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On the Kolmogorov Distance for the Least Squares Estimator in the Fractional Ornstein-Uhlenbeck Process

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Abstract. The paper shows that the distribution of the normalized least squares estimator of the drift parameter in the fractional Ornstein-Uhlenbeck process observed over $[0, T]$ converges to the standard normal distribution with an uniform optimal error bound of the order $O(T^{-1/2})$ for $0.5 \leq H \leq 0.63$ and of the order $O(T^{4H-3})$ for $0.63 < H < 0.75$ where H is the Hurst exponent of the fractional Brownian motion driving the Ornstein-Uhlenbeck process. For the normalized quasi-least squares estimator, the error bound is of the order $O(T^{-1/4})$ for $0.5 \leq H \leq 0.69$ and of the order $O(T^{4H-3})$ for $0.69 < H < 0.75$.

1. Introduction

The fractional Ornstein-Uhlenbeck process, is an extension of Ornstein-Uhlenbeck process with fractional Brownian motion (fBm) driving term. In finance it is known as fractional Vasicek model, and is being extensively used these days as one-factor short-term interest rate model which takes into account the long memory effect of the interest rate. The model parameter is usually unknown and must be estimated from data.

Parameter estimation in stochastic differential equations is studied in Bishwal [1]. For the standard Ornstein-Uhlenbeck process, sufficiency and Rao-Blackwellization was studied in Bishwal [4] where also a time transformation to reduce the general problem to a fixed time case and the asymptotics were studied in large parameter case. For the fractional Ornstein-Uhlenbeck process, Berry-Esseen inequalities of minimum contrast estimators based on continuous and discrete observations was studied in Bishwal [2]. Hu *et al.* [11] studied parameter estimation for the fractional Ornstein-Uhlenbeck process of general Hurst parameter. Bishwal [5] studied Berry-Esseen inequalities for the fractional Black-Karasinski model of term structure of interest rates. Using fractional Levy process as the driving term which include jumps, maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [3]. Parameter estimation in partially observed stochastic differential system was studied in Bishwal [6].

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Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis on which is defined the Ornstein-Uhlenbeck process X_t satisfying the Itô stochastic differential equation

$$dX_t = \theta X_t dt + dW_t^H, \quad t \geq 0, \quad X_0 = 0 \tag{1.1}$$

where $\{W_t^H\}$ is a fractional Brownian motion with $H > 1/2$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta < 0$ is the unknown parameter to be estimated on the basis of continuous observation of the process $\{X_t\}$ on the time interval $[0, T]$.

Recall that a fractional Brownian motion (fBM) has the covariance

$$\widetilde{C}_H(s,t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right], \quad s,t > 0. \tag{1.2}$$

For $H > 0.5$ the process has long range dependence or long memory and the process is self-similar. For $H \neq 0.5$, the process is neither a Markov process nor a semimartingale. For $H = 0.5$, the process reduces to standard Brownian motion.

Note that the solution of the equation (1.1) is given by

$$X_t = \int_0^t e^{\theta(t-s)} dW_s^H. \tag{1.3}$$

Let the realization $\{X_t, 0 \leq t \leq T\}$ be denoted by X_0^T. Let P_θ^T be the measure generated on the space (C_T, B_T) of continuous functions on $[0, T]$ with the associated Borel σ-algebra B_T generated under the supremum norm by the process X_0^T and P_0^T be the standard Wiener measure. Applying Girsanov type formula for fBm, when θ is the true value of the parameter, P_θ^T is absolutely continuous with respect to P_0^T and the Radon-Nikodym derivative (likelihood) of P_θ^T with respect to P_0^T based on X_0^T is given by

$$L_T(\theta) := \frac{dP_\theta^T}{dP_0^T}(X_0^T) = \exp\left\{\theta \int_0^T Q_t dZ_t - \frac{\theta^2}{2}\int_0^T Q_t^2 dv_t\right\}. \tag{1.4}$$

Consider the score function, the derivative of the log-likelihood function, which is given by

$$Y_T(\theta) := \int_0^T Q_t dZ_t - \theta \int_0^T Q_t^2 dv_t. \tag{1.5}$$

A solution of $Y_T(\theta) = 0$ provides the maximum likelihood estimate (MLE)

$$\theta_T := \frac{\int_0^T Q_t dZ_t}{\int_0^T Q_t^2 dv_t.} \tag{1.6}$$

Kleptsyna and Le Breton [13] showed that θ_T is strongly consistent. Using the Fourier method, Bishwal [2] proved a Berry-Esseen type theorem for the estimator θ_T which gives the rate of weak convergence in asymptotic normality.

Using the fractional Itô formula, the score function $Y_T(\theta)$ can be written as

$$Y_T(\theta) = \frac{1}{2}\left[\frac{\lambda_H}{(2-2H)} Z_T \int_0^T t^{2H-1} dZ_t - T\right] - \theta \int_0^T Q_t^2 dv_t. \tag{1.7}$$

Consider the contrast function

$$K_T(\theta) := -\frac{TH\Gamma(H)}{2} - \theta \int_0^T Q_t^2 dv_t \tag{1.8}$$

and the minimum contrast estimate (MCE)

$$\bar{\theta}_T := \frac{-TH\Gamma(H)}{2\int_0^T Q_t^2 dv_t}. \tag{1.9}$$

The least squares estimator (LSE) of θ minimizes

$$\int_0^T |\dot{X}_t - \theta X_t|^2 dt \tag{1.10}$$

and is given by

$$\widehat{\theta}_T := \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt.} = \theta - \frac{\int_0^T X_t dW_t^H}{\int_0^T X_t^2 dt.} \tag{1.11}$$

Based on ergodicity, quasi least squares estimate (QLSE)

$$\widetilde{\theta}_T := \left(\frac{-TH\Gamma(2H)}{\int_0^T X_t^2 dt}\right)^{\frac{1}{2H}} \tag{1.12}$$

