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Strong Continuity of Composition Semigroups on the Generalized Bloch Spaces of the Upper Half Plane

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Abstract. We investigate strong continuity of composition semigroups on the generalized Bloch spaces of the upper half plane. These composition semigroups are induced by automorphisms of the upper half plane as classified into three distinct groups in [3].

1. Introduction

Consider $\mathcal{H}(\Omega)$ as the Fréchet space of analytic functions $f : \Omega \to \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω. A function $f \in \mathcal{H}(\mathbb{D})$ is in the Bloch space of the unit disc $\mathcal{B}(\mathbb{D})$ if

$$\|f\|_{\mathcal{B}_1(\mathbb{D})} := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$$

and in the little Bloch space of the unit disc $\mathcal{B}_0(\mathbb{D})$ if

$$\lim_{|z|\longrightarrow 1}(1-|z|^2)|f'(z)| = 0.$$

For $f \in \mathcal{B}(\mathbb{D})$, we define the norm on $\mathcal{B}(\mathbb{D})$ by

$$\|f\|_{\mathcal{B}(\mathbb{D})} := |f(0)| + \|f\|_{\mathcal{B}_1(\mathbb{D})},$$

where $\|.\|_{\mathcal{B}_1(\mathbb{D})}$ is a seminorm on $\mathcal{B}(\mathbb{D})$.
Bloch space of the upper half plane $\mathcal{B}(\mathbb{U})$ is a set of analytic functions $f \in \mathcal{H}(\mathbb{U})$ such that

$$\|f\|_{\mathcal{B}_1(\mathbb{U})} := \sup_{\omega\in\mathbb{U}} \Im(\omega)|f'(\omega)| < \infty.$$

For $f \in \mathcal{B}(\mathbb{U})$, we define the norm on $\mathcal{B}(\mathbb{U})$ by

$$\|f\|_{\mathcal{B}(\mathbb{U})} := |f(i)| + \|f\|_{\mathcal{B}_1(\mathbb{U})},$$

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where $\|.\|_{\mathcal{B}_1(\mathbb{U})}$ is a seminorm on $\mathcal{B}(\mathbb{U})$.

Let $\alpha > 0$ be a real number, we define the generalized Bloch space of the unit disc, $\mathcal{B}^{\alpha}(\mathbb{D})$ as the space of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_1^{\alpha}(\mathbb{D})} := \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} |f'(z)| < \infty.$$

For $f \in \mathcal{B}^{\alpha}(\mathbb{D})$, we define the norm on $\mathcal{B}^{\alpha}(\mathbb{D})$ by

$$\|f\|_{\mathcal{B}^{\alpha}(\mathbb{D})} := |f(0)| + \|f\|_{\mathcal{B}_1^{\alpha}(\mathbb{D})}. \tag{1}$$

We also define the corresponding generalized little Bloch space of the unit disc as the space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^{\alpha} |f'(z)| = 0,$$

with the same norm given by (1). Here, $\mathcal{B}^{\alpha}(\mathbb{D})$ and $\mathcal{B}_{\circ}^{\alpha}(\mathbb{D})$ are both Banach spaces with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{D})}$. The generalized little Bloch space of the unit disc, $\mathcal{B}_{\circ}^{\alpha}(\mathbb{D})$ is the closure of the set of polynomials in the norm topology of $\mathcal{B}^{\alpha}(\mathbb{D})$. For more details see [17, 18]. The space $\mathcal{B}(\mathbb{D})$ has been studied by many authors because of its intrinsic interest since its introduction [1, 4, 8, 10, 13, 14, 18]. In [17], the generalized Bloch spaces of the open unit disc, $\mathcal{B}^{\alpha}(\mathbb{D})$ are defined and proved to be Banach spaces with respect to their norm. Zhu [17] further established generalized little Bloch spaces of the unit disc $\mathcal{B}_{\circ}^{\alpha}(\mathbb{D})$, as closed, separable subspaces of $\mathcal{B}^{\alpha}(\mathbb{D})$. There is scanty literature on the properties of the generalized Bloch spaces of the upper half plane $\mathcal{B}^{\alpha}(\mathbb{U})$, including whether they are Banach spaces. Composition semigroups on Bloch spaces of the unit disc have been studied in literature, see for instance [2, 11, 12] and references therein. On strong continuity of composition semigroups, Siskakis [12] proved that no nontrivial composition semigroups are strongly continuous on the Bloch space of the unit disc $\mathcal{B}(\mathbb{D})$. The corresponding study of composition semigroups defined on the Bloch spaces of the upper half plane has not yet been exhausted. Moreover, existing works on the half plane, see [7, 13], have neither exhausted the investigation of properties of these semigroups nor considered these generalizations. In this paper therefore, we investigate the properties of the generalized Bloch spaces of the upper half plane as Banach spaces and extend the study of semigroups of composition operators to the setting of the generalized Bloch spaces of the upper half plane.

2. Preliminaries and Definitions

Let $\mathbb{C}$ be the complex plane. The set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is called the *open unit disc*. On the other hand, the set $\mathbb{U} := \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ denotes the upper half of the complex plane $\mathbb{C}$, where $\Im(\omega)$ is the imaginary part of $\omega \in \mathbb{C}$. The function $\psi(z) = \frac{i(1+z)}{1-z}$ is referred to as the *Cayley transform* and maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$, with the inverse $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$ mapping the upper half plane $\mathbb{U}$, onto the unit disc, $\mathbb{D}$. We refer to [16] for details. Let $\alpha > 0$ be a real number. A function $f \in \mathcal{H}(\mathbb{U})$ belongs to the generalized Bloch space of the upper half plane, $\mathcal{B}^{\alpha}(\mathbb{U})$ if

$$\|f\|_{\mathcal{B}^{\alpha}_{1}(\mathbb{U})} := \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |f'(\omega)| < \infty$$

with the norm given by

$$\|f\|_{\mathcal{B}^{\alpha}(\mathbb{U})} := |f(i)| + \|f\|_{\mathcal{B}^{\alpha}_{1}(\mathbb{U})}.$$

The corresponding generalized little Bloch space of the upper half plane, $\mathcal{B}^{\alpha}_{0}(\mathbb{U})$is defined as

$$\mathcal{B}^{\alpha}_{\circ}(\mathbb{U}) := \{f \in \mathcal{H}(\mathbb{U}) : \lim_{\Im(\omega) \longrightarrow 0} \Im(\omega)^{\alpha} |f'(\omega)| = 0\}$$

having the same norm as $\mathcal{B}^{\alpha}(\mathbb{U})$. There is little literature on the properties of the generalized Bloch spaces of the upper half plane as Banach spaces. Let X be a Banach space. A one-parameter family $(T_t)_{t\geq 0}$ is a *semigroup* of bounded linear operators on X, if
(i) $T_o = I$ (Identity operator on X), and
(ii) $T_{t+s} = T_t \circ T_s$ for every $t, s, \geq 0$ (Semigroup property).
A semigroup $(T_t)_{t\geq 0}$ of bounded linear operators on X is *strongly continuous* if

$$\lim_{t \to 0^+} \|T_t x - x\| = 0 \text{ for all } x \in X.$$

The *infinitesimal generator* denoted by Γ of $(T_t)_{t\geq 0}$ is defined by

$$\Gamma x := \lim_{t \to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t}(T_t x)\Big|_{t=0} \text{ for each } x \in \text{dom}(\Gamma),$$

where dom(Γ) denotes the domain of Γ given by

$$\text{dom}(\Gamma) = \left\{x \in X : \lim_{t \to 0^+} \frac{T_t x - x}{t} \text{ exists}\right\}.$$

