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Seventh Order Derivative-Free Methods for Non-differentiable Operator Equations

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Abstract. In nonlinear problems where function's derivatives are difficult or expensive to compute, derivative-free iterative methods are good options to find the numerical solution. One of the important parts in the development of such methods is to study their convergence properties. In this paper, we review the concepts of local and semi-local convergence for a derivative-free method for nonlinear equations. In the earlier study of the considered method, the convergence analysis was carried out assuming the existence of higher order derivatives while no derivative is used in the method. Such assumptions certainly restrict its applicability. The present study further provides the estimate of convergence radius and bounds on the error for the given method. Thus, the applicability of the method clearly seems to be extended over the wider class of problems. We also review some of the recent developments in this area. The results presented in this paper can be useful for practitioners and researchers in developing and analyzing derivative-free numerical algorithms.

1. Introduction

There are several numerical methods such as Newton's method, Broyden's method, secant method and Steffensen's method [3–11, 13, 14, 17] that can be used to approximate x^* of the equation

$$F(x) = 0, \tag{1.1}$$

for $F : \Omega \subset Z \to Z$, F is a continuous operator, acting between Banach space Z and itself. Newton's method is a popular iterative method used to find the roots of a nonlinear equation.

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Iterative solution methods are commonly used when it is not possible to obtain the solution x^* in closed or analytical form. Instead, these methods generate a sequence of approximate solutions that converge towards the true solution x^*.

Steffensen's method [5,9] defined for each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = x_n - B^{-1}F(x_n), \tag{1.2}$$

where $B = B_n = [u_n, x_n; F]$ and $u_n = x_n + F(x_n)$, has been used extensively to generate such a sequence converging quadratically to x^*.

Many iterative approaches have been developed to improve efficiency and order convergence (see [1,2,15,16]). An approach established in [16] that is defined for $x_0 \in \Omega$ by

$$\begin{aligned} u_n =& \ x_n + F(x_n), \quad v_n = x_n - F(x_n), \quad D = D_n = [u_n, v_n; F], \\ y_n =& \ x_n - D^{-1}F(x_n), \\ z_n =& \ y_n - (3I - 2D^{-1}[y_n, x_n; F])D^{-1}F(y_n), \\ x_{n+1} =& \ z_n - \left(\frac{13}{4}I - D^{-1}[z_n, y_n; F]\left(\frac{7}{2}I - \frac{5}{4}D^{-1}[z_n, y_n; F]\right)\right)D^{-1}F(z_n), \end{aligned} \tag{1.3}$$

has received significant attention in this paper. The convergence order seven is shown in [16], when $Z = \mathbb{R}^m$ etc. using assumption on F^i, $i = 1, 2, \dots, 8$ not present in the method, significantly reducing its applicability although it may converge.

Consider the function

$$F(t) = \begin{cases} 7t^3\log(t) + 5t^5 - 5t^4, & t \neq 0 \\ 0, & t = 0 \end{cases} \tag{1.4}$$

Then, in any neighborhood of 0 and 1, say F''' is unbounded. Hence, the results in [16] cannot assure convergence to $t^* = 1$. But the method converges.

In this article we study convergence of the method (1.3) that includes mainly the local and semi-local convergence (not provided in [16]).

Local convergence analysis uses information about the actual solution to determine the rate and radius of convergence of the method. This typically involves estimating the size of the region around the true solution where the method is guaranteed to converge. This type of analysis also usually involves deriving upper bounds on the error norms, which provide an estimate of how close the iterates of the method are to the true solution.

In contrast, in semi-local convergence analysis, the convergence behavior of the method is studied using information from the initial point, typically by deriving sufficient conditions that guarantee convergence of the method. This analysis is usually carried out without any knowledge of the actual solution of the problem.

Generalized Lipschitz-type conditions are often used in both semi-local and local convergence analysis. These conditions involve bounding the difference between the iterates of the method and the true solution using a Lipschitz constant or a related quantity. These conditions can be used to derive sufficient conditions for convergence, as well as to estimate the rate and radius of convergence of the method.

It is crucial to examine how technique (1.3) converges in both the local (Section 2) and the semi-local (Section 3) cases. Moreover, our approach gives a prior error estimates and isolation of the solution results not provided before and in Banach space. This approach also enables a comparison of the convergence criteria of method. If the approach is examined separately, the new convergence criteria may be weaker than those that have been provided. The numerical examples are included in Section 4, and the conclusions are discussed in Section 5.

2. Local convergence

Some real functions assist in the local analysis of the method. Set $T = [0, +\infty)$. Assume:

(H_1) There exist continuous as well as nondecreasing functions (CN) $f_1 : T \to T$, $f_2 : T \to T$, and $w_0 : T \times T \to \mathbb{R}$ so that the equation

$$w_0(f_1(t), f_2(t)) - 1 = 0$$

admits a smallest solution (SS) denoted by $\delta \in T - \{0\}$. Let $T_0 = [0, \delta)$.

