Eur. J. Math. Anal. 3 (2023) 18
doi: 10.28924/ada/ma.3.18

A Modified Algorithms for New Krasnoselskii's Type for Strongly Monotone and Lipschitz Mappings

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Abstract. Let E be a 2 uniformly smooth and convex real Banach space and let a mapping $A : E \to E^*$ be lipschitz and strongly monotone such that $A^{-1}(0) \neq \emptyset$. For an arbitrary $(\{x_1\}, \{y_1\}) \in E$, we define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{cases} y_n = x_n - \theta_n J^{-1}(Ax_n), & n \geq 1 \\ x_{n+1} = y_n - \lambda_n J^{-1}(Ay_n), & n \geq 1 \end{cases}$$

where λ_n and θ_n are positive real number and J is the duality mapping of E. Letting $(\lambda_n, \theta_n) \in (0,1)$, then x_n and y_n converges strongly to ρ^*, a unique solution of the equation $Ax = 0$. We also applied our algorithm in convex minimization and also proved the convergence of it in L_p, ℓ_p *or* $W^{m,p}$. At the end we proposed the algorithm of it in $L_p(\Omega)$ and its inverse $L^q(\Omega)$.

1. Introduction

Definition 1.1. A map $A : E \to E^*$ is called monotone if for each $x, y \in E$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0$$

A is called strongly monotone if there exists $k \in (0,1)$ such that for each $x, y \in E$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq k\|x - y\|^2$$

A map $A : E \to E$ is called accretive if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0$$

A is called strongly accretive if there exists $k \in (0,1)$ such that for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq k\|x - y\|^2$$

Received: 23 Apr 2023.

Key words and phrases. Krasnoselskii-type algorithm; monotone operators; Lipschitz mappings.

A map $A : E \to E^*$ is called Lipschitzian, if for each constant $L > 0$ and for all $x, y \in E$, the following inequality holds:

i):

$$\langle Ax - Ay\rangle \leq L\|x - y\|$$

ii):

$$L^2(d_2 - 1) < k^2$$

Many physical problems in applications can be modeled in the following form: find $x \in H$ such that

$$0 \in Ax \tag{1.1}$$

where A is a monotone operator on a real Hilbert space H. Typical examples where monotone operators occur and satisfy the inclusion $0 \in Ax$ include the equilibrium state of evolution equations and critical points of some functionals and convex optimization, linear programing, monotone inclusions and elliptic differential equations defined on Hilbert spaces (see e.g., Browder [2], Mustafa [19], Stephen [26], Sina [24], Mendy et al, [17] and Chidume [3]). For precisely, the classical convex optimization problem: let $h : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. The sub-differential of h at $x \in H$; is defined by $\partial h : H \to 2^H$

$$\partial(x) = \{x^* \in h : h(y) - h(x) \geq \langle y - x, x^*\rangle, \forall y \in h\}. \tag{1.2}$$

Clearly, $\partial h : H \to 2^H$is monotone operator on H, and $0 \in \partial(x_0)$ if and only if x_0 is a minimizer of h. In the case of setting $\partial(x) \equiv A$; solving the inclusion $0 \in Ax$ is solving for a minimizer of h.

There have been fruitful works on approximating zero point of A in Hilbert spaces (see e.g., Takahashi and Ueda [31], Song and Chen [25], and Cho et al. [9]). The proximal point algorithm (PPA) is recognized as a powerful and successful algorithm in finding a numerical solution of monotone operators equation $0 \in Ax$ which was introduced by Martinet [13] and studied further by Rockafellar [22] and a host of other authors. That is, given $x_k \in H$;

$$x_{n+1} = J_{\lambda_n} x_n. \tag{1.3}$$

where $J_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of operator A. Since Rockafellar [22] only obtained the weak convergence of the algorithm 1.3 as $\lambda_n \to \infty$; so he proposed two open questions for obtaining the strong convergence of the proximal point algorithm: (1) Does the proximal point algorithm always converge weakly? (2) Can the proximal point algorithm be modified to guarantee strong convergence? In studying the strong convergence, many authors have modified the proximal point algorithm (PPA) to guarantee strong convergence under different settings, see e.g., Takahashi [29], Reich [20], Lehdili and Moudafi [12], Chidume et al. [6], and the references therein.

