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On a Generalization of (L^1_ω, L^p_ω)-Multipliers

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Abstract. This paper deals with a generalized aspect of multipliers for the pair (L^1_ω, L^p_ω) of Beurling spaces. Using the Fourier transform related to a Beurling weight, we give a characterization of the aforementioned multipliers. We also prove the identification of the space of the multipliers for the pair (L^1_ω, L^p_ω) with the Beurling space L^p_ω when $1 < p < \infty$.

1. Introduction

Multipliers are intensively studied by many researchers. They appear in several fields of mathematics and in various contexts, namely : mobile communication, signal processing, stochastic process, partial differential equation etc. From a theoretical point of view, we refer to the source [9] for more details about multipliers for commutative Banach algebras.
Like in [4], we are interested in the multipliers on a certain large class of Banach spaces related to a locally compact abelian group. Namely, multipliers of Beurling spaces are concerned. Some interesting publications about multipliers associated with locally compact groups are [1,6,11,12,14,18]. In [4], we study the multipliers on the weighted group algebra $\mathcal{L}^1_\omega(G)$ which is the Banach space $L^1_\omega(G)$ endowed with a generalized convolution product $*_\omega$ which depends on the weight ω. This generalized convolution product first appeared in [10]. The authors in [4] characterized the multipliers on this weighted group algebra.

The present paper is the continuation of the study started in [4]. We consider a generalization of the multipliers for the pair $(L^1_\omega(G), L^p_\omega(G))$. That is, the linear maps $T : L^1_\omega(G) \longrightarrow L^p_\omega(G))$ that commute with a certain class of generalized translation operators denoted here by Γ^s_ω. If $\omega \equiv 1$, then we recover the classical concept of multipliers. Via the weight Fourier transform, we obtain, among other results, a characterization of the multipliers for the pair $(L^1_\omega(G), L^p_\omega(G))$.

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The paper is organized as follows. In Section 2, the definition of Beurling spaces and some results from [4,7,10] are recalled. In Section 3, we state our main results.

2. Preliminaries

2.1. The Beurling spaces. Let G be a group whose neutral element is denoted by e. A *Beurling weight* on G is a continuous fonction $\omega : G \to (0, \infty)$ such that $\forall x, y \in G$,

$$\omega(xy) \leqslant \omega(x)\omega(y),$$
$$\omega(x) \geqslant 1,$$
$$\omega(e) = 1.$$

For instance, for each $\alpha \geq 0$, the function ω_α defined by

$$\omega_\alpha(x) = (1 + \|x\|)^\alpha,$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$, is a Beurling weight on $(\mathbb{R}^n, +)$.

Integration on G is taken with respect to a left Haar measure. Beurling spaces are defined to be

$$L^p_\omega(G) = \left\{ f : G \to \mathbb{C} \ : \ \int_G |f(x)|^p \omega(x) dx < \infty \right\}, 1 \leqslant p < +\infty.$$

The case where $p = \infty$ is defined in an obvious way by essential boundedness. The mapping

$$f \longmapsto \|f\|_{p,\omega} = \left(\int_G |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}}$$

is a norm on $L^p_\omega(G)$.

It is well-known in the mathematical litterature that $L^1_\omega(G)$ is a Banach algebra under the convolution product $*$ defined by

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy.$$

The following sufficient condition for $L^p_\omega(G)$, $1 < p < \infty$, to be a Banach algebra under the convolution product $*$ can be found in [7] : the space $L^p_\omega(G)$, $1 < p < \infty$ is Banach algebra under the convolution product $*$ if $\omega^{\frac{1}{1-p}} * \omega^{\frac{1}{1-p}} \leqslant \omega^{\frac{1}{1-p}}$. For a general background and history on Beurling spaces, we refer to [13,15].

2.2. A Generalized convolution product. In [10], the author introduced a new convolution product on $L^1_\omega(G)$ which has the particularity to depend of the weight ω. That is,

$$f *_\omega g(x) = \int_G f(y) g(y^{-1}x) \frac{\omega(y)\omega(y^{-1}x)}{\omega(x)} dy.$$

If $\omega \equiv 1$, then one recovers the usual convolution

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) dy.$$

Hence, the convolution product $*_\omega$ is a generalization of the usual convolution product. It was shown that $L^1_\omega(G)$ is a Banach algebra under this new convolution product [10]. We denote by $\mathcal{L}^1_\omega(G)$ this new Banach algebra ; in other words $\mathcal{L}^1_\omega(G) = (L^1_\omega(G), \|\cdot\|_{1,\omega}, *_\omega)$.