The LSE and the QLSE are strongly consistent and asymptotically norma as $T \to \infty$

$$\sqrt{T}(\widehat{\theta}_T - \theta) \to^{\mathcal{D}} \mathcal{N}(0, \theta\sigma_H^2), \quad \sqrt{T}(\widetilde{\theta}_T - \theta) \to^{\mathcal{D}} \mathcal{N}(0, \frac{\theta\sigma_H^2}{4H^2}) \tag{1.13}$$

where

$$\sigma_H^2 := (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)\Gamma(2H)}\right). \tag{1.14}$$

Observe that $H = 1/2, \quad \sigma_H^2 = 2$. In this case the LSE and the MLE are identical. Since $\widetilde{\theta}_T$ is a consistent estimator of θ, we can derive the self normalized limit distributions immediately:

$$(\frac{T}{\sigma_H^2\widetilde{\theta}_T})^{1/2}(\widehat{\theta}_T - \theta) \to^{\mathcal{D}} \mathcal{N}(0,1), \quad 2H(\frac{T}{\sigma_H^2\widetilde{\theta}_T})^{1/2}(\widetilde{\theta}_T - \theta) \to^{\mathcal{D}} \mathcal{N}(0,1). \tag{1.15}$$

Define

$$M_T := \int_0^T X_t \, dW_t^H \text{ and } \quad I_T := \int_0^T X_t^2 dt, \quad N_T := \theta^{2H} I_T - TH\Gamma(2H). \tag{1.16}$$

$$V_{H,\theta} := \theta^{-2H} H\Gamma(2H). \tag{1.17}$$

Observe that

$$\left(\frac{T}{-\sigma_H^2\theta}\right)^{1/2}(\widehat{\theta}_T - \theta) = \frac{\left(\frac{-\sigma_H^2\theta}{T}\right)^{1/2} M_T}{\left(\frac{\sigma_H^2\theta}{T}\right) I_T} \tag{1.18}$$

Applying Taylor's formula to the function $x^{-\frac{1}{2H}}$ at the point $V_{H,\theta}$, we have

$$\left(\frac{I_T}{T}\right)^{-\frac{1}{2H}} = V_{H,\theta}^{-\frac{1}{2H}} - \frac{1}{2H} V_{H,\theta}^{-\frac{1+2H}{2H}}\left(\frac{I_T}{T} - V_{H,\theta}\right) + \frac{1+2H}{8H^2}\varpi_T^{-\frac{1+4H}{2H}}\left(\frac{I_T}{T} - V_{H,\theta}\right)^2 \tag{1.19}$$

where ϖ_T is a random point between $V_{H,\theta}$ and $\frac{I_T}{T}$. Further

$$\widetilde{\theta}_T - \theta = -\frac{\theta^{1+2H}}{2H^2\Gamma(2H)}\left(\frac{I_T}{T} - V_{H,\theta}\right) + \frac{(1+2H)(H\Gamma(2H))^{\frac{1}{2H}}}{8H^2}\varpi_T^{-\frac{1+4H}{2H}}\left(\frac{I_T}{T} - V_{H,\theta}\right)^2. \quad (1.20)$$

Thus

$$2H\left(\frac{T}{-\sigma_H^2\theta}\right)^{1/2}(\widetilde{\theta}_T - \theta) = \frac{\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} N_T}{\left(\frac{\sigma_H^2\theta}{4TH^2}\right) I_T}. \quad (1.21)$$

We study the large deviations, moderate deviations and Berry-Esseen bounds of the LSE and the QLSE in this paper. We will use the following optimal fourth moment theorem from Nourdin and Peccati [14] in the sequel. See also Douissi *et al.* [15].

Theorem 1.1 *(Skewness Kurtosis Inequality) Let $(X_n)_{n\geq 1}$ be a sequence of random variables in fixed Wiener chaos of order $q \geq 2$ such that $Var(X_n) = 1$. Assume X_n converges to normal distribution which is equivalent to $\lim_n E(X_n)^4 = 3$, which is also known as the Fourth Moment Theorem. Then we have the following optimal rate for $d_{TV}(X_n, \mathcal{N})$ known as the Optimal Fourth Moment Theorem: There exist two constants $c, C > 0$ depending only on the sequence $(X_n)_{n\geq 1}$ but not on n, such that*

$$c \max\{E(X_n^4) - 3,\ |E(X_n^3)|\} \leq d_{TV}(X_n, \mathcal{N}) \leq C \max\{E(X_n^4) - 3,\ |E(X_n^3)|\}. \quad (1.22)$$

Let $\Phi(\cdot)$ denote the standard normal distribution function. Throughout the paper, C denotes a generic constant (which does not depend on T and x). We have not tried to estimate the constant in the bound on normal approximation.

Hu *et al.* [11] obtained limiting normal distribution of the LSE and the QLSE for the memory range $\frac{1}{2} < H \leq \frac{3}{4}$ with the rate $\sqrt{T}$ for $\frac{1}{2} < H < \frac{3}{4}$ and $\sqrt{T}(\log T)^{-1/2}$ for the case $H = \frac{3}{4}$, and limiting Rosenblatt distribution for the memory range $\frac{3}{4} < H < 1$.

We only consider the memory range $\frac{1}{2} < H < \frac{3}{4}$. Jiang *et al.* [12] used self-normalization along with the splitting method for the LSE and the QLSE in fractional Ornstein-Uhlenbeck process and obtained the rate $T^{-1/2}\log T$ for the range $\frac{1}{2} \leq H \leq \frac{5}{8}$ for the LSE and $T^{-1/4}\log T$ for the range $\frac{1}{2} \leq H \leq \frac{11}{16}$ for the QLSE. They obtained the rate T^{4H-3} for the range $\frac{5}{8} < H < \frac{3}{4}$ for the LSE and the same rate T^{4H-3} for the range $\frac{11}{16} < H < \frac{3}{4}$ for the QLSE.

In this paper we improve the first rate to $T^{-1/2}$ for the MLE for the range $\frac{1}{2} \leq H \leq \frac{5}{8}$ and $T^{-1/4}$ for the range $\frac{1}{2} \leq H \leq \frac{11}{16}$ for the QLSE using the squeezing method as in Chapter 1 in Bishwal [1]. The main contribution of the paper is thus improvement in the rate my removing the $\log T$ term.