We define a group of bounded linear operators as

$$(T_t)_{t \in \mathbb{R}} = \begin{cases} T_t, & t \geq 0, \\ T_{-t}, & t \geq 0. \end{cases}$$

if both $(T_t)_{t\geq 0}$ and $(T_{-t})_{t\geq 0}$ are semigroups on X. For more details see [5,6,9]. Suppose $\varphi : \Omega \to \Omega$ is a self analytic map. The composition operator induced by φ on $\mathcal{H}(\Omega)$ is defined as

$$C_{\varphi}(f) = f \, o \, \varphi,$$

for all $f \in \mathcal{H}(\Omega)$. On the other hand, given $t \geq 0$ we define a semigroup as a family $(\varphi_t)_{t\geq 0}$ of self analytic maps on Ω satisfying the following properties

(i) $\varphi_0(z) = z$ (Identity map on Ω).
(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s, \forall\, t, s \geq 0$ (Semigroup property).
(iii) $\varphi_t \to \varphi_0$ uniformly on compact subsets of Ω as $t \to 0$.

Composition semigroup induced by φ_t on $\mathcal{H}(\Omega)$ is defined as

$$C_{\varphi_t}(f) = f\,o\,\varphi_t, \text{ for all } f \in \mathcal{H}(\Omega).$$

3. Generalized Bloch Spaces of the Upper Half Plane

In this section, we study properties of the generalized Bloch spaces as Banach spaces. We also relate functions in the generalized Bloch space of the upper half plane $\mathbb{U}$ to their counterparts in the unit disc $\mathbb{D}$. Following [17,18], it's well known that $\mathcal{B}^{\alpha}(\mathbb{D})$ and $\mathcal{B}_0^{\alpha}(\mathbb{D})$ are Banach spaces with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{D})}$. Moreover the set of analytic polynomials $\mathbb{C}[z] := \left\{\sum_{n=0}^{\infty} a_n z^n : z \in \mathbb{C}\right\}$ is dense in $\mathcal{B}_0^{\alpha}(\mathbb{D})$. These results are not explicitly clear from the literature in the setting of the upper half plane $\mathbb{U}$.

In the following theorem, we establish the completeness of $\mathcal{B}^{\alpha}(\mathbb{U})$ with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$.

Theorem 3.1. *$\mathcal{B}^{\alpha}(\mathbb{U})$ is a Banach space with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$*

Proof. It's clear that $\left(\mathcal{B}^{\alpha}(\mathbb{U}), \|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}\right)$ is a normed space. Now we prove that the space $\mathcal{B}^{\alpha}(\mathbb{U})$ is complete in $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$. Let $(f_k)_k$ denote a Cauchy sequence in $\mathcal{B}^{\alpha}(\mathbb{U})$. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_k - f_l\|_{\mathcal{B}^{\alpha}(\mathbb{U})} < \epsilon,\ \forall\, k, l > N$. Hence by the definition of the norm, we have for all $\forall\, k, l > N$,

$$|f_k(i) - f_l(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |f_k'(\omega) - f_l'(\omega)| < \epsilon,$$

which means that

$$|f_k(i) - f_l(i)| < \epsilon \text{ and } (\Im(\omega))^{\alpha} |f_k'(\omega) - f_l'(\omega)| < \epsilon,$$

for $\omega \in \mathbb{U}$. So, $(f_k(i))_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$. By the completeness of $\mathbb{C}$, $(f_k(i))_k$ converges to a limit, say u_0. Similarly, $\left(f_k'(\omega)\right)_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$ and therefore converges to a limit, say g.

Since $|f_k'(\omega) - f_l'(\omega)| < \frac{\epsilon}{\Im(\omega)^{\alpha}}$ and $f_k'(\omega) \to g$ uniformly on compact subsets of $\mathbb{U}$, then $g \in \mathcal{H}(\mathbb{U})$.

Now, take f such that $f'(\omega) = g(\omega)\,\forall\,\omega \in \mathbb{U}$ and $f(i) = u_0$.

Thus, $\forall\,\epsilon > 0$, $\exists\, N$ such that $\forall k, l > N$,

$$\Im(\omega)^{\alpha} |f_k'(\omega) - f_l'(\omega)| < \epsilon,\ \forall\,\omega \in \mathbb{U}.$$

Taking limits as $l \to \infty$, then $\forall k > N$,

$$\Im(\omega)^{\alpha} |f_k'(\omega) - f'(\omega)| < \epsilon,\ \forall\,\omega \in \mathbb{U}.$$

It follows that

$$\|f_k - f\|_{\mathcal{B}^{\alpha}(\mathbb{U})} = |f_k(i) - f(i)| + \sup_{\omega\in\mathbb{U}} \Im(\omega)^{\alpha} |f_k'(\omega) - f'(\omega)| < \epsilon$$

and so $\|f_k - f\|_{\mathcal{B}^\alpha(\mathbb{U})} \to 0$ as $k \to \infty$.
Now, it remains to show that $f \in \mathcal{B}^\alpha(\mathbb{U})$. We have

$$\begin{aligned} \Im(\omega)^\alpha |f'(\omega)| &= \Im(\omega)^\alpha |f'(\omega) - f_k'(\omega) + f_k'(\omega)| \\ &\leq \Im(\omega)^\alpha |f'(\omega) - f_k'\omega| + \Im(\omega)^\alpha |f_k'(\omega)| \\ &< \epsilon + \Im(\omega)^\alpha |f_k'(\omega)| < \infty \end{aligned}$$

since $(f_k)_k \subset \mathcal{B}^\alpha(\mathbb{U})$.
Now, taking supremum over all $\omega \in \mathbb{U}$ in the above equation, we have that

$$\sup_{\omega\in\mathbb{U}} \Im(\omega)^\alpha |f'(\omega)| < \infty$$

which implies that $f \in \mathcal{B}^\alpha(\mathbb{U})$, as desired. □

As an immediate consequence, we have

Corollary 3.2. *$\mathcal{B}(\mathbb{U})$ is a Banach space with respect to the norm $\| \,.\, \|_{\mathcal{B}(\mathbb{U})}$*