For $s \in G$, define the operator Γ^s_ω by

$$\Gamma^s_\omega f(x) = \frac{\tau_s M_\omega f(x)}{\omega(x)},\ f \in L^1_\omega(G),$$

where M_ω is the multiplication operator defined by

$$(M_\omega f)(x) = \omega(x) f(x)$$

and τ_s is the translation operator defined by

$$(\tau_s f)(x) = f(s^{-1}x).$$

The operator Γ^s_ω appears first in [4] for the study of the multipliers for the algebra $\mathcal{L}^1_\omega(G)$. A linear map $T : \mathcal{L}^1_\omega(G) \to \mathcal{L}^1_\omega(G)$ is called a multiplier if T commutes with the operators Γ^s_ω for all $s \in G$. Since the operator Γ^s_ω is a generalization of the translation operator τ_s, the latter notion of multiplier covers the classical one related to commutation with translations.

The natural next step is to investigate the multipliers for the pair $(L^1_\omega(G), L^p_\omega(G))$. This is the main purpose of the present article.

We denote by $M^1_\omega(G)$ the Banach space of all complex bounded regular Borel measures μ on G such that

$$\|\mu\|_\omega = \int_G \omega(x) d|\mu|(x) < \infty. \tag{1}$$

We write $M^1(G)$ in the case where $\omega \equiv 1$. For $\mu, \nu \in M^1_\omega(G)$, define $\mu *_\omega \nu$ by

$$\mu *_\omega \nu(f) = \int_G \int_G f(xy) \frac{\omega(x)\omega(y)}{\omega(xy)} d\mu(x) d\nu(y),\ f \in \mathcal{C}_c(G, \omega^{-1})$$

where $\mathcal{C}_c(G, \omega^{-1})$ is the set of complex functions f defined on G such that $f\omega^{-1}$ is of compact support. Also, define

$$\mu *_\omega f(x) = \int_G f(y^{-1}x) \frac{\omega(y)\omega(y^{-1}x)}{\omega(x)} d\mu(y)$$

for $f \in L^1_\omega(G)$ and $\mu \in M^1_\omega(G)$. Then, the Banach space $M^1_\omega(G)$ is a unital Banach algebra with respect to the convolution product $*_\omega$ and $L^1_\omega(G)$ is a closed ideal of $M^1_\omega(G)$ [10, Theorem 5.1].

2.3. **Some useful facts.** Let G be a locally compact abelian group with Pontryagin dual group $\widehat{G}$. We denote by $\widehat{M^1(G)}$ the collection of all the Fourier-Stieltjes transforms of elements of $M^1(G)$. That is,

$$\widehat{M^1(G)} = \{\widehat{\mu} : \mu \in M^1(G)\}$$

where $\widehat{\mu}$ is defined by

$$\widehat{\mu}(\gamma) = \int_G \overline{\gamma(x)} d\mu(x), \gamma \in \widehat{G}.$$

For a function $f \in L^1_\omega(G)$, the Fourier transform of f, denoted $\mathcal{F}f$ or $\widehat{f}$, is defined by

$$(\mathcal{F}f)(\gamma) := \widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}dx$$

The following theorems will play an important role.

Theorem 2.1 ([3] or [17]). *Let G be a locally compact abelian group and let φ be a complex function on $\widehat{G}$. Then, the following assertions are equivalent.*

(1) $\varphi \in \widehat{M^1(G)}$ *and* $\|\varphi\|_\infty \leqslant C$.

(2) φ *is continuous and there exists a constant* $C > 0$ *such that*

$$\left|\sum_{i=1}^{n} c_i\varphi(\gamma_i)\right| < C\left\|\sum_{i=1}^{n} c_i\gamma_i(\cdot)\right\|_\infty \tag{2}$$

for all positive integer n and all choices of $c_i \in \mathbb{C}$ and $\gamma_i \in \widehat{G}$, $i = 1, 2, \cdots, n$.

Moreover, if $\varphi = \widehat{\mu}$, then $\|\mu\|$ is the smallest constant C for which (2) holds.