Note the critical points:

$\frac{1}{2} = 0.50$, $\frac{5}{8} = 0.63$, $\frac{2}{3} = 0.67$, $\frac{11}{16} = 0.69$, $\frac{3}{4} = 0.75$. Also $0.63 + 0.06 = 0.69, 0.69 + 0.06 = 0.75$.

Remark on the Critical point $\frac{5}{8}$: For the discrete observations case, Es-Sebaiy and Viens [7] pointed out that if $0 < H < \frac{5}{8}$, then the fourth moment is of the order n^{-1} and if $\frac{5}{8} < H < \frac{3}{4}$, then the fourth moment is of the order $n^{2(4H-3)}$ where n is the number of observations. The Berry-Esseen rate for $\widehat{\theta}$ is shown to be of the order $n^{-1/4}$ for $0 < H < \frac{5}{8}$ and of the order $n^{-(4H-3)/2}$ if $\frac{5}{8} < H < \frac{3}{4}$. for $H = \frac{3}{4}$, the rate is $(\log n)^{-1/4}$.

The proofs also need large deviation results for the stochastic integral and the energy integral. These integrals can be represented by multiple Wiener integrals. Then their expectations and variances as well as the fourth moment of their Malliavin derivatives can be estimated.

First we calculate bounds on the moments. Let

$$\varphi_T(s,t) := e^{-\theta|t-s|}, \quad \psi_T(s,t) := e^{-2\theta T+\theta(s+t)}, \quad g_T(s,t) := e^{-\theta(t-s)} I_{[0,t]}(s), \tag{1.23}$$

$$V_{H,\theta} := \theta^{-2H} H\Gamma(2H), \quad C_{H,\theta} := \theta^{1-4H}(4H-1)H^2\left(\Gamma^2(2H) + \frac{\Gamma(2H)\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)}\right). \tag{1.24}$$

Observe that

$$X_t = I_1(g_T(\cdot,t)), \tag{1.25}$$

$$M_T = \int_0^T X_t dW_t^H = \int_0^T \int_0^t e^{\theta(t-s)} dW_s^H dW_t^H = \frac{1}{2} e^{\theta|t-s|} dW_s^H dW_t^H = \frac{1}{2} I_2(\varphi_T), \tag{1.26}$$

$$I_T = \int_0^T X_t^2 dt = \frac{1}{2\theta} I_2(\varphi_T) + \frac{1}{2\theta} I_2(\psi_T) + \int_0^T \|g_T(\cdot,t)\|_{\mathcal{H}}^2 dt \tag{1.27}$$

where I_1 and I_2 are first and second Wiener chaos respectively. Furthermore,

$$\int_0^T \|g_T(\cdot,t)\|_{\mathcal{H}}^2 dt = V_{H,\theta} T + o(T). \tag{1.28}$$

For $\frac{1}{2} < H < \frac{3}{4}$,

$$E(X_t X_s) \le C|t-s|^{2H-2}, \tag{1.29}$$

$$\|\varphi_T\|_{\mathcal{H}}^2 = 2T(c_{H,T} + (o(1)), \quad \|\psi_T\|_{\mathcal{H}}^2 = O(1). \tag{1.30}$$

For $H = \frac{1}{2}$, by the isometry of the Itô integral, we obtain

$$\|\varphi_T\|_{\mathcal{H}}^2 = 2\int_0^T \int_0^t e^{2\theta(t-s)} dt ds = \frac{T}{\theta} + \frac{e^{2\theta T} - 1}{2\theta^2} = 2T(C_{H,T} + (o(1)). \tag{1.31}$$

$$\|\psi_T\|_{\mathcal{H}}^2 = e^{-4\theta T} \int_0^T \int_0^T e^{-2\theta(t+s)} dt ds = \frac{(e^{-2\theta T} - 1)^2}{4\theta^2} = O(1). \tag{1.32}$$

For $\frac{1}{2} < H < \frac{3}{4}$,, using Lemma 5.3 in Hu and Nualart [10], we have

$$\|\psi_T\|_{\mathcal{H}}^2 \le \frac{\Gamma^2(2H)}{(2H-1)^2} \theta^{-4H}. \tag{1.33}$$

Let $\Upsilon_T = T$ for $H = \frac{1}{2}$ and $\Upsilon_T = T^{8H-4}$ for $\frac{1}{2} < H < \frac{3}{4}$.

We obtain the variances bounds on the Malliavin derivative of M_T and I_T.

$$E(\|DM_T\|_{\mathcal{H}}^2 - E\|DM_T\|_{\mathcal{H}}^2) \le C\Upsilon_T, \tag{1.34}$$

$$E(\|DI_T\|^2_{\mathcal{H}} - E\|DI_T\|^2_{\mathcal{H}}) \leq C\Upsilon_T \tag{1.35}$$

where D is the Malliavin derivative operator.

We have the bound on the fourth moment

$$E(\|DI_2(\varphi_T)\|^2_{\mathcal{H}} - E\|DI_2(\varphi_T)\|^2_{\mathcal{H}})^2 \leq C\Upsilon_T. \tag{1.36}$$

For $\frac{1}{2} < H < \frac{3}{4}$,, we have the bound on the fourth moment

$$E(\|DI_2(\varphi_T)\|^2_{\mathcal{H}} - E\|DI_2(\varphi_T)\|^2_{\mathcal{H}})^2 \leq CT^{8H-4}. \tag{1.38}$$

We have the bound on the fourth moment

$$E(\|DI_2(\psi_T)\|^2_{\mathcal{H}} - E\|DI_2(\psi_T)\|^2_{\mathcal{H}})^2 \leq C. \tag{1.39}$$

$$D_s I_2(\psi_T) = -2e^{-2\theta T+\theta s}\int_0^T e^{\theta t} dW_t^H. \tag{1.40}$$

$$E\|D_s I_2(\psi_T)\|^4_{\mathcal{H}} = 16e^{-8\theta T}\left(\int_0^T e^{\theta t} dW_t^H\right)^4\left(\int_0^T e^{2\theta t} dt\right)^2 = 48e^{-8\theta T}\left(\int_0^T e^{2\theta t} dt\right)^4. \tag{1.41}$$

$$E\|D_s I_2(\psi_T)\|^2_{\mathcal{H}} = 4e^{-4\theta T}\left(\int_0^T e^{2\theta t} dt\right)^2. \tag{1.42}$$