Proof. Follows immediately by taking $\alpha = 1$ in Theorem 3.1. □

Under the norm $\| \,.\, \|_{\mathcal{B}^\alpha(\mathbb{U})}$, the space $\mathcal{B}_0^\alpha(\mathbb{U})$ also becomes a Banach space as in the following theorem,

Theorem 3.3. *$\mathcal{B}_0^\alpha(\mathbb{U})$ is a Banach space with respect to the norm $\| \,.\, \|_{\mathcal{B}^\alpha(\mathbb{U})}$.*

Proof. Following Theorem 3.1, we need to show that every sequence in $\mathcal{B}_0^\alpha(\mathbb{U})$ convergent in $\mathcal{B}^\alpha(\mathbb{U})$ has its limit in $\mathcal{B}_0^\alpha(\mathbb{U})$.
Let $(f_n) \subset \mathcal{B}_0^\alpha(\mathbb{U})$ and $g \in \mathcal{B}^\alpha(\mathbb{U})$ be such that $f_n \to g$ as $n \to \infty$. We need to prove that $g \in \mathcal{B}_0^\alpha(\mathbb{U})$. Since f_n, g are holomorphic on compact subsets of $\mathbb{U}$, and $f_n \to g$, we have $f_n' \to g'$ uniformly. Now that $f_n \subset \mathcal{B}_0^\alpha(\mathbb{U})$, we have

$$\lim_{\Im(\omega)\to 0} (\Im(\omega))^\alpha |f_n'(\omega)| = 0, \forall\, n. \tag{2}$$

Since $\lim_{n\to\infty} f_n' = g'$, we have

$$\lim_{\Im(\omega)\to 0} (\Im(\omega))^\alpha |g'(\omega)| = \lim_{\Im(\omega)\to 0} (\Im(\omega))^\alpha |\lim_{n\to\infty} f_n'(\omega)|$$

which is equivalent to

$$\lim_{\Im(\omega)\to 0} (\Im(\omega))^\alpha |g'(\omega)| = \lim_{n\to\infty} \left(\lim_{\Im(\omega)\to 0} (\Im(\omega))^\alpha |f_n'(\omega)| \right).$$

Following equation (2), we see that

$$\lim_{\Im(\omega)\to 0} (\Im(\omega))^\alpha |g'(\omega)| = 0.$$

So, $g \in \mathcal{B}_0^\alpha(\mathbb{U})$, completing the proof. □

As a consequence, we have the following,

Corollary 3.4. *$\mathcal{B}_0(\mathbb{U})$ is a Banach space with respect to the norm $\|.\|_{\mathcal{B}(\mathbb{U})}$*

Proof. Follows immediately by taking $\alpha = 1$ in Theorem 3.3. $\square$

In the next results, we generate a relationship between functions in the generalized Bloch space of the upper half plane $\mathbb{U}$ and their counterparts in the unit disc $\mathbb{D}$

Proposition 3.5. *Let $f \in \mathcal{B}^\alpha(\mathbb{U})$ and ψ be the Cayley transform, then $f \in \mathcal{B}^\alpha(\mathbb{U})$ if and only if $f \circ \psi \in \mathcal{B}^\alpha(\mathbb{D})$*

Proof. It suffices to prove that $\|f\|_{\mathcal{B}^\alpha_1(\mathbb{U})} < \infty$ if and only if $\|f \circ \psi\|_{\mathcal{B}^\alpha_1(\mathbb{D})} < \infty$. Let f be a function in $\mathcal{B}^\alpha(\mathbb{U})$. Then by definition,

$$\|f\|_{\mathcal{B}^\alpha_1(\mathbb{U})} = \sup_{\omega \in \mathbb{U}} \Im(\omega)^\alpha |f'(\omega)| < \infty.$$

Now, by changing variables, let $\omega = \psi(z)$, where ψ is the Cayley transform. Then

$$\begin{aligned} \Im(\omega) &= \frac{\omega - \overline{\omega}}{2i} \\ &= \frac{\psi(z) - \overline{\psi(z)}}{2i}. \end{aligned}$$

Using $\psi(z) = \frac{i(1+z)}{1-z}$ and $\overline{\psi(z)} = \frac{-i(1+\overline{z})}{1-\overline{z}}$, we have

$$\begin{aligned} \Im(\omega) &= \frac{\frac{i(1+z)}{1-z} - \frac{-i(1+\overline{z})}{1-\overline{z}}}{2i} \\ &= \frac{i(1+z)(1-\overline{z}) + i(1+\overline{z})(1-z)}{2i(1-z)(1-\overline{z})} \\ &= \frac{i(2-2\overline{z}z)}{2i(1-z)(1-\overline{z})} \\ &= \frac{1-|z|^2}{|1-z|^2}. \end{aligned}$$

We get the absolute of $\psi'(z) = \frac{2i}{(1-z)^2}$ as

$$|\psi'(z)| = \frac{2}{|1-z|^2}. \tag{3}$$

Now, by definition we have

$$\|f\|_{\mathcal{B}^\alpha_1(\mathbb{U})} = \sup_{z \in \mathbb{D}} \left(\frac{1-|z|^2}{|1-z|^2}\right)^\alpha |f'(\psi(z))|.$$

From equation (3), we have $|1-z|^2 = \frac{2}{|\psi'(z)|}$, therefore

$$\|f\|_{\mathcal{B}^\alpha_1(\mathbb{U})} = \frac{1}{2^\alpha} \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha |\psi'(z)|^\alpha |f'(\psi(z))|.$$

Since, $(f \circ \psi)'(z) = f'(\psi(z))\psi'(z)$, we have
$|\psi'(z)|^\alpha |f'(\psi(z))| = |\psi'(z)(f \circ \psi)'(z)||\psi'(z)^{\alpha-1}|$ and hence

$$\begin{aligned} \|f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} &= \frac{1}{2^\alpha} \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |\psi'(z)(f \circ \psi)'(z)||\psi'(z)^{\alpha-1}| \\ &= \frac{1}{2^\alpha} |\psi'(z)^{\alpha-1}| \|f \circ \psi\|_{\mathcal{B}_1^\alpha(\mathbb{D})}, \end{aligned}$$

which is finite if and only if $\|f \circ \psi\|_{\mathcal{B}_1^\alpha(\mathbb{D})}$ is finite. This completes the proof. □

An immediate consequence is the following,

Corollary 3.6. *Let $f \in \mathcal{B}(\mathbb{U})$ and ψ be the Cayley transform, then*

$$\|f\|_{\mathcal{B}_1(\mathbb{U})} = \frac{1}{2} \|f \circ \psi\|_{\mathcal{B}_1(\mathbb{D})} \tag{4}$$

In particular, a function $f \in \mathcal{B}(\mathbb{U})$ if and only if $f \circ \psi \in \mathcal{B}(\mathbb{D})$.

