1-\rho}\frac{d}{ds}\Big)^{1+[\Re(\gamma)]}\int_0^s \frac{\tau^{\rho-1}\varphi(\tau)}{(s^\rho-\tau^\rho)^{\gamma-[\Re(\gamma)]}}d\tau \quad (s>0), \tag{1.3}$$

and

$$({}_\rho D_-^\gamma \varphi)(s) := \Big(-s^{1-\rho}\frac{d}{ds}\Big)^{1+[\Re(\gamma)]}\Big({}_\rho I_-^{1-\gamma+[\Re(\gamma)]}\varphi\Big)(s)$$
$$= \frac{\rho^{\gamma-[\Re(\gamma)]}}{\Gamma(1-\gamma+[\Re(\gamma)])}\Big(-s^{1-\rho}\frac{d}{ds}\Big)^{1+[\Re(\gamma)]}\int_s^\infty \frac{\tau^{\rho-1}\varphi(\tau)}{(\tau^\rho-s^\rho)^{\gamma-[\Re(\gamma)]}}d\tau \quad (s>0). \tag{1.4}$$

Definition 1.2. [2] The generalized $K-$Gamma function $\Gamma_k(y)$ is defined by

$$\Gamma_k(y) = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{y}{k}-1}}{(y)_{n,k}} \quad (k>0;\ y\in\mathbb{C}\setminus k\mathbb{Z}^-), \tag{1.5}$$

where $(y)_{n,k}$ is the $k-$Pochhammer symbol given as

$$(y)_{n,k} := \begin{cases} \frac{\Gamma_k(y+nk)}{\Gamma_k(y)} & (k\in\mathbb{R};\ y\in\mathbb{C}\setminus\{0\}) \\ y(y+k)(y+2k)...(y+(n-1)k) & (n\in\mathbb{N}^+;\ y\in\mathbb{C}) \end{cases} \tag{1.6}$$

and for $\Re(y)>0$, the $K-$Gamma function $\Gamma_k(y)$ is defined by the integral

$$\Gamma_k(y) = \int_0^\infty x^{y-1}e^{-\frac{x^k}{k}}dx. \tag{1.7}$$

This gives a relation with Euler's Gamma function as

$$\Gamma_k(y) = k^{\frac{y}{k}-1}\Gamma(\frac{y}{k}). \tag{1.8}$$

Also, in [8], we have

$$\Gamma(1-y)\Gamma(y) = \frac{\pi}{\sin(y\pi)}. \tag{1.9}$$

Definition 1.3. [14] The Beta function $B(\upsilon,\omega)$ is defined as

$$B(\upsilon,\omega) = \int_0^1 z^{\upsilon-1}(1-z)^{\omega-1}dz, \quad \Re(\upsilon)>0, \quad \Re(\omega)>0,$$
$$= \frac{\Gamma(\upsilon)\Gamma(\omega)}{\Gamma(\upsilon+\omega)} \tag{1.10}$$

Furthermore, we have

$$\int_{\hat{x}}^\infty (z-\hat{x})^{\upsilon-1}(z-\hat{y})^{\omega-1}dz = (\hat{x}-\hat{y})^{\upsilon+\omega-1}B(\upsilon,1-\upsilon-\omega),$$
$$\hat{x}>\hat{y}, \quad 0<\Re(\upsilon)<1-\Re(\omega). \tag{1.11}$$

Recently, the Generalized $K-$Wright function introduced by (Gehlot and Prajapati [3]) is defined as follows:

Definition 1.4. For $k \in \mathbb{R}^+$; $z \in \mathbb{C}$; $p_i, q_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$ $(\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ and $(p_i + \alpha_i r)$, $(q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, the generalized $k-$Wright function ${}_n\Phi_m^k$ is defined by

$$ {}_n\Phi_m^k(z) = {}_n\Phi_m^k\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\(q_j,\beta_j)_{1,m}\end{matrix}\middle| z\right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n} \Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m} \Gamma_k(q_j+\beta_j r)} \frac{z^r}{r!}, \tag{1.12}$$

with the convergence conditions described as

$$\Delta = \sum_{j=1}^{m}(\frac{\beta_j}{k}) - \sum_{i=1}^{n}(\frac{\alpha_i}{k}); \mu = \prod_{i=1}^{n}|\frac{\alpha_i}{k}|^{-\frac{\alpha_i}{k}} \prod_{j=1}^{m}|\frac{\beta_j}{k}|^{\frac{\beta_j}{k}}; \nu = \sum_{j=1}^{m}(\frac{q_j}{k}) - \sum_{i=1}^{n}(\frac{p_i}{k}) + \frac{n-m}{2}.$$

Lemma 1.1. [3] For $k \in \mathbb{R}^+$; $z \in \mathbb{C}$; $p_i, q_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$ $(\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ and $(p_i + \alpha_i r)$, $(q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$

(1) If $\Delta > -1$, then series (1.12) is absolutely convergent for all $z \in \mathbb{C}$ and generalized $k-$Wright function ${}_n\Phi_m^k(z)$ is an entire function of z.