Therefore

$$E(\|DI_2(\psi_T)\|^2_{\mathcal{H}} - E\|DI_2(\psi_T)\|^2_{\mathcal{H}})^2 = E\|DI_2(\psi_T)|^4_{\mathcal{H}} - (E\|DI_2(\psi_T)\|^2_{\mathcal{H}})^2 = \frac{2(1-e^{-2\theta T})^4}{\theta^4}. \tag{1.43}$$

Similarly for the case $\frac{1}{2} < H < \frac{3}{4}$, it can be shown that

$$E(\|DI_2(\psi_T)\|^2_{\mathcal{H}} - E\|DI_2(\psi_T)\|^2_{\mathcal{H}})^2 \leq \frac{32\Gamma^4(2H)}{(2H-1)^4}\theta^{-8H}. \tag{1.44}$$

First we have the Berry-Esseen bounds for the stochastic integral and adjusted energy integral. By using the Optimal Fourth Moment theorem (Skewness-Kurtosis Inequality) from Stein-Malliavin theory, we have:

For $\frac{1}{2} \leq H \leq 5/8$, we have

$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{C_{H,\theta}^{-1}}{T}\right)^{1/2} M_T \leq x\right\} - \Phi(x)\right|\\ &\leq C\left\{E\left(\|D\left(\frac{C_{H,\theta}^{-1}}{T}\right)^{1/2} M_T\|^2_{\mathcal{H}} - E\|D\left(\frac{C_{H,\theta}^{-1}}{T}\right)^{1/2} M_T\|^2_{\mathcal{H}}\right)^2\right\}^{1/2} \leq CT^{-1/2}.\end{aligned} \tag{1.45}$$

For $\frac{5}{8} < H < \frac{3}{4}$,

$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{C_{H,\theta}^{-1}}{T}\right)^{1/2} M_T \leq x\right\} - \Phi(x)\right|\\ &\leq C\left\{E\left(\|D\left(\frac{C_{H,\theta}^{-1}}{T}\right)^{1/2} M_T\|^2_{\mathcal{H}} - E\|D\left(\frac{C_{H,\theta}^{-1}}{T}\right)^{1/2} M_T\|^2_{\mathcal{H}}\right)^2\right\}^{1/2} \leq C\ T^{4H-3}.\end{aligned} \tag{1.46}$$

For $\frac{1}{2} \leq H \leq 5/8$, we have

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} \left(\widetilde{\theta}_T I_T - \frac{T}{-\sigma_H^2} \right) \leq x \right\} - \Phi(x) \right| \\ \leq\ & C \left\{ E\left(\|D \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} \left(\widetilde{\theta}_T I_T - \frac{T}{-\sigma_H^2} \right) \|_{\mathcal{H}}^2 - E\|D \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} \left(\widetilde{\theta}_T I_T - \frac{T}{-\sigma_H^2} \right) \|_{\mathcal{H}}^2 \right)^2 \right\}^{1/2} \\ \leq\ & CT^{-1/2}. \end{aligned} \tag{1.47}$$

For $\frac{5}{8} < H < \frac{3}{4}$,

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} \left(\widetilde{\theta}_T I_T - \frac{T}{-\sigma_H^2} \right) \leq x \right\} - \Phi(x) \right| \\ \leq\ & C \left\{ E\left(\|D \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} \left(\widetilde{\theta}_T I_T - \frac{T}{-\sigma_H^2} \right) \|_{\mathcal{H}}^2 - E\|D \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} \left(\widetilde{\theta}_T I_T - \frac{T}{-\sigma_H^2} \right) \|_{\mathcal{H}}^2 \right)^2 \right\}^{1/2} \\ \leq\ & C\, T^{4H-3}. \end{aligned} \tag{1.48}$$

For $\frac{1}{2} \leq H \leq \frac{5}{8}$, we have for $|x| \leq 2(\log T)^{1/2}$,

$$\sup_{y\in\mathbb{R}} \left| P\left\{ \left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right)^{1/2} M_T - \left(\left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right) I_T - 1 \right) x \leq y \right\} - \Phi(y) \right| \leq CT^{-1/2}. \tag{1.49}$$

For $\frac{5}{8} < H < \frac{3}{4}$, we have for $|x| \leq 2(\log T)^{1/2}$,

$$\sup_{y\in\mathbb{R}} \left| P\left\{ \left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right)^{1/2} M_T - \left(\left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right) I_T - 1 \right) x \leq y \right\} - \Phi(y) \right| \leq CT^{4H-3}. \tag{1.50}$$

For $\frac{1}{2} \leq H \leq \frac{11}{16}$, we have

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} N_T \leq x \right\} - \Phi(x) \right| \\ &\leq C \left\{ E\left(\|D \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} N_T \|_{\mathcal{H}}^2 - E\|D \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} N_T \|_{\mathcal{H}}^2 \right)^2 \right\}^{1/2} \leq CT^{-1/2}. \end{aligned} \tag{1.51}$$

For $\frac{11}{16} < H < \frac{3}{4}$,

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} N_T \leq x \right\} - \Phi(x) \right| \\ &\leq C \left\{ E\left(\|D \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} N_T \|_{\mathcal{H}}^2 - E\|D \left(\frac{C_{H,\theta}^{-1}}{T} \right)^{1/2} N_T \|_{\mathcal{H}}^2 \right)^2 \right\}^{1/2} \leq C\, T^{4H-3}. \end{aligned} \tag{1.52}$$

For $\frac{1}{2} \leq H \leq \frac{11}{16}$, we have for $|x| \leq 2(\log T)^{1/2}$,

$$\sup_{y\in\mathbb{R}} \left| P\left\{ 2H\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} N_T - \left(\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right) I_T - 1\right) x \leq y \right\} - \Phi(y) \right| \leq CT^{-1/4}. \quad (1.53)$$

For $\frac{11}{16} < H < \frac{3}{4}$, we have for $|x| \leq 2(\log T)^{1/2}$,

$$\sup_{y\in\mathbb{R}} \left| P\left\{ 2H\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} N_T - \left(\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right) I_T - 1\right) x \leq y \right\} - \Phi(y) \right| \leq CT^{4H-3}. \quad (1.54)$$

2. Main Results

We need the next two lemmas from Jiang *et al.* [12] on large deviations to obtain bounds on the tail probabilities of the estimators. The first lemma is on large deviations for stochastic integral.