Proof. This follows immediately from Proposition 3.5 by taking $\alpha = 1$. □

4. Composition Semigroups on the Generalized Little Bloch Space of the Upper Half Plane

In [3], the non trivial automorphisms of the upper half plane $\mathbb{U}$ were classified according to the location of their fixed points into three distinct classes namely; scaling, translation and rotation groups. In this section, we determine composition semigroups induced by these automorphism groups of the upper half plane $\mathbb{U}$, on the generalized Bloch space of the upper half plane $\mathcal{B}^\alpha(\mathbb{U})$. We then employ the theory of linear operators on Banach spaces to investigate the semigroup properties of the induced composition semigroup. For any given semigroup φ_t, the induced operator semigroup C_{φ_t} is known to be strongly continuous on the little Bloch space. On the other hand, no non trivial composition semigroup is strongly continuous on the Bloch space, see [11]. Therefore, we shall determine the composition semigroup induced by these automorphism groups on the generalized little Bloch space of the upper half plane, $\mathcal{B}_0^\alpha(\mathbb{U})$. Further, we show that composition semigroups induced by scaling and translation groups are strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{U})$. We also establish strong continuity of composition semigroups induced by rotation group on $\mathcal{B}_0^\alpha(\mathbb{D})$. The infinitesimal generator is identified and its domain stated.

4.1. **Scaling group.** The automorphisms of this group are of the form $\varphi_t(z) = k^t z$, where $z \in \mathbb{U}$ and $k, t \in \mathbb{R}$ with $k \neq 0$. As noted in [3], the semigroup properties of the induced composition operators will differ significantly depending on whether $0 < k < 1$ or $k > 1$. Thus for $0 < k < 1$, we consider without loss of generality, the analytic self maps $\varphi_t : \mathbb{U} \longrightarrow \mathbb{U}$ of the form

$$\varphi_t(z) = e^{-t} z, \; z \in \mathbb{U}. \tag{5}$$

The composition semigroup induced by equation (5) on $\mathcal{B}_0^\alpha(\mathbb{U})$ is given by

$$C_{\varphi_t} f(z) = (f \circ \varphi_t)(z) = f\left(e^{-t} z\right)$$

It can be easily proved that $(C_{\varphi_t})_{t\in\mathbb{R}}$ is a group on $\mathcal{B}_0^\alpha(\mathbb{U})$.

In what follows, we prove that the composition semigroup given by equation (4.1) fails to be an isometry on $\mathcal{B}_0^\alpha(\mathbb{U})$.

Proposition 4.1. *The operator C_{φ_t} fails to be an isometry on $\mathcal{B}_0^\alpha(\mathbb{U})$.*

Proof. By the definition of the norm, we have for all $f \in \mathcal{B}_0^\alpha(\mathbb{U})$

$$\begin{aligned}
\|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{U})} &= |C_{\varphi_t} f(i)| + \sup_{\omega\in\mathbb{U}} \Im(\omega)^\alpha |\left(C_{\varphi_t} f\right)'(\omega)| \\
&= |f(e^{-t}i)| + \sup_{\omega\in\mathbb{U}} \Im(\omega)^\alpha |e^{-t} f'(e^{-t}\omega)|.
\end{aligned}$$

Now by change of variables:
Let $z = e^{-t}\omega$, then $\omega = e^t z$, and $\Im(\omega) = e^t \Im(z)$. Therefore,

$$\begin{aligned}
\|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{U})} &= |f(e^{-t}i)| + \sup_{z\in\mathbb{U}} e^{t\alpha} \Im(z)^\alpha |e^{-t} f'(z)| \\
&= |f(e^{-t}i)| + e^{(\alpha-1)t} \sup_{z\in\mathbb{U}} \Im(z)^\alpha |f'(z)| \\
&\neq |f(i)| + \sup_{z\in\mathbb{U}} \Im(z)^\alpha |f'(z)| = \|f\|_{\mathcal{B}^\alpha(\mathbb{U})},
\end{aligned}$$

which completes the proof. □

Next, we prove that the operator C_{φ_t} given by (4.1) is strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{U})$.

Theorem 4.2. *$(C_{\varphi_t})_{t\in\mathbb{R}}$ is strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{U})$.*

Proof. To prove strong continuity of $(C_{\varphi_t})_{t\in\mathbb{R}}$, it suffices to show that $\|C_{\varphi_t} f - f\|_{\mathcal{B}^\alpha(\mathbb{U})} \to 0$ as $t \to 0$. That is, $|\left(C_{\varphi_t} f - f\right)(i)| + \|C_{\varphi_t} f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0$ as $t \to 0$. This is equivalent to $|\left(C_{\varphi_t} f - f\right)(i)| \to 0$ and $\|C_{\varphi_t} f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0,$ as $t \to 0$. For the former, we have

$$\begin{aligned}
|\left(C_{\varphi_t} f - f\right)(i)| &= |C_{\varphi_t} f(i) - f(i)| \qquad (6)\\
&= |f(\varphi_t(i)) - f(i)| \\
&= |f(e^{-t}i) - f(i)| \to 0 \text{ as } t \to 0,
\end{aligned}$$

as desired. We now prove that $\|C_{\varphi_t} f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0$ as $t \to 0$. Recall that $\psi : \mathbb{D} \to \mathbb{U}$, $\varphi_t : \mathbb{U} \to \mathbb{U}$ and $\psi^{-1} : \mathbb{U} \to \mathbb{D}$. We can therefore have $\mathbb{D} \xrightarrow{\psi} \mathbb{U} \xrightarrow{\varphi_t} \mathbb{U} \xrightarrow{\psi^{-1}} \mathbb{D}$. Now, let $\mathcal{X}_t = \psi^{-1} \circ \varphi_t \circ \psi : \mathbb{D} \to \mathbb{D}$. If $(\varphi_t)_{t\geq 0}$ is an automorphism of the upper half plane $\mathbb{U}$, then $(\mathcal{X}_t)_{t\geq 0}$ is an automorphism of the unit disc $\mathbb{D}$. Since $\mathcal{X}_t = \psi^{-1} \circ \varphi_t \circ \psi$, it follows that $\|C_{\varphi_t} f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0$ as $t \to 0$ if and only if $\|C_{\mathcal{X}_t} f^* - f^*\|_{\mathcal{B}^\alpha(\mathbb{D})} \to 0$ as $t \to 0$

Cayley transform is given by $\psi(z) = \frac{i(1+z)}{1-z}$. We therefore have

$$\begin{aligned}\psi^{-1}\circ\varphi_{-t}\circ\psi(z) &= \psi^{-1}\left(\varphi_t\left(\psi(z)\right)\right).\\ &= \psi^{-1}\left(\varphi_t\left(\frac{i(1+z)}{1-z}\right)\right)\\ &= \psi^{-1}\left(e^{-t}\left(\frac{i(1+z)}{1-z}\right)\right).\end{aligned}$$