(H_2) There exist (CN) functions $w : T_0 \to T$, $w_1 : T_0 \times T_0 \times T_0 \to T$, and $w_2 : T_0 \times T_0 \times T_0 \to T$ such that the equations

$$h_i(t) - 1 = 0, \qquad i = 1, 2, 3$$

have (SS) solutions denoted by $\delta_i \in T_0 - \{0\}$, provided that

$$h_1(t) = \frac{w_1(f_1(t), f_2(t), t)}{1 - w_0(f_1(t), f_2(t))},$$

$$h_2(t) = \Big[\frac{w_1(h_1(t)t, f_1(t), f_2(t))}{1 - w_0(f_1(t), f_2(t))} + \frac{2w_2(t, h_1(t)t, f_1(t), f_2(t))(1 + w(t))}{(1 - w_0(f_1(t), f_2(t)))^2}\Big]h_1(t),$$

$$h_0(t) = \frac{1}{4}\Big[5\Big(\frac{w_2(h_1(t)t, h_2(t)t, f_1(t), f_2(t))}{1 - w_0(f_1(t), f_2(t))}\Big)^2 + 4\frac{w_2(h_1(t)t, h_2(t)t, f_1(t), f_2(t))}{1 - w_0(f_1(t), f_2(t))}\Big],$$

$$h_3(t) = \Big[\frac{w_1(f_1(t), f_2(t), h_2(t)t)}{1 - w_0(f_1(t), f_2(t))} + \frac{h_0(t)(1 + w(h_2(t)t))}{1 - w_0(f_1(t), f_2(t))}\Big]h_2(t).$$

Consider, the parameter δ^* given as

$$\delta^* = \min\{\delta_i\}. \tag{2.5}$$

Let $T_1 = [0, \delta^*)$. These definitions imply

$$0 \le w_0(f_1(t), f_2(t)) < 1 \tag{2.6}$$

and

$$0 \le h_i(t) < 1, \tag{2.7}$$

for all $t \in T_1$.

Let $B(\bar{x}, r)$, $\bar{B}[\bar{x}, r]$ abbreviate open and closed balls in S_1, respectively so that the center is $\bar{x}$ and the radius is some $r > 0$. The preceding real functions are associated to the divided difference $[.,.;F]$ as:

(H_3) There exists an invertible operator $L \in \mathfrak{L}(Z)$ so that for each $x \in \Omega$, $u = x + F(x)$, $v = x - F(x)$

$$||L^{-1}([u, v; F] - L)|| \leq w_0(||u - x^*||, ||v - x^*||),$$
$$||u - x^*|| \leq f_1(||x - x^*||),$$
$$||v - x^*|| \leq f_2(||x - x^*||).$$

Let $B_0 = B(x^*, \delta)$.

(H_4)

$$||L^{-1}([x, x^*; F] - L)|| \leq w(||x - x^*||),$$
$$||L^{-1}([x, y; F] - [z, x^*; F])|| \leq w_1(||x - x^*||, ||y - x^*||, ||z - x^*||)$$

and

$$||L^{-1}([u, v; F] - [y, x; F])|| \leq w_2(||x - x^*||, ||y - x^*||, ||u - x^*||, ||v - x^*||),$$

for each $x, y, z, u, v \in B_0$.

(H_5) $B[x^*, \delta^*] \subset \Omega$.

The local analysis is based on the conditions $(H_1) - (H_5)$ under the preceding notations.

Theorem 2.1 *Assume the conditions $(H_1) - (H_5)$ are validated. If $x_0 \in B(x^*, \delta^*) - \{x^*\}$, then the following items are valid*

$$\{x_n\} \subset B(x^*, \delta^*), \tag{2.8}$$

$$\|y_n - x^*\| \leq h_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|\|x_n - x^*\| < \delta^*, \tag{2.9}$$

$$\|z_n - x^*\| \leq h_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|\|x_n - x^*\|, \tag{2.10}$$

$$\|x_{n+1} - x^*\| \leq h_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|\|x_n - x^*\| \tag{2.11}$$

and the sequence $\{x_n\}$ is convergent to x^*.

Proof. These items are shown by mathematical induction. By the condition (H_3), estimate (2.7) for $u_0 = x_0 + F(x_0)$, $v_0 = x_0 - F(x_0)$ it follows

$$\begin{aligned} \|L^{-1}([u_0, v_0; F] - L)\| &\leq w_0(\|u_0 - x^*\|, \|v_0 - x^*\|) \leq w_0(f_1(\|x_0 - x^*\|), f_2(\|x_0 - x^*\|)) \\ &\leq w_0(f_1(u), f_2(v)) < 1. \end{aligned} \tag{2.12}$$

Then, the existence of $[u_0, v_0; F]^{-1}$ is assured by the estimate (2.12) and the perturbation Lemma on linear operators with inverses attributed to Banach [6]. We also have

$$\|[u_0, v_0; F]^{-1}L\| \leq \frac{1}{1 - w_0(f_1(\|u_0 - x^*\|), f_2(\|u_0 - x^*\|)}. \tag{2.13}$$

