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Two Point Iterative Schemes for Nondifferentiable Equations in Banach Space

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Abstract. The local as well as the semi-local convergence analysis is established for a certain single step-two point iterative scheme defined on a Banach space setting. These schemes converge to a locally unique solution of a nonlinear equation. Both types of convergence are based on w-type continuity and majorizing functions and sequences. An auxiliary fixed linear operator is utilized to assure the existence of inverses of the linear operators involved as well as the initial points of the iterative scheme. The local analysis provides the radius of convergence, error estimates and information on the uniqueness of the solution. Moreover, the semi local analysis provides sufficient convergence conditions, error estimates and uniqueness of the solution results. Numerical examples further validate the theoretical results.

1. Introduction

Using Mathematical modelling, a plethora of applications from diverse disciplines of science and engineering reduce to determining solutions denoted by x^* of a nonlinear equation like

$$F(x) = 0. \tag{1.1}$$

Here, $F : D \subset B \to B$ is a continuous operator, B stands for a Banach space and D is an open set in B. The analytic form of the solution for (1.1) can be found only in special cases. That explains why researchers and practitioners resort to iterative schemes, when a sequence is generated approximating x^* under certain conditions.

Numerous studies exist in the local as well as the semi-local convergence analysis of iterative schemes [1–16]. Recently, there has been a surge in the development of schemes for solving

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equations or systems of equations involving nondifferentiable operators. Looking towards this direction, we develop the Two-point Iterative Scheme (TIPS):

For $x_0 \in D$ and each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = x_n - A(x_n, x_{n-1})^{-1} F(x_n), \tag{1.2}$$

where $A : D \times D \to \mathcal{L}(B)$, the space of bounded linear operators from B into B. TPIS specialties to popular schemes.

Case 1: (Secant Scheme [1–4, 12–15])

Set $A(x, y) = [x, y; F]$, where $[\cdot, \cdot; F] : D \times D \to (\mathcal{B})$ is a divided difference of order one for the operator F [6–8]. Under this choice the scheme (1.2) specializes to

$$x_{n+1} = x_n - [x_n, x_{n-1}; F]^{-1} F(x_n). \tag{1.3}$$

Case 2: (Kurchatov's Scheme [12, 13])

Set $A(x, y) = [2x - y, y; F]$. Then, the scheme (1.2) becomes:

$$x_{n+1} = x_n - [2x_n - x_{n-1}, x_{n-1}; F]^{-1} F(x_n). \tag{1.4}$$

Case 3: (Steffensen's Scheme [2, 8, 15])

Set $A(x, y) = [x + F(x), y; F]$. Then, the scheme (1.2) becomes:

$$x_{n+1} = x_n - [x + F(x), y; F]^{-1} F(x_n). \tag{1.5}$$

Case 4: (Picards's Scheme [1, 3, 8–10, 15, 16])

Set $A(x, y) = I$, where I stands for the identity operator on B.

Case 5: (Newton's Scheme [1–3, 5, 6, 9, 15, 16])

Set $A(x, y) = F'(x)$, where F' stands for the Fréchet derivative of the operator F.

The local as well as the semi-local convergence results for the aforementioned schemes involve assumptions on derivatives which do not appear on some of these methods (except Case 5).

Therefore, these results cannot be used to solve nondifferentiable operator equations (see e.g., Example 4.2). Other conditions involve approximations to the divided difference and the selection of an initial point x_0 so that the first iteration is computable.

In the present article, the local and semi-local convergence of the scheme (1.2) is investigated under w-continuity-type conditions. Moreover, by introducing a certain linear operator P, then invertibility of the linear operator A is assured in a certain subset of D from which the initial point is selected. The rest of the article is structured as follows: The local and the semi-local convergence of the scheme (1.2) appear in Section 2 and section 3, respectively. The examples can be found in Section 4. The article is completed the conclusions in Section 5.

2. Convergence I: Local

Some real functions are introduced that play a role in the local convergence analysis of the scheme (1.2). Let $T = [0, \infty)$. Assume:

(A_1) There exists a continuous and nondecreasing function (CNF) $w_0 : T \times T \to T$ such that the equation

$$w_0(t, t) - 1 = 0$$

has a smallest solution $r \in T_0 - \{0\}$.