Let E be a real normed space, E^* its topological dual space. The map $J : E \to 2^{E^*}$ defined by

$$J_x : \Big\{x^* \in E^* : \quad \langle x, x^*\rangle = \|x\|.\|x^*\| = \|x\|^2 = \|x^*\|^2\Big\}.$$

is called the normalized duality map on E. where $\langle , \rangle$ denotes the generalized duality pairing between E and E^*.

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, monotonicity and accretivity coincide. For an accretive-type operator A,

solutions of the equation $Ax = 0$, in many cases, represent the equilibrium state of some dynamical system (see, for example, [29], page 116). To approximate a solution of $Ax = 0$, assuming existence, where $A : E \to E$ is of accretive type, Browder [2] defined an operator $T : E \to E$ by $T := I - A$, where I is the identity map on E. He called such an operator pseudo-contractive. It is trivial to observe that zeros of A correspond to fixed points of T. For Lipschitz strongly pseudo-contractive maps, Chidume [6] proved the following theorem.

Theorem 1.1. *(Chidume, [7]. Let $E = Lp, 2 \leq p < 8$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \to K$ be a strongly pseudo-contractive and Lipschitz map. For arbitrary $x_0 \in K$, let a sequence $\{x_n\}$ be defined iteratively by $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n, n \geq 0$, where $\{\lambda_n\} \subset (0, 1)$ satisfies the following conditions: ,$(i) \sum_{n=1}^{\infty} \lambda_n = \infty$,$(ii) \sum_{n=1}^{\infty} \lambda_n^2 \leq \infty$. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

By setting $T := I - A$ in Theorem 1.1, the following theorem for approximating a solution of $Ax = 0$ where A is a strongly accretive and bounded operator can be proved.

Unfortunately, the success achieved in using geometric properties developed from the mid-1980s to early 1990s in approximating zeros of accretive-type mappings has not carried over to approximating zeros of monotone-type operators in general Banach spaces. Part of the problem is that since A maps E to E^*, for $x_n \in E, Ax_n$ is in E^*. Consequently, a recursion formula containing x_n and Ax_n may not be well defined. Attempts have been made to overcome this difficulty by introducing the inverse of the normalized duality mapping in the recursion formulas for approximating zeros of monotone-type mappings.Examples Chidume [4], [5], Moudafi [18], Reich [21], Takahashi [30], Zegeye [37], Djitte [17], Mendy [[15], [10]]

Motivated by approximating zeros of monotone mappings, Chidume et al. [8] proposed a Krasnoselskii-type scheme and proved a strong convergence theorem in $L_p, 2 \leq p < \infty$. In fact, they obtained the following result.

Theorem 1.2. *(Chidume et al. [8]). Let $X = L_p, 2 \leq p < \infty$, and $A : X \to X^*$ be a Lipschitz map. Assume that there exists a constant $k \in (0, 1)$ such that A satisfies the condition*

$$\langle Ax - Ay, x - y \rangle \geq k\|x - y\|^{\frac{p}{p-1}} \tag{1.4}$$

and that $A^{-1}(0) \neq \emptyset$. For arbitrary $x_1 \in X$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n Ax_n) \qquad n \geq 0$$

where $\lambda_n \in (0, \delta_p)$ and δ_p is some positive constant. Then the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

In [8], the authors posed the following open problem. If $E = L_p, 2 \leq p < \infty$, attempts to obtain strong convergence of the Krasnoselskii-type sequence defined for $x_0 \in E$ by

$$x_{n+1} = J^{-1}(Jx_n - \lambda_n Ax_n) \qquad n \geq 0$$

to a solution of the equation $Ax = 0$, where A is strongly monotone and Lipschitz, have not yielded any positive result.