Theorem 2.2. *([9, page 252]) Let G be a locally compact abelian group. Then, for each compact $K \subset \widehat{G}$ and $\varepsilon > 0$, given an open set U containing K, there exists a function $f \in L^1(G)$ such that $0 \leqslant \widehat{f}(\gamma) \leqslant 1$ if $\gamma \in \widehat{G}$, $\widehat{f}(\gamma) = 1$ if $\gamma \in K$, $\widehat{f}(\gamma) = 0$ if $\gamma \notin U$ and $\|f\| \leqslant \varepsilon + 1$. In particular, given any open set $U \subset \widehat{G}$ with compact closure, it is possible to find $f \in L^1(G)$ such that $\widehat{f}(\gamma) = 1$ if $\gamma \in U$.*

Theorem 2.3. *([5, Theorem 3.2]) Let G be a locally compact group. Let $f \in L^p_\omega(G)$, $1 \leqslant p < \infty$. Then, $\forall s \in G$,*

$$[\omega(s)]^{\frac{1-p}{p}} \|f\|_{p,\omega} \leqslant \|\Gamma^s_\omega f\|_{p,\omega} \leqslant \left[\omega(s^{-1})\right]^{\frac{p-1}{p}} \|f\|_{p,\omega}. \tag{3}$$

3. Multipliers for the pair $(L^1_\omega(G), L^p_\omega(G))$

In this section, we study a generalization of the concept of multipliers. Here, the multipliers are defined with respect to the generalized translation operators Γ^s_ω. Throughout this section, we assume that G is a locally compact abelian group. A look at Theorem 2.3 shows that $f \in L^p_\omega(G)$ if and only if $\Gamma^s_\omega f \in L^p_\omega(G)$. That is, the spaces $L^p_\omega(G)$ are stable under the action of the operators Γ^s_ω. Therefore, we are able to define a concept of multiplier in the framework of this study.

Definition 3.1. *A linear operator $T : L^1_\omega(G) \longrightarrow L^p_\omega(G)$ is said to be a* multiplier *if T commutes with all the operators Γ^s_ω, $s \in G$. That is,*

$$\forall s \in G,\ T\Gamma^s_\omega = \Gamma^s_\omega T.$$

We denote by $\mathcal{M}^{1,p}_\omega(G)$ the set of such multipliers. We denote by $\|T\|$ the operator norm of $T \in \mathcal{M}^{1,p}_\omega(G)$.

We will use the fact that for $1 < p < \infty$, the following identification holds [7] :

$$(L^p_\omega(G))' = L^q_w(G)$$

with $\frac{1}{p} + \frac{1}{q} = 1$ and $w = \omega^{-\frac{q}{p}}$. From that, one may deduce that for $1 < p < \infty$, the space $L^p_\omega(G)$ is a reflexive space.

For $f \in L^1_\omega(G)$, define the Fourier transform of f by

$$\mathcal{F}_\omega(f)(\gamma) = \int_G f(x)\overline{\gamma(x)}\omega(x)dx,\ \gamma \in \widehat{G}.$$

In [4], the following convolution result was proved.

$$\forall f, g \in L^1_\omega(G),\ \mathcal{F}_\omega(f *_\omega g) = \mathcal{F}_\omega(f)\mathcal{F}_\omega(g).$$

Set

$$\mathcal{F}_\omega(L^1_\omega(G)) = \left\{\mathcal{F}_\omega(f)\ :\ f \in L^1_\omega(G)\right\}.$$

Let us remark that functions in $\mathcal{F}_\omega(L^1_\omega(G))$ are continuous and vanished at infinity by the Riemann–Lebesgue theorem. We fit out the space $\mathcal{F}_\omega(L^1_\omega(G))$ with the norm defined by

$$\|\mathcal{F}_\omega(f)\| = \|f\|_{1,\omega}, f \in L^1_\omega(G).$$

Then, we have the following result.