Set $T_0 = [0, r_0)$ and $D_1 = D \cup U(x^*, R)$. Moreover, there exists CNF $w : T_0 \to T$ such that the equation $h(t) - 1 = 0$ has a smallest solution $r \in (0, r_0)$, where

$$h(t) = \frac{w(t, t)}{1 - w_0(t, t)}.$$

Let $T_1 = [0, r)$.

$$0 \leq w_0(t, t) < 1 \tag{2.6}$$

and

$$0 \leq h(t) < 1 \tag{2.7}$$

From now on we assume that $x^* \in D$ is a solution of the equation $F(x) = 0$ and the divided difference $[*, *; F]$ exists on $D \times D$. The functions w_0 and w are connected to the operators on the scheme (1.2).

(A_2) There exists an invertible operator P such that for each $x, y \in D$

$$\|P^{-1}(A(x, y) - P)\| \leq w_0(\|x - x^*\|, \|y - x^*\|).$$

(A_3)

$$\|P^{-1}(A(x, y) - [x, x^*; F])\| \leq w(\|x - x^*\|, \|y - x^*\|) \text{ for each } x, y \in D_1$$

and

(A_4) $U[x^*, r] \subset D$.

Next, the local convergence of the scheme (1.2) is provided based on the conditions $(A_1) - (A_4)$ and the developed terminology.

THEOREM 2.1. *Assume that the conditions (A_1)–(A_4) are validated. If the initial points $x_{-1}, x_0 \in U(x^*, r) - \{x^*\}$, then the sequence $\{x_n\}$ generated by the scheme (1.2) is well defined in $U(x^*, r)$ for each $n = 0, 1, 2, 3, \ldots$ and is convergent to the solution x^* of the equation $F(x) = 0$, so that*

$$\|x_{n+1} - x^*\| \leq h(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r \tag{2.8}$$

where the radius of convergence r is defined in (A_1) and the function h is also given in (A_1).

Proof. By hypothesis, $x_{-1}, x_0, (A_2)$ and (A_3) we obtain in turn that:

$$\|P^{-1}(A(x_0, x_{-1}) - P)\| \leq w_o(\|x_0 - x^*\|, \|x_{-1} - x^*\|) \leq w_0(r, r) < 1. \tag{2.9}$$

The estimate (2.9) and the Banach Lemma on invertible operators [1–3, 8, 9] assure the existence of $A(x_0, x_{-1})^{-1}\mathcal{L}(B)$ and

$$\|A(x_0, x_{-1})^{-1}P\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|, \|x_{-1} - x^*\|)}. \tag{2.10}$$

Moreover, the iterate x_1 is well defined by the first subset of the scheme (1.2). Then, we can write:

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - A(x_0 - x_{-1})^{-1}F(x_0) \\ &= A(x_0 - x_{-1})(A(x_0, x_{-1}) - [x_0, x^*; F])(x_0 - x^*) \end{aligned} \tag{2.11}$$

Using (2.7), (A_3), (2.10), (A_2) and (2.11) we get in turn that:

$$\|x_1 - x^*\| = \frac{w(\|x_0 - x^*\|, \|x_{-1} - x^*\|)\|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|, \|x_{-1} - x^*\|)} \leq h(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \tag{2.12}$$

Thus, the iterate $x_1 \in U(x^*, r)$ and the item (2.8) holds for $n = 0$. Simply replace x_{-1}, x_0, x_1 by x_{m-1}, x_m, x_{m+1} in the preceding calculations to terminate the induction for items (2.8). Then, from the estimation:

$$\|x_{m+1} - x^*\| \leq c\|x_m - x^*\| < r, \tag{2.13}$$

where $c = h(\|x_0 - x^*\|) \in [0, 1)$. We conclude that $\lim_{m\to\infty} x_m = x^*$ and that the iterate $x_{m+1} \in U(x^*, r)$. □

The uniqueness of the solution region is given in the next result.

PROPOSITION 2.2. *Assume: There exists a solution $z \in U(x^*, R_1)$ of the equation $F(x) = 0$ for some $R_1 \geq 0$; the condition (A_2) holds in the ball $U(x^*, R_1)$ for A being $[\cdot, \cdot; F]$, and there exists $R_2 \geq R_1$ such that:*

$$w_0(R_1, R_2) < 1. \tag{2.14}$$

Define the region $D_2 = D \cup U[x^*, R_2]$. Then, x^* is the only solution of the equation $F(x) = 0$ in the region D_2.