Following the works of Chidume et al [8], and motivation of finding the zeros of the monotone type mapping, several strong convergence results have been established by various authors (see e.g [17], [10], [15], [23], [16]).

Following this great work, in 2023, Mendy [16] constructed the following two-step proximal algorithm for the zero point of monotone mapping and proof a strong convergency of the sequences $\{x_n\}$ and $\{y_n\}$ to a unique point $x^* \in A^{-1}(0)$.

$$\begin{cases} y_{n+1} = J^{-1}(Jx_n - \lambda_n Ax_n), & n \geq 0 \\ x_{n+1} = J^{-1}(Jy_{n+1} - \lambda_{n+1} Ay_{n+1}), & n \geq 0 \end{cases} \tag{1.5}$$

In this paper, we study the two step size of the new Krasnoselskii-type algorithm introduced by Sene et al. [23] and prove a strong convergence theorem to approximate the unique zero of a Lipschitz and strongly monotone mapping 2$-$uniformly smooth and convex real Banach space for $p \geq 2$. This class of Banach spaces contains all Lp-spaces, $2 \leq p < \infty$ and Sobolev space. Then we apply our results to the convex minimization problem. Finally, our method of proof generalized and extended various authors in this way of work.

2. **Preliminaries**

Let E be a normed linear space. E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exist for each $x, y \in S_E$ (Here $S_E := \{x \in E : ||x|| = 1\}$ is the unit sphere of E). E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S_E$, and E is Fréchet differentiable if it is smooth and the limit is attained uniformly for $y \in S_E$.

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*.

A normed linear space E is said to be strictly convex if:

$$\|x\| = \|y\| = 1, \ x \neq y \ \Rightarrow \ \left\|\frac{x+y}{2}\right\| < 1.$$

The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by:

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{1}{2}\|x+y\| \ : \|x\| = \|y\| = 1, \ \|x-y\| \geq \epsilon\right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$. For $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0,2]$. Observe that every p-uniformly convex space is uniformly convex.

Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or \ l_p) \ or \ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases}$$

Remark 1. Note also that duality mapping exists in each Banach space.We recall from [11] some of the examples of this mapping in $\ell_p, L_p, W^{m,p}-$spaces, $1 < p < \infty$

- $\ell_p : Jx = \|x\|_{\ell_p}^{2-p} y \in \ell_q, x = (x_1, x_2, ..., x_n, ...), y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, ..., x_n|x_n|^{p-2}, ...)$
- $L_p : Ju = \|u\|_{L_p}^{2-p}|u|^{p-2}u \in L_q$
- $W^{m,p} : Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|} (-1)^{|\alpha|} D^\alpha(|D^\alpha u|^{p-2} D^\alpha u) \in W^{-m,p}$

In L_p, ℓ_p *and* $W^{m,p}$ spaces for $1 < p < \infty$ are $q-$uniformly smooth real Banach spaces with q, as

$$q = \min\{2, p\} \quad and \quad d_q \geq 1 \tag{2.2}$$

is given by

$$d_q = \begin{cases} \frac{1+\tau^{q-1}}{(1+\tau)^{q-1}}, & if \quad 1 < p < 2; \\ p-1, & if \quad 2 \leq p < \infty. \end{cases} \tag{2.3}$$

and $\tau(0,1)$ as the unique solution of the equation

$$(q-2)t^{q-1} + (q-1)t^{q-2} - 1 = 0$$

It is well known that

- E is smooth if and only if J is single-valued.
- If E is uniformly smooth then J is uniformly continuous on bounded subsets of E.
- If E is reflexive and strictly convex dual then J^{-1} is single-valued, one-to-one, surjective, uniformly continuous on bounded subsets and it is the duality mapping from E^* into E and $J^{-1}J = I_E$ and $JJ^{-1} = I_E$.
- J^{-1} is uniformly continuous if and only if it has a modulus of continuity.