(2) If $\Delta = -1$, then series (1.12) is absolutely convergent for all $|z| < \mu$ and of

$$|z| = \mu, \Re(\mu) > \frac{1}{2}.$$

2. Properties of Katugampola Fractional Integral and Derivative

In this section, we investigate some properties of the Katugampola fractional integrals and derivatives (1.1), (1.2) and (1.3), (1.4) for the power function $\varphi(s) = s^{\alpha-1}$ and the exponential function $e^{-\lambda\ s^\rho}$.

Lemma 2.1. Let $\rho > 0, \Re(\gamma) \geqq 0$ *and* $n = 1 + [\Re(\gamma)]$

(1) *If* $\Re(\alpha) > 0$, then

$$({}_\rho I_{0+}^\gamma \tau^{\alpha-1})(s) = \frac{\rho^{-\gamma}\Gamma(1+\frac{\alpha-1}{\rho})}{\Gamma(1+\frac{\alpha-1}{\rho}+\gamma)} s^{\rho\gamma+(\alpha-1)} \quad (\Re(\gamma) \geq 0;\ \Re(\alpha) > 0) \tag{2.1}$$

$$({}_\rho D_{0+}^\gamma \tau^{\alpha-1})(s) = \frac{\rho^{\gamma-n}\Gamma(1+\frac{\alpha-1}{\rho})}{\Gamma(1+\frac{\alpha-1}{\rho}-\gamma)} s^{(\alpha-1)-\rho\gamma} \quad (\Re(\gamma) \geqq 0;\ \Re(\alpha) > 0). \tag{2.2}$$

(2) *If* $\alpha \in \mathbb{C}$, then

$$({}_\rho I_{-}^\gamma \tau^{\alpha-1})(s) = \frac{\rho^{-\gamma}\Gamma(\frac{1-\alpha}{\rho}-\gamma)}{\Gamma(\frac{1-\alpha}{\rho})} s^{\rho\gamma+(\alpha-1)} \quad (\Re(\gamma) \geq 0;\ \Re(\gamma+\alpha) < 1) \tag{2.3}$$

$$({}_\rho D_{-}^\gamma \tau^{\alpha-1})(s) = \frac{\rho^{\gamma-n}\Gamma(\frac{1-\alpha}{\rho}+\gamma)}{\Gamma(\frac{1-\alpha}{\rho})} s^{(\alpha-1)-\rho\gamma} \quad (\Re(\gamma) \geqq 0;\ \Re(\gamma+\alpha-[\Re(\gamma)]) < 1). \tag{2.4}$$

(3) *If* $\Re(\lambda) > 0$, then

$$({}_\rho I_{-}^\gamma e^{-\lambda\tau^\rho})(s) = (\lambda\rho)^{-\gamma} e^{-\lambda\ s^\rho} \qquad (\Re(\gamma) \geq 0) \tag{2.5}$$

$$({}_\rho D_-^\gamma e^{-\lambda\tau^\rho})(s) = (\lambda\rho)^\gamma e^{-\lambda\ s^\rho} \qquad (\Re(\gamma) \geqq 0). \tag{2.6}$$

Proof. To prove this Lemma, let the substitution $x = \frac{\tau^\rho}{s^\rho}$ in parts (1) and (2).

(1) Firstly, by the equation (1.1) and the given substitution, we have

$$\begin{aligned}({}_\rho I_{0+}^\gamma \tau^{\alpha-1})(s) &= \frac{\rho^{-\gamma}s^{\rho\gamma+\alpha-1}}{\Gamma(\gamma)}\int_0^1 \frac{x^{\frac{\alpha-1}{\rho}}}{(1-x)^{1-\gamma}}dx\\ &= \frac{\rho^{-\gamma}s^{\rho\gamma+\alpha-1}}{\Gamma(\gamma)}\ B\big(\gamma, 1+\frac{\alpha-1}{\rho}\big).\end{aligned}$$

Now, using equation (1.10), we obtain the result (2.1).