Substituting $\psi^{-1}(z) = \frac{z-i}{z+i}$, we obtain

$$\psi^{-1}\circ\varphi_{-t}\circ\psi(z) = \frac{e^{-t}(\frac{i(1+z)}{1-z}) - i}{e^{-t}(\frac{i(1+z)}{1-z}) + i}.$$

Simplifying the fraction, we have

$$\psi^{-1}\circ\varphi_{-t}\circ\psi(z) = \frac{z + e^{-t}z - 1 + e^{-t}}{-z + e^{-t}z + 1 + e^{-t}}.$$

Now, by factorizing z and dividing both the numerator and denominator by $(1+e^{-t})$, we obtain

$$\psi^{-1}\circ\varphi_{-t}\circ\psi(z) = \frac{z - \frac{(1-e^{-t})}{(1+e^{-t})}}{1 - \frac{(1-e^{-t})}{1+e^{-t}}z}.$$

Let $b_t = \frac{1-e^{-t}}{1+e^{-t}}$, and substitute to obtain

$$\begin{aligned}\psi^{-1}\circ\varphi_{-t}\circ\psi(z) &= \frac{z-b_t}{1-b_t z}\\ &:= \mathcal{X}_t(z).\end{aligned}$$

Further, we apply density of polynomials in $\mathcal{B}_0^\alpha(\mathbb{D})$ to prove that for $f^* \in \mathcal{B}_0^\alpha(\mathbb{D})$, we have $\|C_{\mathcal{X}t}f^* - f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})} \to 0$ as $t \to 0$.

By the definition of the norm, we have

$$\lim_{t\to 0^+}\|C_{\mathcal{X}t}f^* - f^*\|_{\mathcal{B}^\alpha(\mathbb{D})} = \lim_{t\to 0^+}|(C_{\mathcal{X}t}f^* - f^*)(0)| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha|(C_{\mathcal{X}t}f^* - f^*)'(z)|.$$

Let $f^*(z) = z^n$ and $z \in \mathbb{D}$.

We need to show that $\|\left(C_{\mathcal{X}_t}f^* - f^*\right)\|_{\mathcal{B}_1^\alpha(\mathbb{D})} \to 0,$ as $t \to 0$.

Since

$$C_{\mathcal{X}_t}z^n - z^n = (\mathcal{X}_t(z))^n - z^n, n \geq 1,$$

differentiating $(\mathcal{X}_t(z))^n - z^n$ with respect to z, we obtain

$$\begin{aligned}(C_{\mathcal{X}_t}f^* - f^*)'(z) &= n(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - nz^{n-1}\\ &= n[(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - z^{n-1}].\end{aligned}$$

Substituting for

$$\mathcal{X}_t(z) = \frac{z-b_t}{1-b_t z}$$

and

$$\begin{aligned}\mathcal{X}_t'(z) &= \frac{(1-b_tz)1-(z-b_t)(-b_t)}{(1-b_tz)^2}\\ &= \frac{(1-b_t^2)}{(1-b_tz)^2},\end{aligned}$$

we obtain

$$\begin{aligned}(C_{\mathcal{X}_t}f^*-f^*)'(z) &= n\left[\left(\frac{z-b_t}{1-b_tz}\right)^{n-1}\frac{(1-b_t^2)}{(1-b_tz)^2}-z^{n-1}\right]\\ &= n\left[\frac{(z-b_t)^{n-1}(1-b_t^2)}{(1-b_tz)^{n-1}(1-b_tz)^2}-z^{n-1}\right]\\ &= n\left[\frac{(z-b_t)^{n-1}(1-b_t^2)-z^{n-1}(1-b_tz)^{n+1}}{(1-b_tz)^{n+1}}\right].\end{aligned}$$

It therefore follows that $\lim_{t\to0^+}\|C_{\mathcal{X}t}f^*-f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})}$ is equivalent to

$$\lim_{t\to0^+}\left((\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|n\left[\frac{(z-b_t)^{n-1}(1-b_t^2)-z^{n-1}(1-b_tz)^{n+1}}{(1-b_tz)^{n+1}}\right]\right|\right).$$

Now, let $b_t\to0$ as $t\to0$, we obtain

$$\begin{aligned}\lim_{t\to0^+}\|C_{\mathcal{X}t}f^*-f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\left|n[z^{n-1}-z^{n-1}]\right|\\ &= 0.\end{aligned}$$

Since $\lim_{t\to0^+}\|(C_{\mathcal{X}_t}f^*-f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})}=0$, it follows that

$$\lim_{t\to0^+}\left(\|C_{\varphi_t}f-f\|_{\mathcal{B}_1^\alpha(\mathbb{U})}\right)=0.$$

Therefore $\|C_{\varphi_t}f-f\|_{\mathcal{B}^\alpha(\mathbb{U})}=|\varphi_tf(i))-f(i)|+\|C_{\varphi_t}f-f\|_{\mathcal{B}_1^\alpha(\mathbb{U})}\to0$ as t $\to0$, as desired. □

In the next proposition, we compute the infinitesimal generator and determine the domain of the composition semigroup in equation (4.1).

Proposition 4.3. *The infinitesimal generator* Γ *of* $(C_{\varphi_t})_{t\geq0}$ *on* $\mathcal{B}_0^\alpha(\mathbb{U})$ *is given by* $\Gamma f(z)=-zf'(z)$ *with the domain dom* $(\Gamma)=\{f\in\mathcal{B}_0^\alpha(\mathbb{U}):zf'(z)\in\mathcal{B}_0^\alpha(\mathbb{U})\}$.

Proof. Using the definition of the infinitesimal generator Γ of $(C_{\varphi_t})_{t\geq0}$, for $f\in\mathcal{B}_0^\alpha(\mathbb{U})$ we have

$$\begin{aligned}\Gamma f(z) &= \lim_{t\to0^+}\frac{C_{\varphi_t}f(z)-f(z)}{t}\\ &= \lim_{t\to0^+}\frac{f\left(e^{-t}z\right)-f(z)}{t}\\ &= \frac{\partial}{\partial t}f(e^{-t}z)\Big|_{t=0}\\ &= -zf'(z).\end{aligned}$$

This implies that $\Gamma f(z) = -zf'(z)$ and therefore $\mathrm{dom}(\Gamma) \subseteq \{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : zf' \in \mathcal{B}_0^\alpha(\mathbb{U})\}$. To prove reverse inclusion, we let $f \in \mathcal{B}_0^\alpha(\mathbb{U})$ be such that $zf' \in \mathcal{B}_0^\alpha(\mathbb{U})$. Then for $z \in \mathbb{U}$,

$$\begin{aligned}\frac{C_{\varphi_t}f(z) - f(z)}{t} &= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}(C_{\varphi_s}f(z))ds \\ &= \frac{1}{t}\int_0^t (-e^{-s}zf'(e^{-s}z))ds \\ &= \frac{1}{t}\int_0^t C_{\varphi_s}F(z)ds, \text{ where } F(z) = -zf'(z).\end{aligned}$$