Proof. If $z \neq x^*$, then the divided difference $E = [x^*, z; F]$ is well defined. Using (A_2) and (2.14), we get in turn that

$$\|P^{-1}(E - P)\|| \leq w_0(\|x^* - x^*\|, \|z - x^*\|) \leq w_0(0, R_2) < 1.$$

Thus, $E^{-1} \in \mathcal{L}(B)$. Moreover, from the identity

$$z - x^* = E^{-1}(F(z) - F(x^*)) = E^{-1}(0) = 0.$$

Hence, we conclude that $z = x^*$. □

3. Convergence II: Semi-local

A certain real sequence is developed that is shown in Theorem 3.1 to be majorizing for the scheme (1.2). Assume:

$(H1)$: There exists a continuous and nondecreasing function $v_0 : T \to T$ such that $v_0(t,t) - 1 = 0$ has a unique positive solution denoted by δ.
Let $T_2 = [0, \delta)$.

$(H2)$: There exists a CNF $v : T_2 \times T_2 \to T$. Define a sequence $\{\gamma_n\}$ for $\gamma_{-1} = 0, \gamma_0 = \alpha$, some $\gamma_1 \geq \alpha$ by:

$$\gamma_{n+2} = \gamma_{n+1} + \frac{v(\gamma_{n+1} - \gamma_n, \gamma_n - \gamma_{n+1})(\gamma_{n+1} - \gamma_n)}{1 - v_0(\gamma_n - \gamma_0, \gamma_{n+1} - \gamma_0)}.$$

$(H3)$: $v_0(\gamma_n - \gamma_0, \gamma_{n+1} - \gamma_0) < 1$ and $\gamma_n \leq \gamma < \delta$.

Clearly, by the definition of the sequence γ_n and $(H3)$, this sequence is nondecreasingly convergent to its unique least upper bound denoted by γ^*. The functions v_0, v and the limit point γ^* are connected to the operators on the scheme as follows:

$(H4)$: There exists an invertible operator $P \in \mathcal{L}(B)$ such that $P^{-1} \in \mathcal{L}(B)$ and for each $x, y \in D$

$$\|P^{-1}(A(x,y) - P)\| \leq v_0(\|x - x_0\|, \|y - x_0\|).$$

It follows by (H1) that: if $x_{-1}, x_0 \in D$ with

$$\|x_{-1} - x_0\| \leq \alpha, \quad v_0(\|x_{-1} - x_0\|, \|x_0 - x_0\|) \leq v_0(\alpha, 0) < 1.$$

Thus $A(x_0, x_{-1})^{-1} \in \mathcal{L}(B)$. Let

$$\|A(x_0, x_{-1})^{-1} F(x_0)\| \leq \gamma_1 - \gamma_0.$$

Set $D_3 = D \cup U(x_0, \overline{\gamma})$.

$$\|P^{-1}(A(x,y) - [y, z; F])\| \leq (\|x - y\|, \|y - z\|) \text{ for each } x, y, z \in D_3$$

and

$(H5)$: $U[x_0, \gamma^*] \subset D$.

Next, we present the semi-local convergence analysis of the scheme (1.2) under the conditions $(H1) - (H5)$ and the preceding terminology.

THEOREM 3.1. *Assume that the conditions $(H1) - (H5)$ hold. Then the sequence $\{x_n\}$ generated by the scheme (1.2) is well defined in $U(x_0, \gamma_0)$, remains in $U(x_0, \gamma_0)$ for each $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in U(x_0, \gamma_0)$ of the equation $F(x) = 0$. Moreover, the following items hold:*

$$\|x^* - x_n\| \leq \gamma_{n+1} - \gamma_n \tag{3.15}$$

and

$$\|x^* - x_n\| \leq \gamma^* - \gamma_n, \tag{3.16}$$

where the sequence $\{\gamma_n\}$ *and the point* x^* *are defined in* $(H3)$.

Proof. The inequality (3.15) holds for $n = 0$, since:

$$\|x_1 - x_0\| = \|A(x_0, x_{-1})^{-1}F(x_0)\| \leq \gamma_1 - \gamma_0 < \gamma^* - \gamma_0.$$

□

Thus, the iterate $x_1 \in U(x_0, \gamma_0)$. Let $x_n, x_{n-1} \in U(x_0, \gamma_0)$. Using $(H3)$ and $(H4)$, we have in turn,

$$\|P^{-1}(A(x_n, x_{n-1}) - P)\| \leq v_0(\|x_n - x_0\|, \|x_{n-1} - x_0\|) \leq v_0(\gamma_0, \gamma_0) < 1.$$