Lemma 2.1 (Xu [32]). *. Let $q > 1$ be a real number and E be a Banach space. Then the following assertion are equivalent*

- **i):** *E is $q-$uniformly smooth*
- **ii):** *There exists a constant $d_n > 0$, such that for all $x, y \in E$, then the following holds*

$$\|x+q\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + d_q\|y\|^q. \tag{2.4}$$

3. Main Result

We now prove the following result

Theorem 3.1. *Let E be a 2 uniformly smooth and convex real Banach space and let a mapping $A : E \to E^*$ be lipschitz strongly monotone such that $A^{-1}(0) \neq \emptyset$. For an arbitrary $(\{x_1\}, \{y_1\}) \in E$, we define the sequences $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{cases} y_n = x_n - \theta_n J^{-1}(Ax_n), & n \geq 1 \\ x_{n+1} = y_n - \lambda_n J^{-1}(Ay_n), & n \geq 1 \end{cases} \tag{3.1}$$

where λ_n and θ_n are positive real number and J is the duality mapping of E. Letting $(\lambda_n, \theta_n) \in (0, 1)$, then $\{x_n\}$ and $\{y_n\}$ converges strongly to ρ^*, a unique solution of the equation $Ax = 0$.

Proof. Letting $\rho^* = x^* \in E$ be the unique solution of $Ax = 0$. From inequality 2.4 in lemma 2.1 with 3.1, knowingly that $\|J^{-1}w\| = \|w\|$ for all $w \in E^*$, then we have the following estimates:

$$\begin{aligned} \|x_{n+1} - \rho^*\|^2 &= \|y_n - \rho^* - \lambda_n J^{-1}(Ay_n)\|^2 \\ &= \|\lambda_n J^{-1}(Ay_n)\|^2 - 2\langle y_n - \rho^*, J(\lambda_n J^{-1}(Ay_n))\rangle + d_2\|y_n - \rho^*\|^2 \\ &\leq \lambda_n^2\|(Ay_n)\|^2 - 2\lambda_n\langle y_n - \rho^*, Ay_n)\rangle + d_2\|y_n - \rho^*\|^2 \\ &\leq \lambda_n^2 L^2\|y_n - \rho^*\|^2 - 2\lambda_n k\|y_n - \rho^*\|^2 + d_2\|y_n - \rho^*\|^2 \\ &= \left(\lambda_n^2 L^2 - 2k\lambda_n + d_2\right)\|y_n - \rho^*\|^2 \end{aligned} \tag{3.2}$$

For the fact that $0 < \left(\lambda_n^2 L^2 - 2k\lambda_n + d_2\right) < 1$, we have the following

$$\|x_{n+1} - \rho^*\|^2 \leq \delta(\lambda_1)\|y_n - \rho^*\|^2 \tag{3.3}$$

where $\delta(\lambda_1) = \left(\lambda_n^2 L^2 - 2k\lambda_n + d_2\right)$.

Using 3.1 , Lipschitz property of A, with the same computational we have the following:

$$\begin{aligned}
\|y_n-\rho^*\|^2 &= \|x_n-\rho^*-\theta_n J^{-1}(Ax_n)\|^2 \\
&= \|\theta_n J^{-1}(Ax_n)\|^2 - 2\langle x_n-\rho^*, J(\theta_n J^{-1}(Ax_n))\rangle + d_2\|x_n-\rho^*\|^2 \\
&\leq \theta_n^2\|(Ax_n)\|^2 - 2\theta_n\langle x_n-\rho^*, Ax_n)\rangle + d_2\|x_n-\rho^*\|^2 \\
&\leq \theta_n^2 L^2\|x_n-\rho^*\|^2 - 2\theta_n k\|x_n-\rho^*\|^2 + d_2\|x_n-\rho^*\|^2 \\
&= \Big(\theta_n^2 L^2 - 2k\theta_n + d_2\Big)\|x_n-\rho^*\|^2 \qquad (3.4)
\end{aligned}$$