Theorem 3.2. *The space $\mathcal{F}_\omega(L^1_\omega(G))$ is a Banach algebra for the pointwise multiplication.*

Proof. Let $(\mathcal{F}_\omega(f_n))$ be a Cauchy sequence in $\mathcal{F}_\omega(L^1_\omega(G))$. Let $p, q \in \mathbb{N}$. The equality

$$\|\mathcal{F}_\omega(f_p) - \mathcal{F}_\omega(f_q)\| = \|f_p - f_q\|_{1,\omega}$$

and the fact that $(L^1_\omega(G), \|\cdot\|_{1,\omega})$ is a Banach space show that there exists $f \in L^1_\omega(G)$ such that (f_n) converges to f in $L^1_\omega(G)$. Now, $\|\mathcal{F}_\omega(f_n) - \mathcal{F}_\omega(f)\| = \|f_n - f\|_{1,\omega}$. Thus, $(\mathcal{F}_\omega(f_n))$ converges to $(\mathcal{F}_\omega(f))$ in $\mathcal{F}_\omega(L^1_\omega(G))$. Thus, the space $\mathcal{F}_\omega(L^1_\omega(G))$ is a Banach space. Moreover,

$$\begin{aligned} \|\mathcal{F}_\omega(f)\mathcal{F}_\omega(g)\| &= \|\mathcal{F}_\omega(f *_\omega g)\| \\ &= \|f *_\omega g\|_{1,\omega} \\ &\leqslant \|f\|_{1,\omega}\|g\|_{1,\omega} = \|\mathcal{F}_\omega(f)\|\|\mathcal{F}_\omega(g)\|. \end{aligned}$$

Thus, the space $\left(\mathcal{F}_\omega(L^1_\omega(G))), \cdot, \|\cdot\|_{1,\omega}\right)$ is a Banach algebra. □

For $f \in L^p_\omega(G)$ and $h \in L^q_w(G)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we set

$$\langle f, h\rangle_\omega = \int_G f(x)h(x^{-1})\omega(x)dx.$$

Theorem 3.3. *Let G be a locally compact abelian group. Let $T : L^1_\omega(G) \longrightarrow L^p_\omega(G)$ be a bounded linear transformation. Then, $T \in \mathcal{M}^{1,p}_\omega(G)$ if and only if there exists a unique element φ such that $Tg = \varphi *_\omega g$ for all $g \in L^1_\omega(G)$, where $\varphi \in M^1_\omega(G)$ if $p = 1$ and $\varphi \in L^p_\omega(G)$ if $1 < p < \infty$.*

Proof. (1) Suppose $p = 1$. Let $T \in \mathcal{M}^{1,1}_\omega(G)$. In [4, Proposition 5.4], it was shown that $T \in \mathcal{M}^{1,1}_\omega(G)$ if and only if there exists a unique function $\mathcal{B}$ defined on $\widehat{G}$ such that $\mathcal{F}_\omega(Tf) = \mathcal{B}\mathcal{F}_\omega(f)$ for all $f \in L^1_\omega(G)$. Clearly, $\mathcal{B}\mathcal{F}_\omega(f) \in \mathcal{F}_\omega(L^1_\omega(G))$. Therefore, the function $\mathcal{B}\mathcal{F}_\omega(f)$ is continuous for all $f \in L^1_\omega(G)$ (the Fourier transform of a function is a continuous function). Moreover, for each open set in $\widehat{G}$ with compact closure, there exists a function $f \in L^1_\omega(G)$ such that $\mathcal{F}_\omega(f)$ is constant on U [9, F.7e]. Thus, $\mathcal{B}$ is continuous on $\widehat{G}$.

Let $\varepsilon > 0$ and let $\gamma_1, \gamma_2, \ldots.., \gamma_n \in \widehat{G}$. Via Theorem 2.2, we can choose $g \in L^1(G)$ such that $\|\mathcal{F}(g)\| = \|g\|_1 < 1 + \varepsilon$ and $\mathcal{F}(g)(\gamma_i) = 1$, $i = 1, 2, 3, \ldots.., n$. Now, set $f = \dfrac{g}{\omega}$. Then, $f \in L^1_\omega(G)$, $\|\mathcal{F}_\omega(f)\| = \|f\|_{1,\omega} < 1 + \varepsilon$ and $\mathcal{F}_\omega(f)(\gamma_i) = 1$, $i = 1, 2, 3, \ldots.., n$.