Thus, the iterate y_0 is well defined and

$$\begin{aligned} y_0 - x^* = & x_0 - x^* - [u_0, v_0; F]^{-1}F(x_0). \\ & [u_0, v_0; F]^{-1}([u_0, v_0; F] - [x_0, x^*; F])(x_0 - x^*). \end{aligned} \tag{2.14}$$

Then, the conditions (H_3), (H_4), (2.5), (2.7) (for $i = 1$), (2.13) and (2.14) give in turn

$$\begin{aligned} \|y_0 - x^*\| \leq & \frac{w_1(\|u_0 - x^*\|, \|v_0 - x^*\|, \|x_0 - x^*\|)\|x_0 - x^*\|}{1 - w_0(f_1(\|x_0 - x^*\|), f_2(\|x_0 - x^*\|))} \\ \leq & h_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \delta^*. \end{aligned} \tag{2.15}$$

Hence, the iterate $y_0 \in B(x^*, \delta^*)$ and the item (2.9) is validated for $n = 0$.

Notice that the iterates z_0 and x_1 are also well defined by the second and the third substep of the method (1.3). In particular, we get

$$\begin{aligned} z_0 - x^* = & y_0 - x^* - D^{-1}F(y_0) + 2D^{-1}(D - [y_n, x_n; F])D^{-1}F(y_n) \\ = & D^{-1}(D - [y_n, x^*; F])(y_n - x^*) + 2D^{-1}([u_0, v_0; F] - [y_0, x_0; F])D^{-1}F(y_0). \end{aligned} \tag{2.16}$$

But we can write by the first substep that

$$F(y_0) = F(y_0) - F(x^*) = [y_0, x^*; F](y_0 - x^*),$$

so by (H_4)

$$\begin{aligned} \|L^{-1}F(y_0)\| = & \|L^{-1}([y_0, x^*; F] - L + L)(y_0 - x^*)\| \\ \leq & (1 + w(\|y_0 - x^*\|))\|y_0 - x^*\|. \end{aligned} \tag{2.17}$$

Consequently, (2.5), (2.7) (for $i = 2$), (H_4), (2.13), (2.16) and (2.17) imply

$$\begin{aligned} \|z_0 - x^*\| \leq & \Big[\frac{w_1(\|y_0 - x^*\|, \|u_0 - x^*\|, \|v_0 - x^*\|)}{1 - w_0(f_1(\|x_0 - x^*\|), f_2(\|x_0 - x^*\|))} \\ & + \frac{2w_2(\|x_0 - x^*\|, \|y_0 - x^*\|, \|u_0 - x^*\|, \|v_0 - x^*\|)(1 + w(\|y_0 - x^*\|))}{(1 - w_0(f_1(\|x_0 - x^*\|), f_2(\|x_0 - x^*\|)))^2}\Big]\|y_0 - x^*\|, \\ \leq & h_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \end{aligned} \tag{2.18}$$

Thus, the iterate $z_0 \in B(x^*, \delta^*)$ and the item (2.10) is validated for $n = 0$. Moreover, the third substep gives

$$\begin{aligned} x_1 - x^* = & z_0 - x^* - D^{-1}F(z_0) - AD^{-1}F(z_0) \\ = & D^{-1}(D - [z_0, x^*; F])(z_0 - x^*) - AD^{-1}F(z_0), \end{aligned} \tag{2.19}$$

where

$$A = -\Big[\frac{9}{4}I - \frac{7}{2}D^{-1}[z_n, y_n; F] + \frac{5}{4}(D^{-1}[z_n, y_n; F])^2\Big]$$
$$= -\frac{1}{4}\big(5(D^{-1}([z_n, y_n; F] - [u_n, v_n; F]))^2 - 4D^{-1}([z_n, y_n; F] - [u_n, v_n; F])\big).$$

Therefore,

$$\|A\| \le \frac{1}{4}\Big[5\Big(\frac{w_2(\|y_0 - x^*\|, \|z_0 - x^*\|, \|u_0 - x^*\|, \|v_0 - x^*\|)}{1 - w_0(f_1(\|x_0 - x^*\|), f_2(\|x_0 - x^*\|))}\Big)^2 + 4\frac{w_2(\|y_0 - x^*\|, \|z_0 - x^*\|, \|u_0 - x^*\|, \|v_0 - x^*\|)}{1 - w_0(f_1(\|x_0 - x^*\|), f_2(\|x_0 - x^*\|))}\Big] = h_0. \tag{2.20}$$

Then, by (2.5), (2.7) (for $i = 3$), (H_4), (2.13) and (2.18)-(2.20)

$$\|x_1 - x^*\| \le \Big[\frac{w_1(\|u_0 - x^*\|, \|v_0 - x^*\|, \|z_0 - x^*\|)}{1 - w_0(f_1(\|x_0 - x^*\|), f_2(\|x_0 - x^*\|))} + \frac{h_0(1 + w(\|z_0 - x^*\|))}{1 - w_0(f_1(\|x_0 - x^*\|), f_2(\|x_0 - x^*\|))}\Big]\|z_0 - x^*\|$$
$$\le h_3(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\|. \tag{2.21}$$