Since $F(z)$ is a function in $\mathcal{B}_0^\alpha(\mathbb{U})$, it remains to show that the limit of $F(z)$ exist in $\mathcal{B}_0^\alpha(\mathbb{U})$. Thus

$$\lim_{t\to 0^+} \frac{C_{\varphi_s}f(z) - f(z)}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t C_{\varphi_s}F(z)ds.$$

By strong continuity of $(C_{\varphi_s})_{s\geq 0}$ we have

$$\frac{1}{t}\int_0^t \|C_{\varphi_s}F - F\|ds \to 0 \text{ as } t \to 0^+.$$

Hence

$$\{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : zf' \in \mathcal{B}_0^\alpha(\mathbb{U})\} \subseteq \mathrm{dom}(\Gamma).$$

This completes the proof. □

4.2. **Translation group.** In this case the automorphisms are of the form $\varphi_t(z) = z + kt$, where $z \in \mathbb{U}$ and $k, t \in \mathbb{R}$ with $k \neq 0$. As noted in [3], we can consider the self analytic maps of $\mathbb{U}$ of the form

$$\varphi_t(z) = z + t. \tag{7}$$

The composition semigroup induced by translation group on $\mathcal{B}_0^\alpha(\mathbb{U})$ is given by

$$C_{\varphi_t}f(z) = f(z + t). \tag{8}$$

The proof of our results given in equation (8) as a group on $\mathcal{B}_0^\alpha(\mathbb{U})$ is basic, we therefore omit the details.

We shall now prove that the composition semigroup in equation (8), fails to be an isometry on $\mathcal{B}_0^\alpha(\mathbb{U})$.

Proposition 4.4. *The operator C_{φ_t} fails to be an isometry on $\mathcal{B}_0^\alpha(\mathbb{U})$.*

Proof. By norm definition, we have

$$\begin{aligned}\|C_{\varphi_t}f\|_{\mathcal{B}^\alpha(\mathbb{U})} &= |C_{\varphi_t}f(i)| + \sup_{z\in\mathbb{U}} \Im(z)^\alpha |\left(C_{\varphi_t}f\right)'(z)| \\ &= |f(i+t)| + \sup_{z\in\mathbb{U}} \Im(z)^\alpha |f'(z+t)|.\end{aligned}$$

Now by change of variables: Let $z + t = \omega$ then $z = \omega - t$, and $\Im(z) = \Im(\omega)$. Therefore,

$$\|C_{\varphi_t}f\|_{\mathcal{B}^\alpha(\mathbb{U})} = |f(i+t)| + \sup_{\omega\in\mathbb{U}} \Im(\omega)^\alpha |f'(\omega)|. \tag{9}$$

The right hand side of equation (9) is not equal to the norm $\|f\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$ for any $t > 0$. This implies that (8) is not an isometry on $\mathcal{B}_0^{\alpha}(\mathbb{U})$. This completes the proof. □

In the following results, we investigate the strong continuity of the composition semigroup in equation (8) on $\mathcal{B}_0^{\alpha}(\mathbb{U})$.

Proposition 4.5. *The operator C_{φ_t} is strongly continuous on $\mathcal{B}_0^{\alpha}(\mathbb{U})$.*

Proof. We need to show that $\|C_{\varphi_t}f - f\|_{\mathcal{B}^{\alpha}(\mathbb{U})} \to 0$ as $t \to 0$. This approach is similar to (7). We omit the details. We compute the automorphism of the unit disc $\mathbb{D}$, denoted by $\mathcal{X}_t$ as follows

$$\begin{aligned} \mathcal{X}_t(z) &= \psi^{-1}\left(\varphi_t\left(\psi(z)\right)\right) \\ &= \psi^{-1}\left(\varphi_t\left(\frac{i(1+z)}{1-z}\right)\right) \\ &= \psi^{-1}\left(\frac{i(1+z)}{1-z} + t\right). \end{aligned}$$

Since the inverse of Cayley transform is given by $\psi^{-1} = \frac{z-i}{z+i}$, we substitute to obtain

$$\begin{aligned} \mathcal{X}_t &= \frac{\frac{i(1+z)}{1-z} - t - i}{\frac{i(1+z)}{1-z} - t + i} \\ &= \frac{\frac{i(1+z)}{1-z} - (t+i)}{\frac{i(1+z)}{1-z} + (i-t)}. \end{aligned}$$

We simplify further by multiplying both the numerator and denominator by $(1-z)$ to obtain

$$\begin{aligned} \mathcal{X}_t(z) &= \frac{i(1+z) + (t-i)(1-z))}{i(1+z) + (t+i)(1-z)} \\ &= \frac{(2i-t)z - t}{(2i+t) - tz}. \end{aligned}$$

By dividing both the numerator and denominator by $2i - t$, we get

$$\mathcal{X}_t = \frac{z + \frac{t}{2i-t}}{\frac{2i+t}{2i-t} - \frac{t}{2i-t}z.}$$

Letting $k_t = \frac{t}{2i-t}$ and $m_t = \frac{2i+t}{2i-t}$. We have

$$\mathcal{X}_t = \frac{z + k_t}{m_t - k_t z}.$$

Next, we apply density of polynomials in $\mathcal{B}_0^{\alpha}(\mathbb{D})$ to prove that for $f^* \in \mathcal{B}_0^{\alpha}(\mathbb{D})$, we have $\|C_{\mathcal{X}t}f^* - f^*\|_{\mathcal{B}_1^{\alpha}(\mathbb{D})} \to 0$ as $t \to 0$.

$$\lim_{t\to 0^+} \|C_{\mathcal{X}t}f^* - f^*\|_{\mathcal{B}_1^{\alpha}(\mathbb{D})} = \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha}|(C_{\mathcal{X}t}f^* - f^*)'(z)|\right).$$

Using density of polynomials in $\mathcal{B}_0^{\alpha}(\mathbb{D})$, let $f^*(z) = z^n$ and $z \in \mathbb{D}$ be such that

$$C_{\mathcal{X}t}z^n - z^n = (\mathcal{X}_t(z))^n - z^n, n \geq 1. \tag{10}$$

Now, differentiating $(\mathcal{X}_t(z))^n - z^n$ with respect to z, we get

$$\begin{aligned}(C_{\mathcal{X}t}f^* - f^*)'(z) &= n(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - nz^{n-1} \\ &= n[(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - z^{n-1}]. \end{aligned} \tag{11}$$

We also differentiate $\mathcal{X}_t = \frac{z+k_t}{m_t - k_t z}$ by quotient rule to obtain

$$\begin{aligned}\mathcal{X}_t'(z) &= \frac{(m_t - k_t z)1 - (z+k_t)(-k_t)}{(m_t - k_t z)^2} \\ &= \frac{m_t + k_t^2}{(m_t - k_t z)^2}. \end{aligned}$$