Secondly, by the equation (1.3), the given substitution and by using the result (2.1), we have

$$\begin{aligned}({}_\rho D_{0+}^\gamma \tau^{\alpha-1})(s) &= \big(s^{1-\rho}\frac{d}{ds}\big)^n\big({}_\rho I_{0+}^{n-\gamma}\tau^{\alpha-1}\big)(s)\\ &= \frac{\rho^{\gamma-n}\Gamma(1+\frac{\alpha-1}{\rho})}{\Gamma(1+\frac{\alpha-1}{\rho}+n-\gamma)}\big(s^{1-\rho}\frac{d}{ds}\big)^n s^{\rho(n-\gamma)+\alpha-1}\\ &= \frac{\rho^{\gamma-n}\Gamma(1+\frac{\alpha-1}{\rho})}{\Gamma(1+\frac{\alpha-1}{\rho}-\gamma)}s^{(\alpha-1)-\rho\gamma}.\end{aligned}$$

(2) Firstly, by the equation (1.2) and the given substitution, we have

$$({}_\rho I_-^\gamma \tau^{\alpha-1})(s) = \frac{\rho^{-\gamma}s^{\rho\gamma+\alpha-1}}{\Gamma(\gamma)}\int_1^\infty x^{\frac{\alpha-1}{\rho}}(x-1)^{\gamma-1}dx.$$

Now, using the equation (1.11) with $\hat{x} = 1$ *and* $\hat{y} = 0$, we obtain

$$({}_\rho I_-^\gamma \tau^{\alpha-1})(s) = \frac{\rho^{-\gamma}s^{\rho\gamma+\alpha-1}}{\Gamma(\gamma)}B\big(\gamma, 1-\gamma-(1+\frac{\alpha-1}{\rho})\big).$$

By using equation (1.10), we obtain the result (2.3).

Secondly, by the equation (1.4), the given substitution and by using the result (2.3), we have

$$\begin{aligned}({}_\rho D_-^\gamma \tau^{\alpha-1})(s) &= \big(-s^{1-\rho}\frac{d}{ds}\big)^n\big({}_\rho I_-^{n-\gamma}\tau^{\alpha-1}\big)(s)\\ &= \frac{(-1)^n\rho^{\gamma-n}\Gamma(\frac{1-\alpha}{\rho}+\gamma-n)}{\Gamma(\frac{1-\alpha}{\rho})}\big(s^{1-\rho}\frac{d}{ds}\big)^n s^{\rho(n-\gamma)+\alpha-1}\\ &= \frac{(-1)^n\rho^{\gamma-n}}{\Gamma(\frac{1-\alpha}{\rho})}\frac{\Gamma(\frac{1-\alpha}{\rho}+\gamma-n)\Gamma(1-[\frac{1-\alpha}{\rho}+\gamma-n])}{\Gamma(1-[\gamma-\frac{\alpha-1}{\rho}])}.\end{aligned} \tag{2.7}$$

Also, by using (1.9), we have

$$\Gamma(\frac{1-\alpha}{\rho}+\gamma-n)\Gamma(1-[\frac{1-\alpha}{\rho}+\gamma-n]) = \frac{\pi}{\sin([\frac{1-\alpha}{\rho}+\gamma-n]\pi)} = \frac{(-1)^n\pi}{\sin([\gamma-\frac{\alpha-1}{\rho}]\pi)} \tag{2.8}$$

and

$$\frac{1}{\Gamma(1-[\gamma-\frac{\alpha-1}{\rho}])} = \frac{\Gamma(\gamma-\frac{\alpha-1}{\rho})}{\Gamma(\gamma-\frac{\alpha-1}{\rho})\Gamma(1-[\gamma-\frac{\alpha-1}{\rho}])} = \frac{\Gamma(\gamma-\frac{\alpha-1}{\rho})}{\pi}\sin([\gamma-\frac{\alpha-1}{\rho}]\pi) \tag{2.9}$$

Substituting relations (2.8) and (2.9) in (2.7), we obtain (2.4).

(3) For this part, let the substitution $x = \tau^\rho - s^\rho$.
Firstly, by the equation (1.2) and the given substitution in this part, we have

$$({}_\rho I_-^\gamma e^{-\lambda\tau^\rho})(s) = \frac{\rho^{-\gamma}}{\Gamma(\gamma)} e^{-\lambda\ s^\rho} \int_0^\infty e^{-\lambda\ x} x^{\gamma-1} dx,$$

then by use the substitution $\vartheta = \lambda\ x$, we obtain

$$({}_\rho I_-^\gamma e^{-\lambda\tau^\rho})(s) = \frac{\rho^{-\gamma}}{\Gamma(\gamma)} e^{-\lambda\ s^\rho} \lambda^{-\gamma} \int_0^\infty e^{-\vartheta} \vartheta^{\gamma-1} d\vartheta,$$

since $\int_0^\infty e^{-\vartheta}\vartheta^{\gamma-1} d\vartheta = \Gamma(\gamma)$ [8], then the result is satisfied.
Secondly, by the equation (1.4) and by using the result (2.5), we have

$$\begin{aligned}
({}_\rho D_-^\gamma e^{-\lambda\tau^\rho})(s) &= \big(-s^{1-\rho}\frac{d}{ds}\big)^n \big({}_\rho I_-^{n-\gamma} e^{-\lambda\tau^\rho}\big)(s)\\
&= (-1)^n \big(s^{1-\rho}\frac{d}{ds}\big)^n \big((\lambda\rho)^{\gamma-n} e^{-\lambda\ s^\rho}\big)\\
&= (-1)^n\ s^{(1-\rho)n}\ (\lambda\rho)^{\gamma-n} \big(\frac{d^n}{ds^n} e^{-\lambda\ s^\rho}\big)\\
&= (\lambda\rho)^\gamma e^{-\lambda\ s^\rho}.
\end{aligned}$$