Hence, the items (2.8) and (2.11) are validated for $n = 0$ and the iterate $x_1 \in B(x^*, \delta^*)$. If the preceding calculations are repeated with x_m, y_m, x_{m+1}, replacing x_0, y_0, x_1, respectively, the induction for the items (2.8)-(2.11) is terminated. Furthermore, from estimation

$$\|x_{m+1} - x^*\| \le \mu\|x_m - x^*\| < \|x_m - x^*\|, \tag{2.22}$$

where $\mu = h_3(\|x_0 - x^*\|) \in [0, 1)$, we conclude that $\lim_{m\to\infty} x_m = x^*$ and the iterate $x_{m+1} \in B(x^*, \delta^*)$. □

Remark 2.2 The second and third hypotheses in (H_3) are left as uncluttered as possible. Some possible choices for the functions f_1 and f_2 are specified.

$$u_n - x^* = x_n - x^* + F(x_n) = (I + [x_n, x^*; F])(x_n - x^*)$$
$$= (I + L + LL^{-1}([x_n, x^*; F] - L))(x_n - x^*),$$

so

$$\|u_n - x^*\| \le \big(\|I + L\| + (\|L\|w(\|(x_n - x^*)\|))\big)\|(x_n - x^*)\|.$$

Thus, we can choose

$$f_1(t) = \big(\|I + L\| + \|L\|w(t)\big)t.$$

Notice also that we can set $w(t) = w_0(0, t)$.

Similarly, we define

$$f_2(t) = \big(\|I - L\| + \|L\|w(t)\big)t.$$

In view of the above the second and third conditions in (H_3) can be dropped if (H_5) is replaced by

$(H_5)'$ $B[x^*, \bar{\delta}] \subset \Omega$, where $\bar{\delta} = \max\{\delta^*, f_1(\delta^*)\delta^*, f_2(\delta^*)\delta^*\}$.

Possible choices for L are

$$L = F'(x^*) \quad \text{(The differentiable case)}$$

or

$$L = [.,.;F] \quad \text{(The non-differentiable case).}$$

In practice L should be chosen to optimize the results.

Next, the point x^* is shown to be the only solution of the equation $F(x) = 0$ in a certain set.

Preposition 2.3 *Assume: There exists a solution $x_1^* \in B(x^*, \delta_4)$ of the equation $F(x) = 0$ for some $\delta_4 > 0$; The first assumption in (H_3) is validated on the ball $B(x^*, \delta_4)$ and there exists $\delta_5 \geq \delta_4$ so that*

$$w(\delta_5) < 1.$$

Let $B_1 = \Omega \cap B[x^*, \delta_5]$. Then, the only solution of the equation $F(x) = 0$ in the set B_1 is x^*.

Proof. Let $Q = [x^*, x_1^*]$. By the assumption it follows

$$\begin{aligned} \|L^{-1}(Q - L)\| &\leq w(\|x_1^* - x^*\|) \\ &\leq w(\delta_5) < 1, \end{aligned}$$

thus $Q^{-1} \in \mathfrak{L}(Z)$ and consequently from the approximation

$$x_1^* - x^* = Q^{-1}(F(x_1^*) - F(x^*)) = Q^{-1}(0) = 0,$$

it is concluded that $x_1^* = x^*$. □

Clearly, we can choose $\delta_4 = \delta^*$.

3. Semi-local Analysis

The role of x^* is exchanged by x_0. But there are some more differences.

Assume:

(C_1) There exist (CN) functions $g_1 : T_0 \to T$, $g_2 : T_0 \to T$ and $w_0 : T_0 \times T_0 \to T$ so that the equation

$$w_0(g_1(t), g_2(t)) - 1 = 0$$

has a (SS) denoted by $r_0 \in T_0 - \{0\}$. Set $T_3 = [0, r_0)$.

Define the scaler sequence $\{a_n\}$ for $a_0 = 0$, $b_0 \in [0, r_0)$ and some (CN) functions $g_1 : T_3 \to T$, $g_2 : T_3 \to T$, $w_2 : T_3 \times T_3 \times T_3 \times T_3 \to T$ by

$$c_n = b_n + \Big[\frac{w_2(a_n, b_n, f_1(a_n), f_2(a_n))}{1 - w_0(g_1(a_n), g_2(a_n))} + \frac{2w_2(a_n, b_n, f_1(an), f_2(an))}{(1 - w_0(g_1(a_n), g_2(a_n)))^2}\Big](\beta_n - a_n), \tag{3.23}$$

$$\beta_n = \frac{1}{4}\Big[5\Big(\frac{w_2(a_n, b_n, f_1(a_n), f_2(a_n))}{1 - w_0(g_1(a_n), g_2(a_n))}\Big)^2 + 4\Big(\frac{w_2(a_n, b_n, f_1(a_n), f_2(a_n))}{1 - w_0(g_1(a_n), g_2(a_n))}\Big)^2\Big],$$

$$\gamma_n = w_2(a_n, b_n, g_1(a_n), g_2(a_n))(b_n - a_n),$$

$$a_{n+1} = c_n + \frac{(1 + w_0(b_n, c_n))(c_n - b_n)\beta_n + \gamma_n}{1 - w_0(g_1(a_n), g_2(a_n))},$$

$$\delta_{n+1} = (1 + v_0(a_n, b_n))(a_{n+1} - a_n) + (1 + w_0(g_1(an), g_2(an)))(b_n - a_n)$$

and

$$b_{n+1} = a_{n+1} + \frac{\delta_{n+1}}{1 - w_0(g_1(a_{n+1}), g_2(a_{n+1}))}.$$

Next, general convergence conditions are developed.