Lemma 2.1 For every $\delta > 0$,

$$P\left\{ \left| \frac{M_T}{T} \right| \geq \delta \right\} \leq C \exp\left(-\frac{T^{1/2}\delta}{4C_{H,\theta}^{1/2}} \right).$$

Remark For the case $H = 0.5$, there is a long history of work:

For every $\delta > 0$,

$$P\left\{ \left| \frac{M_T}{T} \right| \geq \delta \right\} \leq C_0 \exp\left(-C_1 T\delta^2 \right).$$

See Gao and Jiang [9].

For any $0 \leq \alpha \leq \theta^2/4$, there exist constants C_3 and C_4 such that

$$E(e^{\alpha I_T}) \leq C_3 e^{C_4\alpha T}. \quad (2.1)$$

See Gao and Jiang [9]. By Chebyshev inequality, we have

$$P(|X_T - E(X_T)| \geq \delta) \leq 2\exp(-\theta\delta^2). \quad (2.2)$$

The second lemma is on large deviations in the ergodic theorem.

Lemma 2.2 For every $\delta > 0$,

$$P\left\{ \left| \frac{I_T}{T} - V_{H,\theta} \right| \geq \delta \right\} \leq C \exp\left(-\frac{T^{1/2}\delta}{4C_{H,\theta}^{1/2}} \right).$$

Observe that by (1.11)

$$\widehat{\theta}_T = \theta - \frac{M_T}{I_T}.$$

Using the elementary inequality

$$P(|\frac{\xi}{\eta}| \geq u) \leq P(|\xi| \geq uv) + P(\eta - 2v| \geq v), \tag{2.3}$$

we have

$$\begin{aligned} & P(|\widehat{\theta}_T - \theta| \geq \delta) \\ \leq \;& P(|I_T - V_{H,\theta}T| \geq \tfrac{1}{2}V_{H,\theta}T) + P(|\widehat{\theta}_T - \theta| \geq \delta,\ |I_T - V_{H,\theta}T| < \tfrac{1}{2}V_{H,\theta}T) \\ \leq \;& P(|I_T - V_{H,\theta}T| \geq \tfrac{1}{2}V_{H,\theta}T) + P(|M_T| \geq \tfrac{1}{2}V_{H,\theta}T\delta). \end{aligned} \tag{2.4}$$

Combining Lemma 2.1 and Lemma 2.2, we obtain

Lemma 2.3 For every $\delta > 0$ and large $T > 0$, we have

$$\text{a)} \quad P(|\widehat{\theta}_T - \theta| \geq \delta) \leq C_0 \exp(-C_1 T^{1/2}\delta)$$

$$\text{b)} \quad P(|\widetilde{\theta}_T - \theta| \geq \delta) \leq C_0 \exp(-C_1 T^{1/2}\delta^{1/2}).$$

To obtain the rate of normal approximation for the LSE and the QLSE, we need the following tail probability estimate of the estimators.

Lemma 2.4

$$\text{(a)} \quad P\left\{\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2} |\widehat{\theta}_T - \theta| \geq 2(\log T)^{1/2}\right\} \leq CT^{-1/2}.$$

$$\text{(b)} \quad P\left\{2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2} |\widetilde{\theta}_T - \theta| \geq 2(\log T)^{1/2}\right\} \leq CT^{-1/4}.$$

Proof : Observe that

$$\begin{aligned} & P\left\{\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2} |\widehat{\theta}_T - \theta| \geq 2(\log T)^{1/2}\right\} \\ = \;& P\left\{\left|\frac{\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} M_T}{\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right) I_T}\right| \geq 2(\log T)^{1/2}\right\} \\ \leq \;& P\left\{\left|\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} M_T\right| \geq (\log T)^{1/2}\right\} + P\left\{\left|\frac{-\sigma_H^2\widetilde{\theta}_T}{T} I_T\right| \leq \frac{1}{2}\right\} \\ \leq \;& \left|P\left\{\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} |M_T| \geq (\log T)^{1/2}\right\} - 2\Phi(-(\log T)^{1/2})\right| \\ & +2\Phi(-(\log T)^{1/2}) + P\left\{\left|\frac{\sigma_H^2\widetilde{\theta}_T}{T} I_T - 1\right| \geq \frac{1}{2}\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right)^{1/2} |M_T| \geq x \right\} - 2\Phi(-x) \right| \\
&\quad +2\Phi(-(\log T)^{1/2}) + P\left\{ \left| \left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right) I_T - 1 \right| \geq \frac{1}{2} \right\} \\
&\leq CT^{-1/2} + C(T\log T)^{-1/2} + C\exp\left(-\frac{T^{1/2}}{8C_{H,\theta}^{1/2}} \right) \\
&\leq CT^{-1/2}.
\end{aligned}
$$

The bounds for the first and the third terms come from Lemma 2.2 and Lemma 2.1 respectively and that for the middle term comes from Feller [8] (p. 166). Proof of (b) is similar. □

Now we are ready to obtain the uniform rate of normal approximation of the distribution of the LSE and the QLSE.