□

Remark 2.1. (a) In Lemma 2.1, if the power function is $\varphi(s) = \big(\frac{s^\rho}{\rho}\big)^{\alpha-1}$, then
(1) *If* $\Re(\alpha) > 0$, then

$$\left({}_\rho I_{0+}^\gamma \big(\frac{\tau^\rho}{\rho}\big)^{\alpha-1} \right)(s) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \big(\frac{s^\rho}{\rho}\big)^{\alpha+\gamma-1} \quad (\Re(\gamma) \geq 0;\ \Re(\alpha) > 0)$$

$$\left({}_\rho D_{0+}^\gamma \big(\frac{\tau^\rho}{\rho}\big)^{\alpha-1} \right)(s) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} \big(\frac{s^\rho}{\rho}\big)^{\alpha-\gamma-1} \quad (\Re(\gamma) \geqq 0;\ \Re(\alpha) > 0).$$

(2) *If* $\alpha \in \mathbb{C}$, then

$$\left({}_\rho I_-^\gamma \big(\frac{\tau^\rho}{\rho}\big)^{\alpha-1} \right)(s) = \frac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-\alpha)} \big(\frac{s^\rho}{\rho}\big)^{\alpha+\gamma-1} \quad (\Re(\gamma) \geq 0;\ \Re(\gamma+\alpha) < 1)$$

$$\left({}_\rho D_-^\gamma \big(\frac{\tau^\rho}{\rho}\big)^{\alpha-1} \right)(s) = \frac{\Gamma(1+\gamma-\alpha)}{\Gamma(1-\alpha)} \big(\frac{s^\rho}{\rho}\big)^{\alpha-\gamma-1} \quad (\Re(\gamma) \geqq 0;\ \Re(\gamma+\alpha-[\Re(\gamma)]) < 1).$$

(b) If $\Re(\alpha) > \Re(\gamma) > 0$, then

$$({}_\rho I_-^\gamma \tau^{-\alpha})(s) = \frac{\rho^{-\gamma}\Gamma(\frac{\alpha}{\rho}-\gamma)}{\Gamma(\frac{\alpha}{\rho})} s^{\rho\gamma-\alpha}. \tag{2.10}$$

3. Katugampola Fractional integration for Generalized $k-$Wright Function

In this section, we establish the Katugampola fractional integration for generalized $k-$Wright function (1.12).

Theorem 3.1. Let $\gamma,\ \alpha\in\mathbb{C}$ such that $\Re(\gamma)>0,\ \Re(\alpha)>0;\ \lambda\in\mathbb{C},\ \rho>0,\ \nu>0$, then for $\Delta>-1$, the Katugampola fractional integration ${}_{\rho}I^{\gamma}_{0+}$ for generalized $k-$Wright function ${}_n\Phi^k_m(z)$ is given as

$$\left({}_{\rho}I^{\gamma}_{0+}\left(\tau^{\frac{\alpha}{k}-1}\ {}_n\Phi^k_m\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\(q_j,\beta_j)_{1,m}\end{matrix}\middle|\ \lambda\,\tau^{\frac{\nu}{k}}\right]\right)\right)(s)$$
$$=(\frac{k}{\rho})^{\gamma}\ s^{\frac{\alpha}{k}+\rho\gamma-1}\ {}_{n+1}\Phi^k_{m+1}\left[\begin{matrix}(p_i,\alpha_i)_{1,n},\ \left(\frac{1}{\rho}(\alpha+(\rho-1)k),\frac{\nu}{\rho}\right)\\(q_j,\beta_j)_{1,m},\ \left(\frac{1}{\rho}(\alpha+(\rho(\gamma+1)-1)k),\frac{\nu}{\rho}\right)\end{matrix}\middle|\ \lambda\,s^{\frac{\nu}{k}}\right].\tag{3.1}$$

Proof. According to Lemma 1.1, a generalized $k-$Wright function in both sides of the equation (3.1) exists for $s>0$. We consider that

$$M\equiv\left({}_{\rho}I^{\gamma}_{0+}\left(\tau^{\frac{\alpha}{k}-1}\ {}_n\Phi^k_m\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\(q_j,\beta_j)_{1,m}\end{matrix}\middle|\ \lambda\,\tau^{\frac{\nu}{k}}\right]\right)\right)(s).$$

Using (1.12), we can write the above equation as

$$M\equiv\left({}_{\rho}I^{\gamma}_{0+}\left(\tau^{\frac{\alpha}{k}-1}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\Gamma_k(q_j+\beta_j r)}\frac{(\lambda\,\tau^{\frac{\nu}{k}})^r}{r!}\right)\right)(s).$$