Lemma 3.1 *Assume there exists* $\mu_1 \in [0, r_0)$ *such that for each* $n = 0, 1, 2, \ldots,$

(C_2) $w_0(g_1(a_n), g_2(a_n)) < 1$ *and* $a_n \leq \mu_1$.

Then, the following items hold

$$0 \leq a_n \leq b_n \leq c_n \leq a_{n+1} \leq \mu_1$$

and there exists $a^* \in (0, \mu_1]$ *such that* $\lim_{n\to\infty} a_n = a^*$.

Proof. The conclusions follow immediately by the formula (3.23) and the condition (C_2). □

Notice that the limit a^* is unique, since it is the unique least upper bound of the sequence $\{a_n\}$.

(C_3) There exist an invertible operator L and a point $x_0 \in \Omega$ such that for each $x, y \in \Omega$, $u = x + F(x)$, $v = x - F(x)$

$$\begin{aligned} \|L^{-1}([u, v; F] - L)\| &\leq w_0(\|u - x_0\|, \|v - x_0\|), \\ \|u - x_0\| &\leq g_1(\|x - x_0\|), \\ \|v - x_0\| &\leq g_2(\|x - x_0\|) \end{aligned}$$

and $w_0(g_1(\|F(x_0)\|), g_2(\|F(x_0)\|)) < 1$.

The existence of $[u_0, v_0; F]^{-1}$ is guaranteed, by the Banach lemma and since

$$L^{-1}\|[u_0, v_0; F] - L\| \leq w_0(\|u_0 - x_0\|, \|v_0 - x_0\|) < 1.$$

(C_4) $\|[u_0, v_0; F]^{-1}F(x_0)\| \leq b_0$. Let $B_2 = B(x_0, \mu_0)$.

(C_5) $\|L^{-1}([x, y; F] - [u, v; F])\| \leq w_2(\|x - x_0\|, \|y - x_0\|, \|u - x_0\|, \|v - x_0\|)$ for each $x, y, u, v \in B_2$.

(C_6) $B[x_0, a^*] \subset \Omega$.

Next, the semi-local convergence is provided for the method (1.3).

Theorem 3.2 *Assume the conditions $(C_1)-(C_6)$ hold. Then, the following items hold*

$$\{x_n\} \subset B(x_0, a^*), \tag{3.24}$$

$$\|y_n - x_n\| \leq b_n - a_n, \tag{3.25}$$

$$\|z_n - y_n\| \leq c_n - b_n, \tag{3.26}$$

$$\|x_{n+1} - z_n\| \leq a_{n+1} - c_n, \tag{3.27}$$

and there exists a solution x^* of the equation $F(x) = 0$ such that

$$\|x^* - x_n\| \leq a^* - a_n. \tag{3.28}$$

Proof. The items (3.24)-(3.27) are shown by induction. Notice that the iterates y_0, z_0, x_1 exists by the invertibility of $[u_0, v_0; F]$ and the method (1.3). The estimate (3.25) is validated for $n = 0$, since by the condition (C_4)

$$\|y_0 - x_0\| = \|[u_0, v_0; F]^{-1}F(x_0)\| \leq b_0 = b_0 - a_0 \leq a^* \tag{3.29}$$

and the iterate $y_0 \in B(x_0, a^*)$.

Then, as in the local convergence case but using x_0, (C) instead of x^*, (H), we obtain from

$$F(y_n) = F(y_n) - F(x_n) - D(y_n - x_n) = ([y_n, x_n; F] - D)(y_n - x_n),$$

so

$$\|L^{-1}F(y_n)\| \leq w_2(\|x_n - x_0\|, \|y_n - x_0\|, \|u_n - x_0\|, \|v_n - x_0\|). \tag{3.30}$$

Hence, by the second substep

$$z_n - y_n = -D^{-1}F(y_n) - 2D^{-1}(D - [y_n, x_n; F])^{-1}D^{-1}F(y_n),$$

and

$$\begin{aligned}\|z_n - y_n\| \leq \Big[&\frac{w_2(\|x_n - x_0\|, \|y_n - x_0\|, \|u_n - x_0\|, \|v_n - x_0\|)}{1 - w_0(f_1(\|x_n - x_0\|), f_2(\|x_n - x_0\|))} \\ &+ 2\Big(\frac{w_2(\|x_n - x_0\|, \|y_n - x_0\|, \|u_n - x_0\|, \|v_n - x_0\|)}{1 - w_0(f_1(\|x_n - x_0\|), f_2(\|x_n - x_0\|))}\Big)^2\Big]\|y_n - x_n\| \leq a_n - b_n\end{aligned} \tag{3.31}$$

and

$$\|z_n - x_0\| \leq \|z_n - y_n\| + \|y_n - x_0\| \leq a_n - b_n + b_n - a_0 = c_n < a^*,$$

thus the item (3.26) holds and the iterate $z_0 \in B(x_0, a^*)$.