So, $A(x_n, x_{n-1})^{-1}\mathcal{L}(B)$ and

$$\|A(x_n, x_{n-1})^{-1}P\| \leq \frac{1}{1 - v_0(\|x_n - x_0\|, \|x_{n-1} - x_0\|)}. \tag{3.17}$$

Moreover, the iterate x_{n+1} is well defined by the scheme (1.2) and (3.17). Further we can writer by the scheme (1.2)

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - A(x_n, x_{n-1})(x_{n+1} - x_n) \\ &= ([x_{n+1}, x_n; F] - A(x_n, x_{n-1}))(x_{n+1} - x_n). \end{aligned} \tag{3.18}$$

In view of $(H4)$ and (3.18) we get:

$$\begin{aligned} \|P^{-1}F(x_{n+1})\| &\leq \|P^{-1}([x_{n+1}, x_n; F] - A(x_n, x_{n-1}))\|\|x_{n+1} - x_n\| \\ &\leq v(\|x_{n+1} - x_n\|, \|x_n - x_{n-1}\|)\|x_{n+1} - x_n\| \\ &\leq v(\|\gamma_{n+1} - \gamma_n\|, \|\gamma_n - \gamma_{n-1}\|)\|\gamma_{n+1} - \gamma_n\|. \end{aligned} \tag{3.19}$$

Then, by the scheme (1.2) for n replaced by $n + 1$, we obtain:

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\leq \|A(x_{n+1}, x_n)^{-1}P\|\|PF(x_{n+1})\| \\ &\leq \frac{v(\gamma_{n+1} - \gamma_n, \gamma_n - \gamma_{n-1})}{1 - v_0(\|x_{n+1} - x_0\|, \|x_n - x_0\|)} \leq \frac{v(\gamma_{n+1} - \gamma_n, \gamma_n - \gamma_{n-1})(\gamma_{n+1} - \gamma_n)}{1 - v_0(\gamma_{n+1}, \gamma_n)} \end{aligned}$$

and

$$\|x_{n+2} - x_0\| \leq \|x_{n+2} - x_{n+1}\| + \|x_{n+1} - x_0\| \leq \gamma_{n+2} - \gamma_{n+1} + \gamma_{n+1} - \gamma_0 < \gamma^* - \gamma_0.$$

Therefore, the induction for the item (3.15) is terminated and $\{x_n\} \subset U(x_0, \gamma^* - \gamma_0)$. But the sequence $\{\gamma_n\}$ is complete. Thus, the sequence $\{x_n\}$ is also complete in a Banach space B. Hence there exists $x^* \in U[x_0, \gamma^* - \gamma_0]$ such that $\lim_{n\to\infty} x_n = x^*$.

By letting $n \to +\infty$ in (3.19) and using the continuity of the operator F, we deduce $F(x^*) = 0$. Then from the estimation:

$$\|x_{n+i} - x_n\| \leq \gamma_{n+i} - \gamma_n, \tag{3.20}$$

the item (3.16) is obtained by letting $i \to +\infty$ in (3.20).

A Uniqueness of the solution region is determined in the next result.

PROPOSITION 3.2. *Assume: There exists a solution $z \in U(x_0, \delta_1)$ of the equation $F(x) = 0$ for some $\delta_1 > 0$; the condition $(H4)$ holds for $[*, *, ; F]$ replacing A on the ball $U(x_0, \delta_1)$ and there exists $\delta_2 \geq \delta_1$ such that*

$$v_0(\delta_1, \delta_2) < 1. \tag{3.21}$$

Define the region $D_4 = D \cap U[x_0, \delta_2]$. Then, the only solution of the equation $F(x) = 0$ in the region D_4 is z.

Proof. As in Proposition 2.2, consider $z_1 \in D_4$ with $F(z_1) = 0$ and define $E_1 = [z, z_1; F]$ for $z \neq z_1$. Then, the application of (H_4) and (3.21) give

$$\|P^{-1}(E_1 - P)\| \leq v_0(\|z - z_0\|, \|z_1 - x_0\|) \leq v_0(\delta_1, \delta_2) < 1.$$

Consequently, it follows again that $z_1 = z_0$. □

REMARK 3.3. (i) *Possible choices but not the only ones for the linear operator P are:*

- *Differential case: $P = F'(x^*)$*

  and

- *Non-Differential case: $P = [x_{-1}, x_0; F]$.*

P should be chosen in general, so the majorant functions are as tight as possible in both the local and semi-local analysis (see the Numerical Section 4 that follows).