For $z_i \in \mathbb{C}$, $i = 1, 2, \ldots.., n$, one has

$$\begin{aligned}\left|\sum_{i=1}^n z_i \mathcal{B}(\gamma_i)\right| &= \left|\sum_{i=1}^n z_i \mathcal{F}_\omega(f)(\gamma_i)\right| = \left|\int_G \left[\sum_{i=1}^n z_i \overline{\gamma_i(x)}\right] f(x)\omega(x)dx\right| \\ &\leqslant \|f\|_{1,\omega} \left\|\sum_{i=1}^n z_i \overline{\gamma_i}\right\|_\infty \\ &< (1+\varepsilon) \left\|\sum_{i=1}^n z_i \overline{\gamma_i}\right\|_\infty .\end{aligned}$$

Since ε is chosen arbitrarily, it follows that

$$\left|\sum_{i=1}^n z_i \mathcal{B}(\gamma_i)\right| < \left\|\sum_{i=1}^n z_i \overline{\gamma_i}\right\|_\infty .$$

We conclude via Theorem 2.1 (with $C = 1$) that there exists a unique bounded measure μ such that $\mathcal{B} = \mathcal{F}(\mu)$. Now, if we set $\varphi = \omega^{-1}\mu$, then $\varphi \in M^1_\omega(G)$ and $\mathcal{B} = \mathcal{F}_\omega(\varphi)$. Therefore,

$$\mathcal{F}_\omega(Tf) = \mathcal{F}_\omega(\varphi)\mathcal{F}_\omega(f) = \mathcal{F}_\omega(\varphi *_\omega f).$$

Since the Fourier transform is injective, it follows that $Tf = \varphi *_\omega f$.

(2) Assume that $1 < p < \infty$. Let $T \in \mathcal{M}^{1,p}_\omega(G)$. The weighted group algebra $(L^1_\omega(G), \|\cdot\|_{1,\omega}, *_\omega)$ has a bounded approximate identity [10, Theorem 2.2]. Let $\{\upsilon_n\}$ be a bounded approximate identity for $(L^1_\omega(G), \|\cdot\|_{1,\omega}, *_\omega)$. Let $g \in L^1_\omega(G)$. Then,

$$\begin{aligned}\|Tg - T\upsilon_n *_\omega g\|_{p,\omega} &= \|Tg - T(\upsilon_n *_\omega g)\|_{p,\omega} \\ &\leqslant \|T\|\|g - \upsilon_n *_\omega g\|_{1,\omega}.\end{aligned}$$

Since $\|g - \upsilon_n *_\omega g\|_{1,\omega}$ tends to 0 whenever n goes to ∞, then $(T\upsilon_n *_\omega g)_n$ converges to Tg in $L^p_\omega(G)$.

Moreover, $\|T v_n\|_{p,\omega} \leqslant \|T\| \|v_n\|_{1,\omega} = \|T\|$. Therefore, $\{T v_n\}$ lies in a norm bounded subset of $L^p_\omega(G) = \left(L^q_w(G)\right)'$. So, by Alaoglu's Theorem ([16, page 299] or [9, Theorem D.4.3.]) and the reflexivity of $L^p_\omega(G)$, we see that there exists a subnet $\{T v_m\}$ of $\{T v_n\}$ and $\varphi \in L^p_\omega(G)$ such that $\{T v_m\}$ converges to φ in the weak*-topology. That is, $\lim_m \langle T v_m, u\rangle_\omega = \langle \varphi, u\rangle_\omega$ for all $u \in L^q_w(G)$. Then, for $h, g \in \mathcal{C}_c(G)$, we have

$$\begin{aligned} \langle Th, g\rangle_\omega &= \lim_m \langle T v_m *_\omega h, g\rangle_\omega \\ &= \lim_m \langle T v_m, (h *_\omega \frac{g}{\omega})\omega\rangle_\omega \\ &= \langle \varphi, (h *_\omega \frac{g}{\omega})\omega\rangle_\omega \\ &= \langle \varphi *_\omega h, g\rangle_\omega . \end{aligned}$$

However, $\mathcal{C}_c(G)$ is norm dense in $L^q_w(G)$. Therefore, $Th = \varphi *_\omega h$ for each $h \in \mathcal{C}_c(G)$. Moreover, $\mathcal{C}_c(G)$ is norm dense in $L^1_\omega(G)$. Thus, $Th = \varphi *_\omega h$ for all $h \in L^1_\omega(G)$.

(3) Conversely, let $1 \leq p < \infty$. Assume that there exists a measure $\mu \in M^1_\omega(G)$ or a function $\varphi \in L^p_\omega(G)$ such that $Th = \varphi *_\omega h$ for all $h \in L^1_\omega(G)$. Then,

$$\begin{aligned} (T\Gamma^s_\omega)h &= T(\Gamma^s_\omega h) \\ &= \varphi *_\omega \Gamma^s_\omega h \\ &= \Gamma^s_\omega(\varphi *_\omega h) \\ &= \Gamma^s_\omega(Th) = (\Gamma^s_\omega T)h. \end{aligned}$$

Thus, $T \in \mathcal{M}^{1,p}_\omega(G)$.