Moreover, by the third substep

$$x_{n+1} - z_n = -BD^{-1}F(z_n), \tag{3.32}$$

where

$$B = -\frac{1}{4}\big(13I - 14D^{-1}[z_n, y_n; F] + 5(D^{-1}[z_n, y_n; F])^2\big)$$
$$= -\frac{1}{4}\big(5(D^{-1}[z_n, y_n; F] - I)^2 - 4(D^{-1}[z_n, y_n; F] - I) + 4I\big),$$

thus

$$\|B\| \leq \frac{1}{4}\Big(5\Big(\frac{w_2(\|x_n - x_0\|, \|y_n - x_0\|, \|u_n - x_0\|, \|v_n - x_0\|)}{1 - w_0(f_1(\|x_n - x_0\|), f_2(\|x_n - x_0\|))}\Big)^2 + 4\Big(\frac{w_2(\|x_n - x_0\|, \|y_n - x_0\|, \|u_n - x_0\|, \|v_n - x_0\|)}{1 - w_0(f_1(\|x_n - x_0\|), f_2(\|x_n - x_0\|))}\Big) + 4\Big) = \bar{\beta} < \beta_n.$$

Consequently, we have

$$\|x_{n+1} - z_n\| \leq \frac{\bar{\beta}_n(1 + w_0(\|y_n - x_0\|, \|z_n - x_0\|))\|z_n - y_n\|}{1 - w_0(f_1(\|x_n - x_0\|), f_2(\|x_n - x_0\|))}$$
$$\leq \frac{\beta_n(1 + w_0(\beta_n, c_n))(c_n - b_n)}{1 - w_0(f_1(a_n), f_2(a_n))} = a_{n+1} - c_n < \beta_n \tag{3.33}$$

and

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - z_n\| + \|z_n - x_0\| \leq a_{n+1} - c_n + c_n - a_0 = a_{n+1} < a^*.$$

Hence, the item (3.27) is validated and iterate $x_{n+1} \in B(x_0, a^*)$. □

Remark 3.3 As in the local case the functions g_1 and g_2 can be expressed in terms of the rest of the conditions.

Assume that there exists a CN function $\varphi_1 : T \to \mathbb{R}$ such that for each $x \in \Omega$

$$\|L^{-1}([x, x_0; F] - L)\| \leq \varphi_1(\|x - x_0\|).$$

Then, from the estimate

$$u_n - x_0 = x_n - x_0 + F(x_n) - F(x_0) + F(x_)$$
$$= (I + L + LL^{-1}([x_n, x_0; F] - L))(x_n - x_0) + F(x_0),$$

so we can choose

$$g_1(t) = (\|1 + L\| + \|L\|\varphi_1(t))t + \|F(x_0)\|$$

and similarly

$$g_2(t) = (\|1 - L\| + \|L\|\varphi_1(t))t + \|F(x_0)\|.$$

Under these choices of g_1 and g_2

$$\|u - x_0\| = g_1(\|x - x_0\|),$$
$$\|v - x_0\| = g_2(\|x - x_0\|)$$

and the second and third conditions in (H_3) can be dropped.

Possible choices for L are

$$L = F'(x_0) \quad \text{(The differentiable case)}$$

or

$$L = [.,.;F] \quad \text{(The non-differentiable case)}.$$

The condition (C_6) can be replaced by

$(C_6)'$ $B[x_0, \bar{a}] \subset \Omega$, where $\bar{a} = \max\{a^*, g_1(a^*), g_2(a^*)\}$.

A uniqueness of the solution set is specified.

Proposition 3.4 *Assume: There exists a solution $d \in B(x_0, \delta_6)$ of the equation $F(x) = 0$ for some $\delta_6 > 0$; the first condition in (H_3) holds in the ball $B(x_0, \delta_6)$ and there exists $\delta_7 \geq \delta_6$ such that*

$$w_0(\delta_6, \delta_7) < 1.$$

Let $B_3 = B[x_0, \delta_7] \cap \Omega$. Then, the point d is the only solution of the equation $F(x) = 0$ in the set B_3.

Proof. Let $d_1 \in B_3$ be such that $F(d_1) = 0$. Define the divided difference $[d, d_1; F]$. Then,

$$\begin{aligned} \|L^{-1}([d, d_1; F] - L)\| &\leq w_0(\|d - x_0\|, \|d_1 - x_0\|) \\ &\leq w_0(\delta_6, \delta_7) < 1, \end{aligned}$$

thus $d_1 = d$. □

Remark 3.5 If all the conditions $(C_1) - (C_6)$ hold, then set $d = x^*$ and $\delta_6 = a^*$.