Recall that

$$
\sigma_H^2 := (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)\Gamma(2H)} \right). \tag{2.5}
$$

Theorem 2.5

a) If $\frac{1}{2} \leq H \leq \frac{5}{8}$

$$
\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta) \leq x \right\} - \Phi(x) \right| \leq CT^{-1/2}.
$$

b) If $\frac{5}{8} < H < \frac{3}{4}$

$$
\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta) \leq x \right\} - \Phi(x) \right| \leq CT^{4H-3}.
$$

c) If $\frac{1}{2} \leq H \leq \frac{11}{16}$

$$
\sup_{x\in\mathbb{R}} \left| P\left\{ 2H \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widetilde{\theta}_T - \theta) \leq x \right\} - \Phi(x) \right| \leq CT^{-1/4}.
$$

d) If $\frac{11}{16} < H < \frac{3}{4}$

$$
\sup_{x\in\mathbb{R}} \left| P\left\{ 2H \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widetilde{\theta}_T - \theta) \leq x \right\} - \Phi(x) \right| \leq CT^{4H-3}.
$$

Proof : First we prove (a). We shall consider two possibilities (i) and (ii).

(i) $|x| > 2(\log T)^{1/2}$.

We shall give a proof for the case $x > 2(\log T)^{1/2}$. The proof for the case $x < -2(\log T)^{1/2}$ runs similarly. Note that

$$\left| P\left\{ \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta) \leq x \right\} - \Phi(x) \right| \leq P\left\{ \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta) \geq x \right\} + \Phi(-x). \quad (2.6)$$

But from Feller [8] (p. 166) we have

$$\Phi(-x) \leq \Phi(-2(\log T)^{1/2}) \leq CT^{-1}. \quad (2.7)$$

Moreover, by Lemma 2.4 (a), we have

$$P\left\{ \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta) \geq 2(\log T)^{1/2} \right\} \leq CT^{-1/2}. \quad (2.8)$$

Hence

$$\left| P\left\{ \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta) \leq x \right\} - \Phi(x) \right| \leq CT^{-1/2}. \quad (2.9)$$

(ii) $|x| \leq 2(\log T)^{1/2}$.

$$\text{Let} \quad A_T := \left\{ \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} |\widehat{\theta}_T - \theta| \leq 2(\log T)^{1/2} \right\} \quad \text{and} \quad B_T := \left\{ \frac{I_T}{T} > c_0 \right\} \quad (2.10)$$

where $0 < c_0 < \frac{1}{-\sigma_H^2 \theta}$. By Lemma 2.4, we have

$$P(A_T^c) \leq CT^{-1/2}. \quad (2.11)$$

By Lemma 2.1, we have

$$P(B_T^c) = P\left\{ \left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right) I_T - 1 < \sigma_H^2 \theta c_0 - 1 \right\}$$

$$< P\left\{ \left| \left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right) I_T - 1 \right| > 1 - \sigma_H^2 \theta c_0 \right\} \leq C \exp\left(-\frac{T^{1/2}(1 - \sigma_H^2 \theta c_0)}{4C_{H,\theta}^{1/2}} \right). \quad (2.12)$$

Let b_0 be some positive number. On the set $A_T \cap B_T$ for all $T > T_0$ with $4b_0(\log T_0)^{1/2} \left(\frac{\sigma_H^2 \theta}{T} \right)^{1/2} \leq c_0$, we have

$$\left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta) \leq x$$

$$\Rightarrow \quad I_T + b_0 T(\widehat{\theta}_T - \theta) < I_T + \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} \sigma_H^2 b_0 \theta x$$

$$\Rightarrow \quad \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta)[I_T + b_0 T(\theta_T - \theta)] < x[I_T + \left(\frac{T}{-\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} \sigma_H^2 b_0 \theta x]$$

$$\Rightarrow \quad (\widehat{\theta}_T - \theta) I_T + b_0 T(\theta_T - \theta)^2 < \left(\frac{-\sigma_H^2 \widetilde{\theta}_T}{T} \right)^{1/2} I_T x + \sigma_H^2 b_0 \theta x^2$$

$$\Rightarrow \quad -M_T + (\widehat{\theta}_T - \theta)I_T + b_0T(\widehat{\theta}_T - \theta)^2 < -M_T + \left(\frac{\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} I_T x + \sigma_H^2 b_0\theta x^2$$

$$\Rightarrow \quad 0 < -M_T + \left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} I_T x + \sigma_H^2 b_0\theta x^2$$

since

$$\begin{aligned} & I_T + b_0T(\widehat{\theta}_T - \theta) > Tc_0 + b_0T(\widehat{\theta}_T - \theta) \\ > \quad & 2\sigma_H^2 b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} - \sigma_H^2 b_0(\log T)^{1-H}\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} \\ = \quad & \sigma_H^2 b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} > 0. \end{aligned}$$

On the other hand, on the set $A_T \cap B_T$ for all $T > T_0$ with $4b_0(\log T_0)^{1/2}\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T_0}\right)^{1/2} \leq c_0$, we have

$$\begin{aligned} & \left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widehat{\theta}_T - \theta) > x \\ \Rightarrow \quad & I_T - b_0T(\widehat{\theta}_T - \theta) < I_T - \left(\frac{T}{\sigma_H^2\widetilde{\theta}_T}\right)^{1/2} 2b_0\theta x \\ \Rightarrow \quad & \left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widehat{\theta}_T - \theta)[I_T - b_0T(\widehat{\theta}_T - \theta)] > x[I_T - \left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}\sigma_H^2 b_0\theta x] \\ \Rightarrow \quad & (\widehat{\theta}_T - \theta)I_T - b_0T(\widehat{\theta}_T - \theta)^2 > \left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{-1/2} I_T x - \sigma_H^2 b_0\theta x^2 \\ \Rightarrow \quad & -M_T + (\widehat{\theta}_T - \theta)I_T - b_0T(\widehat{\theta}_T - \theta)^2 > -M_T + \left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{-1/2} I_T x - \sigma_H^2 b_0\theta x^2 \\ \Rightarrow \quad & 0 > -M_T + \left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} I_T x - \sigma_H^2 b_0\theta x^2 \end{aligned}$$

since

$$\begin{aligned} & I_T - b_0T(\widehat{\theta}_T - \theta) > Tc_0 - b_0T(\widehat{\theta}_T - \theta) \\ > \quad & 2\sigma_H^2 b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} - \sigma_H^2 b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} \\ = \quad & \sigma_H^2 b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} > 0. \end{aligned}$$