Substituting for $\mathcal{X}_t = \frac{z+k_t}{m_t + k_t z}$ and $\mathcal{X}_t'(z) = \frac{m_t - k_t^2}{(m_t - k_t z)^2}$ in equation (11) we have

$$\begin{aligned}(C_{\mathcal{X}t}f^* - f^*)'(z) &= n[(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - z^{n-1}] \\ &= n\left[\frac{(z+k_t)^{n-1}(m_t - k_t z^2) - z^{n-1}(m_t - k_t z)^{n+1}}{(m_t - k_t z)^{n+1}}\right]. \end{aligned}$$

It therefore follows that as $t \to 0$, we have

$$\begin{aligned}\|C_{\mathcal{X}t}f^* - f^*\|_{\mathcal{B}^\alpha(\mathbb{D})} &= (|(\mathcal{X}_t(0))^n - 0|) \\ &+ (\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \left|n[(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - z^{n-1}]\right| = 0. \end{aligned}$$

Therefore $\|C_{\varphi_t}f - f\|_{\mathcal{B}^\alpha(\mathbb{U})} = |\varphi_t f(i)) - f(i)| + \|C_{\varphi_t}f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0$ as t $\to 0$, as desired. This completes the proof. ☐

In the next theorem, we obtain the infinitesimal generator of the strongly continuous composition semigroup given in equation (8).

Theorem 4.6. *The infinitesimal generator* Γ *of* $(C_{\varphi_t})_{t\geq 0}$ *on* $\mathcal{B}_0^\alpha(\mathbb{U})$ *is given by* $\Gamma f(z)=f'(z)$ *with the domain dom*$(\Gamma) = \{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : f'(z) \in \mathcal{B}_0^\alpha(\mathbb{U})\}$.

Proof. Using the definition of the infinitesimal generator Γ, for $f \in \mathcal{B}_0^\alpha(\mathbb{U})$, we have;

$$\begin{aligned}\Gamma f(z) &= \lim_{t\to 0^+}\frac{f(z+t) - f(z)}{t} \\ &= \frac{\partial}{\partial t}f(z+t)\Big|_{t=0} \\ &= f'(z). \end{aligned}$$

This means that dom$(\Gamma) \subset \{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : f'(z) \in \mathcal{B}_0^\alpha(\mathbb{U})\}$.
It remains to prove the reverse inclusion. Let $f \in \mathcal{B}_0^\alpha(\mathbb{U})$ be such that $f'(z) \in \mathcal{B}_0^\alpha(\mathbb{U})$.
Then for $z \in \mathbb{U}$, we have;

$$\begin{aligned}C_{\varphi_t}f(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}f(z+s)ds \\ &= \int_0^t f'(z)ds. \end{aligned}$$

Letting $F(z) = f'(z)$, we obtain

$$C_{\varphi_t} f(z) - f(z) = \int_0^t F(z) ds.$$

This implies that $F(z) = f'(z)$ is a function of $\mathcal{B}_0^\alpha(\mathbb{U})$. It remains to show that the limit of $F(z)$ exists in $\mathcal{B}_0^\alpha(\mathbb{U})$. Since

$$\frac{C_{\varphi_t} f(z) - f(z)}{t} = \frac{1}{t}\int_0^t F(z) ds,$$

we now take limits as $t \to 0^+$ and invoke strong continuity of $(C_{\varphi_s})_{s\geq 0}$ to obtain

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t \|C_{\varphi_s} F ds - F\| = 0.$$

Hence $\mathrm{dom}(\Gamma) \supseteq \{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : f'(z) \in \mathcal{B}_0^\alpha(\mathbb{U})\}$ which completes the proof. □

5. Rotation Group

The induced composition semigroups for rotation group are defined on the analytic spaces of the unit disk. We shall therefore generate composition semigroups induced by rotation group on the generalized little Bloch space of the disc. The results obtained can then be mapped onto the upper half plane by use of Cayley transform. In this case, the self analytic maps of $\mathbb{D}$ are of the form $\varphi_t(z) = e^{ikt}z$. We consider the composition semigroup induced by the rotation group on $\mathcal{B}_0^\alpha(\mathbb{D})$ given by

$$\begin{aligned} C_{\varphi_t} f(z) &= (f \circ \varphi_t)(z) \\ &= f\left(e^{it}z\right), \end{aligned} \tag{12}$$

for all $f \in \mathcal{B}_0^\alpha(\mathbb{D})$.

It can be easily shown that $(C_{\varphi_t})_{t\geq 0}$ and $\left(C_{\varphi_{-t}}\right)_{t\geq 0}$ are semigroups on $\mathcal{B}_0^\alpha(\mathbb{D})$ thus $(C_{\varphi_t})_{t\in\mathbb{R}}$ defines a group on $\mathcal{B}_0^\alpha(\mathbb{D})$.

Moreover, this group is an isometry, as we prove in the next proposition.

Proposition 5.1. *The operator C_{φ_t} given by (12) is an isometry on $\mathcal{B}_0^\alpha(\mathbb{D})$.*

Proof. We shall prove that for each $t \in \mathbb{R}$, the group $(C_{\varphi_t})_{t\in\mathbb{R}}$ is an isometry on $\mathcal{B}_0^\alpha(\mathbb{D})$. It suffices to prove that

$$\|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{D})} = \|f\|_{\mathcal{B}^\alpha(\mathbb{D})}.$$

It follows from the definition that

$$\begin{aligned} \|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{D})} &= |C_{\varphi_t} f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |(C_{\varphi_t} f)'(z)| \\ &= |(e^{it})f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |e^{it} f'(e^{it}z)| \\ &= |f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |f'(e^{it}z)|. \end{aligned}$$

Now, let $\omega = e^{it}z$ so that $z = e^{-it}\omega$. Then;

$$\begin{aligned}
\|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{D})} &= |f(0)| + \sup_{\omega\in\mathbb{D}} \left(1 - |e^{-it}\omega|^2\right)^\alpha |f'(\omega)|) \\
&= |f(0)| + \sup_{\omega\in\mathbb{D}} (1 - |\omega|^2)^\alpha |f'(\omega)| \\
&= \|f\|_{\mathcal{B}^\alpha(\mathbb{D})}.
\end{aligned}$$

□

Theorem 5.2. *The operator C_{φ_t} given by (12) is strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{D})$.*

Proof. Since polynomials are dense in $\mathcal{B}_0^\alpha(\mathbb{D})$, it suffices to show that $(C_{\varphi_t})_{t\in\mathbb{R}}$ is strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{D})$ that is, for a polynomial $(z^n)_{n\geq 0}$ where $z \in \mathbb{D}$ we obtain

$$\lim_{t\to 0^+} \|C_{\varphi_t} z^n - z^n\|_{\mathcal{B}^\alpha(\mathbb{D})} = 0.$$