Now, using the integration of the series term by term, we obtain

$$M\equiv\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\Gamma_k(q_j+\beta_j r)}\frac{(\lambda)^r}{r!}\left({}_{\rho}I^{\gamma}_{0+}\left(\tau^{\frac{\alpha}{k}+\frac{\nu r}{k}-1}\right)\right)(s).$$

Applying (2.1), the above equation is reduced to

$$M\equiv\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\Gamma_k(q_j+\beta_j r)}\frac{(\lambda)^r}{r!}\frac{\rho^{-\gamma}\Gamma(1+\frac{\frac{\alpha}{k}+\frac{\nu r}{k}-1}{\rho})}{\Gamma(1+\frac{\frac{\alpha}{k}+\frac{\nu r}{k}-1}{\rho}+\gamma)}s^{\frac{\alpha+\nu r}{k}+\rho\gamma-1}.$$

Using (1.8), we obtain

$$M\equiv(\frac{k}{\rho})^{\gamma}\ s^{\frac{\alpha}{k}+\rho\gamma-1}\ {}_{n+1}\Phi^k_{m+1}\left[\begin{matrix}(p_i,\alpha_i)_{1,n},\ \left(\frac{1}{\rho}(\alpha+(\rho-1)k),\frac{\nu}{\rho}\right)\\(q_j,\beta_j)_{1,m},\ \left(\frac{1}{\rho}(\alpha+(\rho(\gamma+1)-1)k),\frac{\nu}{\rho}\right)\end{matrix}\middle|\ \lambda\,s^{\frac{\nu}{k}}\right].$$

□

Theorem 3.2. Let $\gamma,\ \alpha\in\mathbb{C}$ such that $\Re(\gamma)>0,\ \Re(\alpha)>0;\ \lambda\in\mathbb{C},\ \rho>0,\ \nu>0$, then for $\Delta>-1$, the Katugampola fractional integration ${}_{\rho}I^{\gamma}_{-}$ for generalized $k-$Wright function ${}_n\Phi^k_m(z)$ is given as

$$\left({}_{\rho}I^{\gamma}_{-}\left(\tau^{-\frac{\alpha}{k}}\ {}_n\Phi^k_m\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\(q_j,\beta_j)_{1,m}\end{matrix}\middle|\ \lambda\,\tau^{-\frac{\nu}{k}}\right]\right)\right)(s)$$

$$= (\frac{k}{\rho})^{\gamma}\ s^{\rho\gamma-\frac{\alpha}{k}}\ {}_{n+1}\Phi^{k}_{m+1}\left[\begin{matrix}(p_i,\alpha_i)_{1,n},\ (\frac{\alpha}{\rho}-k\gamma,\frac{\nu}{\rho})\\ (q_j,\beta_j)_{1,m},\ (\frac{\alpha}{\rho},\frac{\nu}{\rho})\end{matrix}\ \middle|\ \lambda\ s^{-\frac{\nu}{k}}\right]. \tag{3.2}$$

Proof. According to Lemma 1.1, a generalized $k-$Wright function in both sides of the equation (3.2) exists for $s > 0$. We consider that

$$N \equiv \left(\ {}_{\rho}I^{\gamma}_{-}\left(\tau^{-\frac{\alpha}{k}}\ {}_{n}\Phi^{k}_{m}\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\ (q_j,\beta_j)_{1,m}\end{matrix}\middle|\ \lambda\ \tau^{-\frac{\nu}{k}}\right]\right)\right)(s).$$

Using (1.12), we can write the above equation as

$$N \equiv \sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\ \Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\Gamma_k(q_j+\beta_j r)}\frac{(\lambda)^r}{r!}\left(\ {}_{\rho}I^{\gamma}_{-}\left(\tau^{-\frac{\alpha+\nu r}{k}}\right)\right)(s).$$

Applying (2.10), the above equation is reduced to

$$N \equiv \sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\ \Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\Gamma_k(q_j+\beta_j r)}\frac{(\lambda)^r}{r!}\frac{\rho^{-\gamma}\Gamma(\frac{\frac{\alpha+\nu r}{k}}{\rho}-\gamma)}{\Gamma(\frac{\frac{\alpha+\nu r}{k}}{\rho})}s^{\rho\gamma-\frac{\alpha+\nu r}{k}}.$$

Using (1.8), we obtain

$$N \equiv\ (\frac{k}{\rho})^{\gamma}\ s^{\rho\gamma-\frac{\alpha}{k}}\ {}_{n+1}\Phi^{k}_{m+1}\left[\begin{matrix}(p_i,\alpha_i)_{1,n},\ (\frac{\alpha}{\rho}-k\gamma,\frac{\nu}{\rho})\\ (q_j,\beta_j)_{1,m},\ (\frac{\alpha}{\rho},\frac{\nu}{\rho})\end{matrix}\ \middle|\ \lambda\ s^{-\frac{\nu}{k}}\right].$$

□

4. Katugampola Fractional Differentiation for Generalized $k-$Wright Function

This section deals with the Katugampola fractional differentiation for generalized $k-$Wright function (1.12).