Hence

$$0 < -M_T + \left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2} I_T x - \sigma_H^2 b_0\theta x^2 \Rightarrow \left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widehat{\theta}_T - \theta) \leq x.$$

Letting

$$D^{\pm}_{T,x} := \left\{-M_T + \left(\frac{\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} I_T x \pm \sigma_H^2 b_0\theta x^2 > 0\right\},$$

we obtain

$$D^{-}_{T,x} \cap A_T \cap B_T \subseteq A_T \cap B_T \cap \left\{\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widehat{\theta}_T - \theta) \leq x\right\} \subseteq D^{+}_{T,x} \cap A_T \cap B_T. \tag{2.13}$$

If it is shown that

$$\left|P\left\{D^{\pm}_{T,x}\right\} - \Phi(x)\right| \leq CT^{-1/2} \tag{2.14}$$

for all $T > T_0$ and $|x| \leq 2(\log T)^{1/2}$, then the theorem would follow from (2.11) - (2.14).

We shall prove (2.4) for $D^{+}_{T,x}$. The proof for $D^{-}_{T,x}$ is analogous. Observe that

$$\begin{aligned}
& \left|P\left\{D^{+}_{T,x}\right\} - \Phi(x)\right| \\
= & \left|P\left\{\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} M_T - \left(\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right) I_T - 1\right) x < x + \sigma_H^2\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} b_0\theta x^2\right\} - \Phi(x)\right| \\
\leq & \sup_{y\in\mathbb{R}}\left|P\left\{\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} M_T - \left(\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right) I_T - 1\right) x \leq y\right\} - \Phi(y)\right| \\
& + \left|\Phi\left(x + \left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} b_0\theta x^2\right) - \Phi(x)\right| \\
=: & \Delta_1 + \Delta_2.
\end{aligned} \tag{2.15}$$

(1.50) immediately yields

$$\Delta_1 \leq CT^{-1/2}. \tag{2.16}$$

On the other hand, for all $T > T_0$,

$$\Delta_2 \leq 2\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} b_0\theta x^2(2\pi)^{-1/2}\exp(-\overline{x}^2/2)$$

where

$$|\overline{x} - x| \leq 2\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} b_0\theta x^2.$$

Since $|x| \leq 2(\log T)^{1/2}$, it follows that $|\bar{x}| > |x|/2$ for all $T > T_0$ and consequently

$$\begin{aligned}
\Delta_2 \leq & \ 2\left(\frac{-\sigma_H^2\widetilde{\theta}_T}{T}\right)^{1/2} b_0\theta x^2(2\pi)^{-1/2}x^2\exp(-x^2/8) \\
\leq & \ CT^{-1/2}.
\end{aligned} \tag{2.17}$$

From (2.15) - (2.17), we obtain

$$\left|P\left\{D^{+}_{T,x}\right\}-\Phi(x)\right|\leq CT^{-1/2}.$$

This completes the proof of part (a) of the theorem.

Next we prove (c). Again we shall consider two possibilities (i) and (ii).

(i) $|x|>2(\log T)^{1-/2}$.

We shall give a proof for the case $x>2(\log T)^{1/2}$. The proof for the case $x<-2(\log T)^{1/2}$ runs similarly. Note that

$$\left|P\left\{2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widetilde{\theta}_T-\theta)\leq x\right\}-\Phi(x)\right|\leq P\left\{2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widetilde{\theta}_T-\theta)\geq x\right\}+\Phi(-x).$$

By (2.7) and Lemma 2.4 (b), we have

$$P\left\{2H\left(\frac{T}{\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widetilde{\theta}_T-\theta)\geq 2(\log T)^{1/2}\right\}\leq CT^{-1/4}.$$

Hence

$$\left|P\left\{2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widetilde{\theta}_T-\theta)\leq x\right\}-\Phi(x)\right|\leq CT^{-1/4}.$$

(ii) $|x|\leq 2(\log T)^{1/2}$.

$$\text{Let}\quad A_{1,T}:=\left\{2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}|\widetilde{\theta}_T-\theta|\leq 2(\log T)^{1/2}\right\}\text{ and } B_{1,T}:=\left\{\frac{I_T}{T}>c_0\right\}$$

where $0<c_0<\frac{1}{-\sigma_H^2\theta}$. By Lemma 2.4, we have

$$P(A^c_{1,T})\leq CT^{-1/4}. \tag{2.18}$$

By Lemma 2.1, we have

$$P(B^c_{1,T})=P\left\{\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)I_T-1<\sigma_H^2\theta c_0-1\right\}<P\left\{\left|\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)I_T-1\right|>1-\sigma_H^2\theta c_0\right\}\leq CT^{-1}. \tag{2.19}$$

Let b_0 be some positive number. On the set $A_{1,T}\cap B_{1,T}$ for all $T>T_0$ with $4b_0(\log T_0)^{1/2}\left(\frac{-\sigma_H^2\theta}{4T_0H^2}\right)^{1/2}\leq c_0$, we have

$$2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widetilde{\theta}_T-\theta)\leq x$$

$$\Rightarrow\quad I_T+b_0T(\widetilde{\theta}_T-\theta)<I_T+2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}\sigma_H^2b_0\theta x$$

$$\Rightarrow \quad 2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widetilde{\theta}_T-\theta)[I_T+b_0T(\theta_T-\theta)]<x\left[I_T+2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}\sigma_H^2b_0\theta x\right]$$

$$\Rightarrow \quad (\widetilde{\theta}_T-\theta)I_T+b_0T(\theta_T-\theta)^2<\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}I_Tx+\sigma_H^2b_0\theta x^2$$

$$\Rightarrow \quad -N_T+(\widetilde{\theta}_T-\theta)I_T+b_0T(\widetilde{\theta}_T-\theta)^2<-N_T+\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}I_Tx+\sigma_H^2b_0\theta x^2$$

$$\Rightarrow \quad 0<-N_T+\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}I_Tx+\sigma_H^2b_0\theta x^2$$

since

$$\begin{aligned} & I_T+b_0T(\theta_T-\theta)>Tc_0+b_0T(\theta_T-\theta) \\ > \quad & 4b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}-\sigma_H^2b_0(\log T)^{1-H}\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} \\ = \quad & \sigma_H^2b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}>0. \end{aligned}$$