Again, with the fact that $0 < \Big(\theta_n^2 L^2 - 2k\theta_n + d_2\Big) < 1$, we have the following

$$\|y_n-\rho^*\|^2 \leq \delta(\lambda_2)\|x_n-\rho^*\|^2 \tag{3.5}$$

where $\delta(\lambda_2) = \Big(\theta_n^2 L^2 - 2k\theta_n + d_2\Big)$

Putting 3.5 in 3.3, we have the following

$$\|x_{n+1}-\rho^*\|^2 \leq \delta(\lambda_1)\delta(\lambda_2)\|x_n-\rho^*\|^2 \tag{3.6}$$

$$\|x_{n+1}-\rho^*\| \leq \sqrt{\delta(\lambda_1)\delta(\lambda_2)}\|x_n-\rho^*\| \tag{3.7}$$

$$\|x_{n+1}-\rho^*\| \leq \mu\|x_n-\rho^*\|$$

where $\mu = \sqrt{\delta(\lambda_1)\delta(\lambda_2)}$.

Therefore the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to ρ^*. This complete the proof. □

Corollary 3.1. *Let $E = L_p, 2 \leq p < \infty$, and $A : E \to E^*$ be a Lipschitz strongly monotone mapping such that $A^{-1}(0) \neq \emptyset$. For arbitrary $(x_1, y_1) \in E$, define the sequence $\{x_n\}$ and $\{y_n\}$ iteratively by*

$$\begin{cases} y_n = x_n - \theta_n J^{-1}(Ax_n), & n \geq 1 \\ x_{n+1} = y_n - \lambda_n J^{-1}(Ay_n), & n \geq 1 \end{cases} \tag{3.8}$$

where λ_n and θ_n are positive real number and J is the duality mapping of E. Letting $(\lambda_n, \theta_n) \in (0, 1)$, then x_n and y_n converges strongly to ρ^*, a unique solution of the equation $Ax = 0$.

Proof. Since $E = L_p$ spaces, $2 \leq p < \infty$, are $2-$uniformly smooth and convex real Banach spaces, then the proof follows from Theorem 3.1.

□

4. Convergence in L_p, ℓ_p or $W^{m,p}, 2 \leq p < \infty$

Theorem 4.1. *Let E be a 2 uniformly smooth and convex real Banach space either L_p, ℓ_p or $W^{m,p}, 2 \leq p < \infty$ with it dual E^*. Let a mapping $A : E \to E^*$ be lipschitz and strongly monotone such that $A^{-1}(0) \neq \emptyset$. For an arbitrary $(\{x_1\}, \{y_1\}) \in E$, we define the sequences $\{x_n\}$ and $\{y_n\}$ by 3.1 converges strongly to ρ^*, a unique solution of the equation $Ax = 0$.*

Proof. Since L_p, ℓ_p *or* $W^{m,p}, 2 \leq p < \infty$ are $2-$ uniformly smooth Banach spaces, then with the same computation in 3.1, the proof follows.

□

Corollary 4.1. *Let E be a Banach space either L_p, ℓ_p or $W^{m,p}, 2 \leq p < \infty$ with it dual E^*. Let a mapping $A : E \to E^*$ be lipschitz and strongly monotone such that $A^{-1}(0) \neq \emptyset$. For an arbitrary $(\{x_1\}, \{y_1\}) \in E$, we define the sequences $\{x_n\}$ and $\{y_n\}$ by 3.1 converges strongly to ρ^*, a unique solution of the equation $Ax = 0$.*

Proof. Since L_p, ℓ_p *or* $W^{m,p}, 2 \leq p < \infty$ are $2-$ uniformly smooth Banach spaces, then from Theorem 4.1 with the same computation in 3.1, the proof follows.

□

5. Application to Convex minimization problem

Now, we present a convex minimization problem for a convex function $\nabla : E \to \mathbb{R}$.
The following results are well known.

Remark 2. Let $\Delta : E \to \mathbb{R}$ be a differentiable convex function and $\rho^* \in E$, then the point ρ^* is a minimizer of ∇ on E if and only if $d\nabla(\rho^*) = 0$.