Theorem 4.1. Let $\gamma,\ \alpha \in \mathbb{C}$ such that $\Re(\gamma) > 0$, $\Re(\alpha) > 0$; $\lambda \in \mathbb{C}$, $\rho > 0$, $\nu > 0$, then for $\Delta > -1$, the Katugampola fractional differentiation ${}_{\rho}D^{\gamma}_{0+}$ for generalized $k-$Wright function ${}_{n}\Phi^{k}_{m}(z)$ is given as

$$\left(\ {}_{\rho}D^{\gamma}_{0+}\left(\tau^{\frac{\alpha}{k}-1}\ {}_{n}\Phi^{k}_{m}\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\ (q_j,\beta_j)_{1,m}\end{matrix}\middle|\ \lambda\ \tau^{\frac{\nu}{k}}\right]\right)\right)(s)$$
$$= (\frac{k}{\rho})^{-\gamma}\ s^{\frac{\alpha}{k}-\rho\gamma-1}\ {}_{n+1}\Phi^{k}_{m+1}\left[\begin{matrix}(p_i,\alpha_i)_{1,n},\ (\frac{1}{\rho}(\alpha+(\rho-1)k),\frac{\nu}{\rho})\\ (q_j,\beta_j)_{1,m},\ (\frac{1}{\rho}(\alpha+(\rho(1-\gamma)-1)k),\frac{\nu}{\rho})\end{matrix}\middle|\ \lambda\ s^{\frac{\nu}{k}}\right]. \tag{4.1}$$

Proof. According to Lemma 1.1, a generalized $k-$Wright function in both sides of the equation (4.1) exists for $s > 0$. Let $n = 1 + [\Re(\gamma)]$. Then, we consider that

$$P \equiv \left(\ {}_{\rho}D^{\gamma}_{0+}\left(\tau^{\frac{\alpha}{k}-1}\ {}_{n}\Phi^{k}_{m}\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\ (q_j,\beta_j)_{1,m}\end{matrix}\middle|\ \lambda\ \tau^{\frac{\nu}{k}}\right]\right)\right)(s).$$

Using (1.3), we have

$$P \equiv \left(s^{1-\rho}\frac{d}{ds}\right)^n \left({}_\rho I_{0+}^{n-\gamma} \left(\tau^{\frac{\alpha}{k}-1} \; {}_n\Phi_m^k \left[\begin{matrix} (p_i,\alpha_i)_{1,n} \\ (q_j,\beta_j)_{1,m} \end{matrix} \middle| \; \lambda \, \tau^{\frac{\nu}{k}} \right] \right) \right)(s).$$

Using Theorem 3.1, we obtain

$$P \equiv \left(s^{1-\rho}\frac{d}{ds}\right)^n \left((\frac{k}{\rho})^{n-\gamma} \; s^{\frac{\alpha}{k}+\rho(n-\gamma)-1} \; {}_{n+1}\Phi_{m+1}^k \left[\begin{matrix} (p_i,\alpha_i)_{1,n}, \; \left(\frac{1}{\rho}(\alpha+(\rho-1)k), \frac{\nu}{\rho}\right) \\ (q_j,\beta_j)_{1,m}, \; \left(\frac{1}{\rho}(\alpha+(\rho(n-\gamma+1)-1)k), \frac{\nu}{\rho}\right) \end{matrix} \middle| \; \lambda \, s^{\frac{\nu}{k}} \right] \right).$$

Using (1.12), we can write the above equation as

$$P \equiv \; (\frac{k}{\rho})^{n-\gamma} \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n} \; \Gamma_k(p_i+\alpha_i r)\Gamma_k(\frac{1}{\rho}(\alpha+(\rho-1)k)+\frac{\nu}{\rho}r)}{\prod_{j=1}^{m} \; \Gamma_k(q_j+\beta_j r)\Gamma_k(\frac{1}{\rho}(\alpha+(\rho(n-\gamma+1)-1)k)+\frac{\nu}{\rho}r)} \frac{(\lambda)^r}{r!} \left(s^{1-\rho}\frac{d}{ds}\right)^n \left(\; s^{\frac{\alpha}{k}+\frac{\nu}{k}+\rho(n-\gamma)-1}\right).$$