4. Numerical Tests

In order to validate the theoretical deductions, we take into account the following numerical examples to estimate the real parameters defined in the preceding sections:

Example 1. Let $Z = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $\Omega = B(\xi^*, 1)$ with $\xi^* = (0, 0, 0)^T$. Define the mapping F for $\xi = (\xi_1, \xi_2, \xi_3)^T$, $\xi_i \in \mathbb{R}$ by

$$F(\xi) = \left(\xi_1, e^{\xi_2} - 1, \frac{(e-1)}{2}\xi_3^2 + \xi_3\right)^T.$$

This definition gives that the F' of the mapping F is the Jacobian matrix

$$F'(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\xi_2} & 0 \\ 0 & 0 & (e-1)\xi_3 + 1 \end{bmatrix}.$$

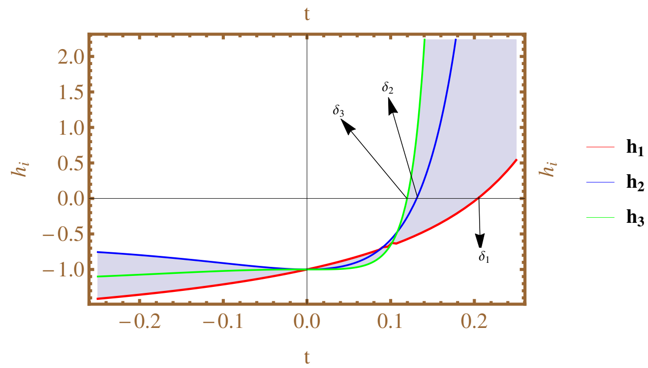
Notice that $F(\xi^*) = O$ and $F'(\xi^*) = I$. Then, the conditions $(H_1) - (H_4)$ are validated if

$$\begin{aligned} w_0(t_1, t_2) &= \frac{1}{2}(e-1)(t_1 + t_2), \\ w(t) &= \frac{1}{2}(e-1)t, \\ w_1(t_1, t_2, t_3) &= \frac{1}{2}(e-1)(t_1 + t_2 + t_3), \\ w_2(t_1, t_2, t_3, t_4) &= \frac{1}{2}(e-1)(t_1 + t_2 + t_3 + t_4) \end{aligned}$$

and the functions f_1 and f_2 are given in Remark 2.2. Then, the radius δ^* using (2.5) is

$$\delta_1 = 0.20415,\ \delta_2 = 0.13109,\ \delta_3 = 0.11960 \ \text{ and } \ \delta^* = 0.11960.$$

Figure 1. Graph of radius of convergence of example 1.



Example 2. Let $Z = C[0,1]$ be the space of continuous functions defined in $[0,1]$ and $\Omega = \bar{B}(l^*, 1)$. Consider the integral equation of the mixed Hammerstein-type [6, 12] by

$$l(d) = \int_0^1 T(d,\omega)\Big(l(\omega)^{3/2} + \frac{l(\omega)^2}{2}\Big)d\omega, \tag{4.34}$$

$$T(d,\omega) = \begin{cases} (1-d)\omega, & \omega \le d, \\ d(1-\omega), & d \le \omega. \end{cases}$$

Notice that $l^*(d) = 0$. Define $H : \Omega \subseteq [0,1] \to C[0,1]$ as

$$H(l)(d) = l(d) - \int_0^1 T(d,\omega)\Big(l(\omega)^{3/2} + \frac{l(\omega)^2}{2}\Big)d\omega.$$

The derivative H' is given by

$$H'(l)q(d) = q(d) - \int_0^1 T(d,\omega)\Big(\frac{3}{2}\ l(\omega)^{1/2} + l(\omega)\Big)d\omega,$$

since $H'(l^*(d)) = 1$, it follows

$$\|H'(\alpha)^{-1}(H'(l) - H'(q))\| \le \frac{5}{16}\|l - q\|. \tag{4.35}$$

In (4.35), switch q by l_0

$$\|H'(\alpha)^{-1}(H'(l) - H'(l_0))\| \le \frac{5}{16}\|l - l_0\|.$$

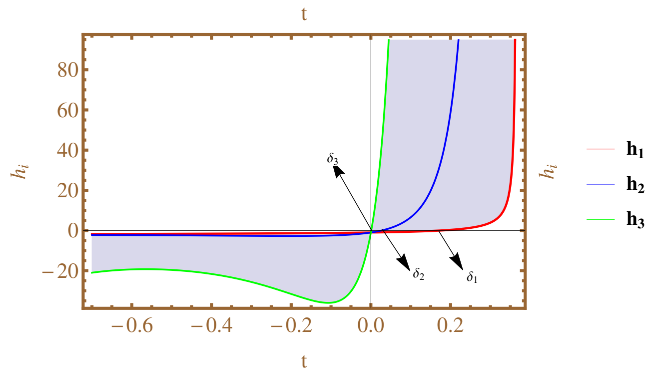
Thus, we take

$$\begin{aligned} w_0(t_1,t_2) &= t_1 + t_2, \\ w(t) &= t, \\ w_1(t_1,t_2,t_3) &= t_1 + t_2 + t_3, \\ w_2(t_1,t_2,t_3,t_4) &= 5. \end{aligned}$$

Hence, we obtain

$$\delta_1 = 0.17539, \ \delta_2 = 0.02678, \ \delta_3 = 0.90819 \times 10^{-3} \text{ and } \ \delta^* = 0.90819 \times 10^{-3}.$$

FIGURE 2. Graph of radius of convergence of example 2.