(ii) *The limit point γ^* in (H_5) is replaced by δ given in (H_1).*

(iii) *Notice that only (H_4) out of conditions $(H_1) - (H_5)$ is used in Proposition 3.2. However, if all conditions are used, let $\delta_1 = \gamma^*$ and $z = x^*$.*

4. Numerical Examples

EXAMPLE 4.1. *Let $B = \mathbb{R} \times \mathbb{R}$ and $D = U[x^*, 1]$ with $x^* = (0, 0, 0)^T$. Define the mapping F on D for $y = (y_1, y_2, y_3)^T$ as:*

$$F(y) = \left(e^{y_1} - 1, y_2, \frac{e-1}{2} y_3^2 + y_3\right)^T.$$

Then, by the definition of the Fréchet derivative F' of F is given by:

$$F'(y) = \begin{bmatrix} e^{y_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (e-1)y_3 + 1 \end{bmatrix}$$

It follows that $F'(x^*) = I$. Then, the convergence conditions $(A3)-(A5)$ are validated, respectively for:

Case 1:

$$w_0(s_1, s_2) = \frac{1}{2}(e-1)(s_1+s_2), w(s_1, s_2) = \frac{1}{2}(e-1)s_1, \text{ and } U[x^*, r] \subset D.$$

Case 2:

$$w_0(s_1, s_2) = \frac{1}{2}(e-1)(s_1+s_2), w(s_1, s_2) = \frac{1}{2}(e-1)(2s_1+s_2), \text{ and } U[x^*, 3r] \subset D.$$

Case 3:

$$w_0(s_1, s_2) = \frac{1}{2}(e-1)(s_1+s_2), w(s_1, s_2) = \frac{1}{2}(e-1)(s_1+2s_2), \text{ and } U[x^*, \bar{r}] \subset D,$$

where

$$\bar{r} = \max\{r, f(r)\} \text{ and } f(t) = \left(\|I+P\| + \frac{e-1}{2}t\right) t \text{ or } f(t) = \left(2 + \frac{e-1}{t}\right) t.$$

Case 4:

$$w_0(s_1, s_2) = 0, w(s_1, s_2) = \frac{1}{2}(e-1)s_1, \text{ and } U[x^*, r] \subset D.$$

Case 5:

$$w_0(s_1, s_2) = \frac{1}{2}(e-1)(s_1+s_2), w(s_1, s_2) = \frac{1}{2}(e-1)s_1, \text{ and } U[x^*, 3r] \subset D.$$

The exact radius r can be then computed immediately using the condition (A_2). As an example, for the case 5, we must solve:

$$\frac{\frac{1}{2}(e-1)t}{1-(e-1)t} = 0,$$

leading to

$$t = r = \frac{2}{3(e-1)}.$$

Concerning the semi-local case and the application of the method (1.2), we present another example:

EXAMPLE 4.2. *Let $B = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The two by two nonlinear and nondifferentiable system to be solved is:*

$$3s_1^2 s_2 + s_2^2 - 1 + |s_1 - 1| = 0,$$
$$s_1^4 + s_1 s_2^3 - 1 + |s_2| = 0.$$

Then, the system can be described as $Q = (Q_1, Q_2)$, where

$$Q_1(s_1, s_2) = 3s_1^2 s_2 + s_1^2 - 1 + |s_1 - 1|$$

and

$$Q_2(s_1, s_2) = s_1^4 + s_1 s_2^3 - 1 + |s_2|.$$

The system becomes $Q(s_1, s_2) = 0$. Then as $A = [*, *; Q]$, which is an 2×2 real matrix defined for $\bar{s} = [s_1, s_2]^T$ and $\tilde{s} = [s_3, s_4]^T$ by:

$$[\tilde{s}, \tilde{s}; Q]_{i,1} = \frac{Q_i[(s_1, s_4) - Q_i(s_1, s_2)]}{s_4 - s_2} \text{ for } s_2 \neq s_4, i = 1, 2.$$

Otherwise, set $[*, *; Q] = 0$. Let us choose $\bar{s}_0 = (5, 5)$ and $\bar{s}_{-1} = (1, 0)$ to be the starters for the scheme (1.2). Then,

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	
1	1	0	5
2	0.989090909090909	0.363636363636364	3.636E-01
3	0.894886945874111	0.329098638203090	3.453E-02
4	0.894655531991499	0.327827544745569	1.271E-03
5	0.894655373334793	0.327826521746906	1.022E-06
6	0.894655373334687	0.327826421746298	6.089E-13
7	0.894655373334687	0.327826421746298	2.701E-20