On the other hand, on the set $A_{1,T}\cap B_{1,T}$ for all $T>T_0$ with $4b_0(\log T_0)^{1/2}\left(\frac{-\sigma_H^2\theta}{4T_0H^2}\right)^{1/2}\le c_0$, we have

$$2H\left(\frac{T}{\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\theta_T-\theta)>x$$

$$\Rightarrow \quad I_T-b_0T(\widetilde{\theta}_T-\theta)<I_T-2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}\sigma_H^2b_0\theta x$$

$$\Rightarrow \quad 2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}(\widetilde{\theta}_T-\theta)[I_T-b_0T(\theta_T-\theta)]>x\left[I_T-2H\left(\frac{T}{-\sigma_H^2\widetilde{\theta}_T}\right)^{1/2}2b_0\theta x\right]$$

$$\Rightarrow \quad (\widetilde{\theta}_T-\theta)I_T-b_0T(\widetilde{\theta}_T-\theta)^2>\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}I_Tx-\sigma_H^2b_0\theta x^2$$

$$\Rightarrow \quad -N_T+(\widetilde{\theta}_T-\theta)I_T-b_0T(\theta_T-\theta)^2>-N_T+\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}I_Tx-\sigma_H^2b_0\theta x^2$$

$$\Rightarrow \quad 0>-N_T+\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}I_Tx-\sigma_H^2b_0\theta x^2$$

since

$$\begin{aligned} & I_T-b_0T(\widetilde{\theta}_T-\theta)>Tc_0-b_0T(\widetilde{\theta}_T-\theta) \\ > \quad & 2\sigma_H^2b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}-\sigma_H^2b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} \\ = \quad & \sigma_H^2b_0(\log T)^{1/2}\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2}>0. \end{aligned}$$

Hence

$$0 < -N_T + \left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} I_T x - \sigma_H^2 b_0\theta x^2 \Rightarrow 2H(\frac{T}{-\sigma_H^2\tilde{\theta}_T})^{1/2}(\theta_T - \theta) \le x.$$

Letting

$$D^{\pm}_{1,T,x} := \left\{-N_T + \left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} I_T x \pm \sigma_H^2 b_0\theta x^2 > 0\right\},$$

we obtain

$$D^{-}_{1,T,x} \cap A_{1,T} \cap B_{1,T} \subseteq A_{1,T} \cap B_{1,T} \cap \left\{2H\left(\frac{T}{-\sigma_H^2\tilde{\theta}_T}\right)^{1/2}(\tilde{\theta}_T - \theta) \le x\right\} \subseteq D^{+}_{1,T,x} \cap A_{1,T} \cap B_{1,T}. \tag{2.20}$$

If it is shown that

$$\left|P\left\{D^{\pm}_{1,T,x}\right\} - \Phi(x)\right| \le CT^{-1/4} \tag{2.21}$$

for all $T > T_0$ and $|x| \le 2(\log T)^{1/2}$, then the theorem would follow from (2.18) - (2.21).

We shall prove (2.21) for $D^{+}_{1,T,x}$. The proof for $D^{-}_{1,T,x}$ is analogous.

Observe that

$$\begin{aligned}
&\left|P\left\{D^{+}_{1,T,x}\right\} - \Phi(x)\right| \\
= &\left|P\left\{\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} N_T - \left(\left(\frac{-\sigma_H^2\theta}{4TH^2}\right) I_T - 1\right)x < x + 2\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} b_0\theta x^2\right\} - \Phi(x)\right| \\
\le &\sup_{y\in\mathbb{R}}\left|P\left\{\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} N_T - \left(\left(\frac{-\sigma_H^2\theta}{4TH^2}\right) I_T - 1\right)x \le y\right\} - \Phi(y)\right| \\
&+\left|\Phi\left(x + \left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} b_0\theta x^2\right) - \Phi(x)\right| \\
=: &\ \Delta_{11} + \Delta_{12}.
\end{aligned} \tag{2.22}$$

(1.53) immediately yields

$$\Delta_{11} \le CT^{-1/4}. \tag{2.23}$$

On the other hand, for all $T > T_0$,

$$\Delta_{12} \le 2\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} b_0\theta x^2(2\pi)^{-1/2}\exp(-\bar{x}^2/2)$$

where

$$|\bar{x} - x| \le 2\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} b_0\theta x^2.$$

Since $|x| \le 2(\log T)^{1/2}$, it follows that $|\bar{x}| > |x|/2$ for all $T > T_0$ and consequently

$$\Delta_{12} \le 2\left(\frac{-\sigma_H^2\theta}{4TH^2}\right)^{1/2} b_0\theta x^2(2\pi)^{-1/2}x^2\exp(-x^2/8) \le CT^{-1/4}. \tag{2.24}$$

From (2.12) - (2.14), we obtain

$$\left|P\left\{D^{+}_{1,T,x}\right\} - \Phi(x)\right| \le CT^{-1/4}.$$

This completes the proof of part (c) of the theorem. Next we demonstrate the proof of (b) and (d).

If $\frac{5}{8} < H < \frac{3}{4}$ by following similar steps, one can show that

$$\sup_{x\in\mathbb{R}} \left| P\left\{ \left(\frac{T}{\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widehat{\theta}_T - \theta) \le x \right\} - \Phi(x) \right| \le C_\theta T^{4H-3}.$$

If $\frac{11}{16} < H < \frac{3}{4}$ by following similar steps, one can show that

$$\sup_{x\in\mathbb{R}} \left| P\left\{ 2H \left(\frac{T}{\sigma_H^2 \widetilde{\theta}_T} \right)^{1/2} (\widetilde{\theta}_T - \theta) \le x \right\} - \Phi(x) \right| \le C_\theta T^{4H-3}.$$

This completes the proof of the theorem. □

Concluding Remark For the case $\frac{1}{2} \le H \le \frac{5}{8}$, our rate is $O(T^{-1/2})$ is optimal.

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