Definition 5.1. A function $\nabla : E \to \mathbb{R}$ is said to be strongly convex if there exists $\gamma > 0$ such that the following condition holds:

$$\nabla(\beta x + (1-\beta)y) \leq \beta \nabla x + (1-\beta)\nabla y - \gamma \|x-y\|^2 \tag{5.1}$$

for every $x, y \in E$ with $x \neq y$ and $\beta \in (0,1)$,

Lemma 5.2. *Let E be normed linear space and $\nabla : E \to \mathbb{R}$ a convex differentiable function. Suppose that ∇ is strongly convex. Then the differential map $d\nabla : E \to E^*$ is strongly monotone, i.e., there exists $k > 0$ such that*

$$\langle d\nabla x - d\nabla y, x - y\rangle \geq k\|x-y\|^2 \; \forall x, y \in E. \tag{5.2}$$

Now we present the following result.

Theorem 5.3. *Let $d\nabla : E^* \to E$ be a L-Lipschitz continuous and strongly monotone mapping such that $d\nabla^{-1}(0) \neq \emptyset$. Let $E = L_p, p \geq 2$ and $\nabla : E \to \mathbb{R}$ be a differentiable, strongly convex real-valued function. For given $x_1, y_1 \in E$, define the sequence $\{x_n\}$ and $\{y_n\}$ as follows:*

$$\begin{cases} y_n = x_n - \theta_n d\nabla x_n), & n \geq 1 \\ x_{n+1} = y_n - \lambda_n d\nabla y_n), & n \geq 1 \end{cases} \tag{5.3}$$

where the sequences $\{\lambda_n\}$ and $\{\theta_n\}$, are in the interval $[0,1]$. Then ∇ has a unique minimizer $\rho^* \in E$ such that if $\Big(\lambda_n, \theta_n\Big) \in [0,1]$, the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to ρ^*.

Proof. From Remark 2 it follows that ∇ has a unique minimizer ρ^* and is obtained by $d\nabla(\rho^*) = 0$. From Lemma 5.2 and using the fact that the differential mapping $d\nabla : E \to E^*$ is Lipschitz, considering the result of Theorem 3.1, we can complete the proof. $\square$

6. The proposed algorithm in $L_p(\Omega)$

Now, From [14], the duality mapping J is known precisely in $L_p(\Omega)$ for $1 < p < \infty$ by

$$Jv = \|v\|_{L_p}^{2-p}|v|^{p-2}v, \forall v \in L_p(\Omega)$$

and if $L_p(\Omega)$ is reflexive, smooth and strictly convex real Banach space, for $1 < p < \infty$, then the duality mapping J is surjective, one-to-one and its inverse J^{-1} is given by

$$Ju = \|\|\|_{L}^{2-q}|u|^{q-2}u, \forall u \in L^q(\Omega)$$

with $\frac{1}{p} + \frac{1}{q} = 1$
Now from 3.1, we defined $x_1, y_1 \in L^q(\Omega)$

$$\begin{cases} y_n = x_n - \theta_n\|Ax_n\|_{L_q}^{2-q}|Ax_n|_{L_q}^{2-q}Ax_n, & n \geq 1 \\ x_{n+1} = y_n - \lambda_n\|Ay_n\|_{L_q}^{2-q}|Ay_n|_{L_q}^{2-q}Ay_n, & n \geq 1 \end{cases} \tag{6.1}$$

Conclusion

In this paper, we proposed and analyzed the strong convergence theorem of two step size of the new Krasnoselskii-type algorithm introduced by Sene et al. [7] and prove a strong convergence theorem to approximate the unique zero of a Lipschitz strongly monotone mapping $2-$uniformly smooth and $p-$uniformly convex real Banach space for $p \geq 2$. This class of Banach spaces contains all Lp-spaces, $2 \leq p < \infty$ and Sobolev space. Then we apply our results to the convex minimization problem. We also complemented and generalized previous worked been done under this setting.

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