Also, the above equation can be written as

$$\begin{aligned} P \equiv \; k^{n-\gamma} \; \rho^{\gamma} \sum_{r=0}^{\infty} & \frac{\prod_{i=1}^{n} \; \Gamma_k(p_i+\alpha_i r)\Gamma_k(\frac{1}{\rho}(\alpha+(\rho-1)k)+\frac{\nu}{\rho}r)}{\prod_{j=1}^{m} \; \Gamma_k(q_j+\beta_j r)\Gamma_k(\frac{1}{\rho}(\alpha+(\rho(n-\gamma+1)-1)k)+\frac{\nu}{\rho}r)} \frac{(\lambda)^r}{r!} \\ & \times \frac{\Gamma(\frac{1}{\rho}(\frac{\alpha}{k}+\frac{\nu r}{k}+(n-\gamma)\rho+\rho-1)}{\Gamma(\frac{1}{\rho}(\frac{\alpha}{k}+\frac{\nu r}{k}-\gamma\rho+\rho-1)} \; s^{\frac{\alpha}{k}+\frac{\nu}{k}-\rho\gamma-1}. \end{aligned}$$

Using (1.8), we obtain

$$P \equiv (\frac{k}{\rho})^{-\gamma} \; s^{\frac{\alpha}{k}-\rho\gamma-1} \; {}_{n+1}\Phi_{m+1}^k \left[\begin{matrix} (p_i,\alpha_i)_{1,n}, \; \left(\frac{1}{\rho}(\alpha+(\rho-1)k), \frac{\nu}{\rho}\right) \\ (q_j,\beta_j)_{1,m}, \; \left(\frac{1}{\rho}(\alpha+(\rho(1-\gamma)-1)k), \frac{\nu}{\rho}\right) \end{matrix} \middle| \; \lambda \, s^{\frac{\nu}{k}} \right].$$

□

Theorem 4.2. Let $\gamma, \; \alpha \in \mathbb{C}$ such that $\Re(\gamma) > 0, \; \Re(\alpha) > 1 + [\Re(\gamma)] - \Re(\gamma); \; \lambda \in \mathbb{C}, \; \rho > 0, \; \nu > 0$, then for $\Delta > -1$, the Katugampola fractional differentiation ${}_\rho D_-^{\gamma}$ for generalized $k-$Wright function ${}_n\Phi_m^k(z)$ is given as

$$\begin{aligned} & \left({}_\rho D_-^{\gamma} \left(\tau^{-\frac{\alpha}{k}} \; {}_n\Phi_m^k \left[\begin{matrix} (p_i,\alpha_i)_{1,n} \\ (q_j,\beta_j)_{1,m} \end{matrix} \middle| \; \lambda \, \tau^{-\frac{\nu}{k}} \right] \right) \right)(s) \\ & \quad = (\frac{k}{\rho})^{-\gamma} \; s^{-\rho\gamma-\frac{\alpha}{k}} \; {}_{n+1}\Phi_{m+1}^k \left[\begin{matrix} (p_i,\alpha_i)_{1,n}, \; \left(\frac{\alpha}{\rho}+k\gamma, \frac{\nu}{\rho}\right) \\ (q_j,\beta_j)_{1,m}, \; \left(\frac{\alpha}{\rho}, \frac{\nu}{\rho}\right) \end{matrix} \middle| \; \lambda \, s^{-\frac{\nu}{k}} \right] \end{aligned} \tag{4.2}$$

Proof. According to Lemma 1.1, a generalized $k-$Wright function in both sides of the equation (4.2) exists for $s > 0$. Let $n = 1 + [\Re(\gamma)]$. Then, we consider that

$$Q \equiv \left({}_\rho D_-^{\gamma} \left(\tau^{-\frac{\alpha}{k}} \; {}_n\Phi_m^k \left[\begin{matrix} (p_i,\alpha_i)_{1,n} \\ (q_j,\beta_j)_{1,m} \end{matrix} \middle| \; \lambda \, \tau^{-\frac{\nu}{k}} \right] \right) \right)(s).$$

Using (1.4), we have

$$Q \equiv \left(-s^{1-\rho}\frac{d}{ds}\right)^n \left({}_\rho I_{-}^{n-\gamma} \left(\tau^{-\frac{\alpha}{k}} \; {}_n\Phi_m^k \left[\begin{matrix} (p_i,\alpha_i)_{1,n} \\ (q_j,\beta_j)_{1,m} \end{matrix} \middle| \; \lambda \, \tau^{-\frac{\nu}{k}} \right] \right) \right)(s).$$

Using Theorem 3.2, we obtain

$$Q \equiv \big(-s^{1-\rho}\frac{d}{ds}\big)^n(\frac{k}{\rho})^{n-\gamma}\ s^{\rho(n-\gamma)-\frac{\alpha}{k}}\ {}_{n+1}\Phi^k_{m+1}\left[\begin{matrix}(p_i,\alpha_i)_{1,n},\ (\frac{\alpha}{\rho}-k(n-\gamma),\frac{\nu}{\rho})\\ (q_j,\beta_j)_{1,m},\ (\frac{\alpha}{\rho},\frac{\nu}{\rho})\end{matrix}\ \middle|\ \lambda\ s^{-\frac{\nu}{k}}\right].$$