Therefore, the solution $s^* = (s_1^*, s_2^*)^T$ of the system is

$$s_1^* = 0.894655373334687 \text{ and } s_2^* = 0.327826421746298.$$

REMARK 4.3. *A more targeted choice for L than the two mentioned already can give a larger radius of convergence. Indeed, assume the conditions instead of (A_3) and (A_4)*

$(A_3)'$

$$\|P^{-1}(F'(x) - P)\| < 1$$

and

$(A_4)'$

$$\|P^{-1}(F'(x^* + \theta(x - x^*)) - F'(x))\| \leq g_4(\theta)\|x - x^*\| \text{ for each } x \in D.$$

Consider an example

$$F(x) = e^x - 1$$

with $D = U[x^*, 1]$. Then $x^* = 0$. Choose $P = \frac{1}{b}e^x$ for $b \in (0, 2)$. Then the Newton's scheme gives:

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - F'(x_n)^{-1}F(x_n) \\ &= s_0 F'(x_n)^{-1}\left(F'(x^* + \theta(x_n - x^*)) - F'(x_n)\right)(x_n - x^*) \end{aligned}$$

Leading by $(A_3)'$ and $(A_4)'$ to

$$\|x_{n+1} - x^*\| \leq \frac{|b|(e-2)\|x_n - x^*\|^2}{1 - |1-b|},$$

where we also used

$$\|P^{-1}(F'(x) - P)\| = \left\| be^{-x}\left(e^x - \frac{1}{b}e^x\right)\right\| = |b-1| < 1.$$

Thus,

$$\|P^{-1}(F'(x) - P)\| \leq \frac{1}{1 - |1-b|},$$

and

$$\|P^{-1}(F'(x^* + \theta(x - x^*)) - F'(x))\| \leq |b|(e-2).$$

The last estimate is obtained, since for $y = x^* + \theta(x - x^*)$

$$e^{-x}(e^{-y} - e^x) = e^y - 1 = 1 + y + \frac{y^2}{2!} + \cdots + \frac{y^k}{k!} + \cdots - 1 = y\left(1 + \frac{y}{2!} + \cdots + \frac{y^{k-1}}{k!} + \ldots\right).$$

Hence,

$$\|e^x(e^y - e^x)\| = (1-\theta)G_n\|x - x^*\|,$$

where,

$$G_n = 1 + \frac{1-\theta}{2!} + \cdots + \frac{(1-\theta)^{n-1}}{n!} + \ldots,$$

and

$$\int_0^1 (1-\theta)^n d\theta = \frac{1}{n+1}.$$

But,

$$\begin{aligned} \int_0^1 (1-\theta)^n G_n d\theta &= \int_0^1 (1-\theta)d\theta t + \int_0^1 \frac{(1-\theta)^2}{2!}d\theta t^2 + \cdots + \int_0^1 \frac{(1-\theta)^n}{n!}d\theta t^{n-1} + \ldots \\ &= \frac{1}{2}t + \frac{1}{3\cdot 2!}t^2 + \cdots + \frac{1}{(n+1)n!}t^{n-1} + \cdots \leq e - 2. \end{aligned}$$

Consequently, it follows from the error estimate that:

$$r_A = \frac{1 - |1-b|}{|b|(e-2)}.$$

Let us compare the new radius with the ones already in the literature developed independently by Rheinboldt [11] and Traub [15]. The condition used is:

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq l\|x - y\|$$

for each $x, y \in D$ to obtain the error estimate:

$$\|x_{n+1} - x^*\| \leq \frac{l\|x_n - x^*\|^2}{2(1 - \|x_n - x^*\|)},$$

and the radius is

$$r_{TR} = \frac{2}{3l}.$$

But $l = e$ for the example, so:

$$r_{TR} = \frac{3}{2e} < r_A, \text{ say for } b = 1.$$

Therefore, the new radius of convergence is larger allowing for a wider choice of initial points. Other choices of P can lead to even larger radius of convergence. We leave the detail to the motivated reader.

5. Conclusion

A finer and more flexible local and semi-local convergence analysis for the scheme (1.2) is developed involving an invertible operator P, which if chosen appropriately leads to weaker convergence conditions, better uniqueness of the solution and a larger radius of convergence than if P is chosen to be as in earlier studies $F'(x^*)$ or $F'(x_0)$ or $[x_0, x_{-1}; F]$. This idea can be extended to multistep and multipoint schemes [1–16]. This is the direction of our future research.

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