Clearly,

$$\lim_{t\to 0^+} \|C_{\varphi_t} z^n - z^n\|_{\mathcal{B}^\alpha(\mathbb{D})} = \lim_{t\to 0^+} |C_{\varphi_t} f(0) - f(0)| + \left(\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |(C_{\varphi_t} z^n - z^n)'|)\right).$$

But

$$C_{\varphi_t} z^n - z^n = (e^{int} - 1)z^n.$$

So its derivative is given by

$$(C_{\varphi_t} z^n - z^n)' = n(e^{int} - 1)z^{n-1},$$

implying that

$$\lim_{t\to 0^+} \|C_{\varphi_t} z^n - z^n\|_{\mathcal{B}^\alpha(\mathbb{D})} = \lim_{t\to 0^+} |e^{it} f(0) - f(0)| + \left(\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |nz^{n-1}||(e^{int} - 1)|)\right).$$

Hence,

$$\lim_{t\to 0^+} \|C_{\varphi_t} z^n - z^n\|_{\mathcal{B}^\alpha(\mathbb{D})} = 0 \,\text{as desired}\,.$$

□

Proposition 5.3. *The infinitesimal generator Γ of (C_{φ_t}) is given by $\Gamma f(z) = izf'(z)$ with the domain $dom(\Gamma) = \{f \in \mathcal{B}_0^\alpha(\mathbb{D}) \,:\, zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{D})\}$.*

Proof. We obtain the infinitesimal generator as follows

$$\begin{aligned}
\Gamma f(z) &= \lim_{t\to 0^+} \frac{C_{\varphi t}(z) - f(z)}{t} \\
&= \frac{\partial}{\partial t} f(e^{it}z)\Big|_{t=0} \\
&= izf'(z).
\end{aligned}$$

It therefore follows that $\mathrm{dom}(\Gamma) \subseteq \{f \in \mathcal{B}_0^\alpha(\mathbb{D})\} : zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{D})\}$. On the other hand, let $f \in \mathcal{B}_0^\alpha(\mathbb{D})\}$ be such that $zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{D})\}$, then for $z \in \mathbb{D}$ we have by the Fundamental theorem of Calculus,

$$\begin{aligned} C_{\varphi_t} f(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}(C_{\varphi_s} f(z))ds \\ &= \int_0^t ie^{is} zf'(e^{is}z)ds \\ &= \int_0^t C_{\varphi_s} F(z)ds, \end{aligned}$$

where $F(z) = izf'(z)$ is a function in $\mathcal{B}_0^\alpha(\mathbb{D})$. Thus

$$\lim_{t\to 0^+} \frac{C_{\varphi_t} f - f}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t C_{\varphi_s} F ds$$

and strong continuity of $(C_{\varphi_s})_{s\geq 0}$ implies that $\|\frac{1}{t}\int_0^t C_{\varphi_s} F ds - F\| \leq \frac{1}{t}\int_0^t \|C_{\varphi_s} F - F\| ds \to 0^+$ as $t \to 0^+$. Thus $\mathrm{dom}(\Gamma) \supseteq \{f \in \mathcal{B}_0^\alpha(\mathbb{D}) : zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{D})\}$, as desired. □

References

[1] R. F. Allen and F. Colonna, Isometries and spectral of multiplication operators on the Bloch space, Bull. Aust. Math. Soc, 79, (2009), 147-160. https://doi.org/10.48550/arXiv.0809.3278.

[2] M. Bagasa, Weighted composition groups on the little Bloch space, J. Funct. Spaces, 2020 (2020), 5480602. https://doi.org/10.1155/2020/5480602.

[3] S. Ballamoole, J.O. Bonyo, T.L. Miller, V. G. Miller, Cesaro-like operators on the Hardy and Bergman Spaces of the Half Plane, Complex Anal. Oper. Theory, 10 (2016), 187-203. https://link.springer.com/article/10.1007/s11785-015-0481-8

[4] C. Bishop, Bounded functions in the little Bloch space, Pac. J. Math. 142 (1990), 209-225. https://doi.org/10.2140/pjm.1990.142.209.

[5] N. Dunford, J.T. Schwartz, Linear operators part I, Interscience Publishers, New York, 1958. https://doi.org/:10.4236/jhepgc.2018.43031.

[6] K.J. Engel, R. Nagel, A short course on operator semigroups, Springer New York, 2006. https://doi.org/10.1007/0-387-36619-9.

[7] E.O. Gori, J.O. Bonyo, Duality of the nonreflexive Bergman space of the upper half plane and composition groups, arXiv:1901.07780 [math.FA], (2019), 1-16. https://doi.org/10.48550/arXiv.1901.07780.

[8] B.D. MacCluer, R. Zhao, Essential norms of composition operators between Bloch type spaces, Rocky Mountain J. Math. 33 (2003), 1437-1458. https://doi.org/10.1216/rmjm/1181075473.

[9] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer New York, New York, NY, 1983. https://doi.org/10.1007/978-1-4612-5561-1.

[10] S. Stevic, Characterizations of composition followed by differentiation between Bloch-type spaces. Appl. Math. Comput, 218 (2011), 4312–4316. https://doi.org/10.1016/j.amc.2011.10.004.

[11] A.G. Siskakis, Semigroups of composition operators and integral operators in the spaces of analytic functions, Ann. Acad. Sci. Fenn. Math. 38 (2013), 67-89. https://doi.org/10.5186/aasfm.2013.3806.

[12] A.G. Siskakis, Semigroups of composition operators on the spaces of analytic functions, A review, Contemp. Math. 213 (1998), 229-252. https://doi.org/10.1007/BF03322189.

[13] S. Stevic, A.K. Sharma, Composition operators between Hardy and Bloch-type spaces of the upper half-plane, Bull. Korean Math. Soc. 44 (2007), 475-482. https://doi.org/:10.4134/BKMS.2007.44.3.475.

[14] J. Zang, X. Fu, Bloch type space on the upper half plane, Bull. Korean Math. Soc. 54 (2017), 1337-1346. https://doi.org/10.4134/BKMS.b160572.

[15] Y. Cheng, T. Zhang, Y. Jiang, Composition operators and the Bloch spaces, in: 2010 Third International Conference on Information and Computing, IEEE, Wuxi, TBD, China, 2010: pp. 297-300. https://doi.org/10.1109/ICIC.2010.170.

[16] K. Zhu, Operator theory in function spaces, Marcel Dekker, Inc., New York and Basel, 1990. https://doi.org/10.1090/surv/138.

[17] K. Zhu, Bloch type spaces of analytic functions, Rocky Mountain J. Math. 23 (1993), 1143-1177. https://www.jstor.org/stable/44237763.

[18] K. Zhu, Spaces of holomorphic functions in the unit ball, Springer-Verlag, New York, 2006. https://doi.org/10.1007/0-387-27539-8.