Using (1.12), we can write the above equation as

$$Q \equiv (-1)^n(\frac{k}{\rho})^{n-\gamma}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n}\ \Gamma_k(p_i+\alpha_i r)\Gamma_k(\frac{\alpha}{\rho}-(n-\gamma)k+\frac{\nu}{\rho}r)}{\prod_{j=1}^{m}\ \Gamma_k(q_j+\beta_j r)\Gamma_k(\frac{\alpha}{\rho}+\frac{\nu}{\rho}r)}\frac{(\lambda)^r}{r!}\big(s^{1-\rho}\frac{d}{ds}\big)^n\big(\ s^{\rho(n-\gamma)-\frac{\alpha}{k}-\frac{\nu}{k}}\big).$$

On simplifying the above equation, we obtain

$$\begin{aligned}Q \equiv (-1)^n k^{n-\gamma}\rho^{\gamma}\sum_{r=0}^{\infty}&\frac{\prod_{i=1}^{n}\ \Gamma_k(p_i+\alpha_i r)\Gamma_k(\frac{\alpha}{\rho}-(n-\gamma)k+\frac{\nu}{\rho}r)}{\prod_{j=1}^{m}\ \Gamma_k(q_j+\beta_j r)\Gamma_k(\frac{\alpha}{\rho}+\frac{\nu}{\rho}r)}\frac{(\lambda)^r}{r!}\\ &\times\frac{\Gamma(1+(n-\gamma)-\frac{\alpha}{\rho k}-\frac{\nu}{\rho k}r)}{\Gamma(1-\gamma-\frac{\alpha}{\rho k}-\frac{\nu}{\rho k}r)}\big(\ s^{-\rho\gamma-\frac{\alpha}{k}-\frac{\nu}{k}}\big).\end{aligned}$$

Using (1.8), we obtain

$$\begin{aligned}Q \equiv (-1)^n\rho^{\gamma}\sum_{r=0}^{\infty}&\frac{\prod_{i=1}^{n}\ \Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m}\ \Gamma_k(q_j+\beta_j r)\Gamma(\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r)}\frac{(\lambda)^r}{r!}\\ &\times\frac{\Gamma(\gamma-n+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r)\Gamma(1-(\gamma-n+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r))}{\Gamma(1-(\gamma+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r))}\big(\ s^{-\rho\gamma-\frac{\alpha}{k}-\frac{\nu}{k}}\big).\end{aligned}\tag{4.3}$$

Using (1.9), we have

$$\begin{aligned}\Gamma(\gamma-n+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r)\Gamma(1-(\gamma-n+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r))&\\ =&\frac{\pi}{\sin[(\gamma+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r)\pi-n\pi]}\\ =&\frac{\pi}{\sin[(\gamma+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r)\pi]\cos(n\pi)}\\ =&\frac{(-1)^n\pi}{\sin[(\gamma+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r)\pi]}\end{aligned}\tag{4.4}$$

and

$$\frac{1}{\Gamma(1-(\gamma+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r))}=\frac{\Gamma(\gamma+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r)\sin[(\gamma+\frac{\alpha}{\rho k}+\frac{\nu}{\rho k}r)\pi]}{\pi}.\tag{4.5}$$

Substituting (4.4) and (4.5) in (4.3) and finally by using (1.8), we obtain

$$Q \equiv (\frac{k}{\rho})^{-\gamma}\ s^{-\rho\gamma-\frac{\alpha}{k}}\ {}_{n+1}\Phi^k_{m+1}\left[\begin{matrix}(p_i,\alpha_i)_{1,n},\ (\frac{\alpha}{\rho}+k\gamma,\frac{\nu}{\rho})\\ (q_j,\beta_j)_{1,m},\ (\frac{\alpha}{\rho},\frac{\nu}{\rho})\end{matrix}\ \middle|\ \lambda\ s^{-\frac{\nu}{k}}\right].$$

□

5. Concluding Remarks

- If $\rho = 1$, then

  Theorems 3.1, 3.2, 4.1 and 4.2, are reduced to Theorems 2, 3, 4 and 5 respectively (see [4]).
- Some general properties of the Katugampola fractional integrals and derivatives for the power function $\varphi(s) = s^{\alpha-1}$ and the exponential function $e^{-\lambda\ s^{\rho}}$ are investigated.
- The Katugampola fractional integration ${}_{\rho}I^{\gamma}_{0+}$ and ${}_{\rho}I^{\gamma}_{-}$ for generalized $k-$Wright function ${}_{n}\Phi^{k}_{m}(z)$ are established.
- The Katugampola fractional differentiation ${}_{\rho}D^{\gamma}_{0+}$ and ${}_{\rho}D^{\gamma}_{-}$ for generalized $k-$Wright function ${}_{n}\Phi^{k}_{m}(z)$ are established.

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