Example 3. Let $Z = C[0,1]$ be the space of continuous functions [12] defined on the interval $[0,1]$ and $\Omega = \bar{B}(0,1)$. Define the function H on Ω by

$$H(\varphi)(l) = \varphi(l) - 10\int_0^1 l\rho\varphi(\rho)^3 d\rho.$$

It follows that

$$H'(\varphi(\xi))(l) = \xi(l) - 30\int_0^1 l\rho\varphi(\rho)^2\xi(\rho)d\rho, \quad \text{for each } \xi \in \Omega.$$

Since $l^* = 0$, so we can set

$$\begin{aligned} w_0(t_1,t_2) &= 2(t_1+t_2), \\ w(t) &= \frac{t}{5}, \\ w_1(t_1,t_2,t_3) &= 2(t_1+t_2+t_3), \\ w_2(t_1,t_2,t_3,t_4) &= 2(t_1+t_2+t_3+t_4). \end{aligned}$$

Hence, we obtain

$$\delta_1 = 0.98449 \times 10^{-1}, \ \delta_2 = 0.62003 \times 10^{-1}, \ \delta_3 = 0.55704 \times 10^{-1} \text{ and } \ \delta^* = 0.55704 \times 10^{-1}.$$

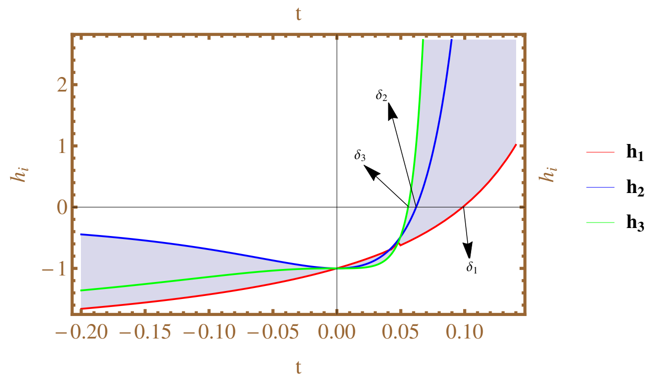
Example 4. Lastly, a nondifferentiable nonlinear system on $\mathbb{R} \times \mathbb{R}$ is solved using the method (1.3), where the divided difference is defined by the 2×2 matrix given for $\bar{t} = (t_1, t_2) \in \mathbb{R} \times \mathbb{R}$, $\tilde{t} = (t_3, t_4) \in \mathbb{R} \times \mathbb{R}$, and $F = (F_1, F_2)$ by

$$[\bar{t}, \tilde{t}; F]_{i,1} = \frac{F_i(t_3,t_4) - F_i(s_1,s_4)}{t_3 - t_1} \quad t_3 \neq t_1$$

and

$$[\bar{t}, \tilde{t}; F]_{i,2} = \frac{F_i(s_1,s_4) - F_i(s_1,s_2)}{s_4 - s_2}. \quad s_4 \neq s_2.$$

Figure 3. Graph of radius of convergence of example 3.



Otherwise, we set $[.,.;.] = O$. 7 Let us consider the nonlinear and nondifferentiable system as

$$3t_1^2 t_2 + t_2^2 - 1 + |t_1 - 1| = 0,$$
$$t_1^4 + t_1 t_2^3 - 1 + |t_2| = 0.$$

Then, we set $F = (F_1, F_2)$, where

$$F_1(t_1, t_2) = F_1 = 3t_1^2 t_2 + t_2^2 - 1 + |t_1 - 1| = 0$$

and

$$F_2(t_1, t_2) = F_2 = t_1^4 + t_1 t_2^3 - 1 + |t_2| = 0.$$

Choose initial points $(5, 5)$ and $(1, 0)$. Then, using the aforementioned divided difference and the method (1.3) we obtain the solution $x^* = (x_1^*, x_2^*)$ after three iterations with $x_1^* = 0.894655074966771$ and $x_2^* = 0.327826643198819$.

5. Conclusion

The focus of this paper is to provide a comprehensive analysis of the local and semilocal convergence of a derivative-free seventh-order method in Banach space. It is noteworthy that the convergence has been investigated in earlier studies by assuming the existence of some higher order derivatives, which in fact are not used in the iterative method. Contrary to this, our approach only considers the first-order divided differences that are actually present in the iterative process. This unique feature makes the method applicable to a wider range of functions, thereby expanding its utility. In the analysis, we present an error estimate and convergence ball that bounds the iterates, providing further benefits to the analysis of convergence. In addition, the sufficient conditions are developed to show the uniqueness of solution in the given domain. To verify the theoretical results, we have conducted numerical tests on several problems, demonstrating the effectiveness of this approach. Moreover, this idea has the potential to be extended to other methods, making it a valuable contribution in the field of the theory of iterative functions [1–17].

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