(4) Concerning the uniqueness statement, let us consider φ and ψ be such that $Th = \varphi *_\omega h = \psi *_\omega h$ for all $h \in L^1_\omega(G)$. Then, using the Fourier transform, we obtain $\mathcal{F}_\omega(\varphi)\mathcal{F}_\omega(h) = \mathcal{F}_\omega(\psi)\mathcal{F}_\omega(h)$. So, $\mathcal{F}_\omega(\varphi) = \mathcal{F}_\omega(\psi)$. Finally, $\varphi = \psi$ by the injectivity of the Fourier transform.

□

Theorem 3.4. *Let G be a locally compact abelian group. Then, $\mathcal{M}^{1,1}_\omega(G)$ is isometrically isomorphic to $M^1_\omega(G)$.*

Proof. We have seen in Theorem 3.3 that $T \in \mathcal{M}^{1,1}_\omega(G)$ if and only if there exists a unique measure $\mu \in M^1_\omega(G)$ such that $Tf = \mu *_\omega f$ for all $f \in L^1_\omega(G)$. Then, the mapping $T \longmapsto \mu$ defines a bijection from $\mathcal{M}^{1,1}_\omega(G))$ onto $M^1_\omega(G)$. Moreover,

$$\begin{aligned} \|Tf\|_{1,\omega} &= \|\mu *_\omega f\|_{1,\omega} \\ &= \int_G \left| \int_G f(y^{-1}x) \frac{\omega(y^{-1}x)\omega(y)}{\omega(x)} d\mu(y) \right| \omega(x) dx \\ &\leqslant \int_G \int_G \left| f(y^{-1}x) \right| \frac{\omega(y^{-1}x)\omega(y)}{\omega(x)} \omega(x) d|\mu|(y) dx \end{aligned}$$

$$\leqslant \left(\int_G |f(x)|\omega(x)dx\right)\left(\int_G \omega(y)d|\mu|(y)\right)$$
(invariance of the Haar measure)
$$\leqslant \|f\|_{1,\omega}\|\mu\|_\omega.$$

Then, $\|T\|_{1,\omega} \leqslant \|\mu\|_\omega$.

In the converse, for $\gamma_1, \cdots, \gamma_n \in \widehat{G}$, $z_1, \cdots, z_n \in \mathbb{C}$, and $\varepsilon > 0$, let us choose $f \in L^1_\omega(G)$ such that $\|\mathcal{F}_\omega(f)\| = \|f\|_{1,\omega} < 1 + \varepsilon$ and $\mathcal{F}_\omega(f)(\gamma_i) = 1$, $i = 1, 2, 3, \ldots.., n$. Then,

$$\begin{aligned} \left|\sum_{i=1}^n z_i \mathcal{F}_\omega(\mu)(\gamma_i)\right| &= \left|\sum_{i=1}^n z_i \mathcal{F}_\omega(\mu)(\gamma_i)\mathcal{F}_\omega(f)(\gamma_i)\right| \\ &= \left|\sum_{i=1}^n z_i \mathcal{F}_\omega(\mu *_\omega f)(\gamma_i)\right| \\ &= \left|\sum_{i=1}^n z_i \mathcal{F}_\omega(Tf)(\gamma_i)\right| \\ &\leqslant \|T\|(1+\varepsilon)\left\|\sum_{i=1}^n z_i \overline{\gamma_i}\right\|_\infty . \end{aligned}$$

Since ε is arbitrary, then $\|T\| \geqslant \|\mu\|_\omega$ by the use of Theorem 2.1 applied with $\mathcal{F}_\omega$ instead of $\mathcal{F}$. □

Theorem 3.5. *Let G be a locally compact abelian group. Let $1 \leqslant p < \infty$. If $f \in L^p_\omega(G)$, then the mapping $s \longmapsto \Gamma^s_\omega f$ is continuous from G into $L^p_\omega(G)$.*

Proof. The set of complex continuous functions on G with compact support $\mathcal{C}_c(G)$ is dense in $L^p_\omega(G)$ under the norm $\|\cdot\|_{p,\omega}$. Let $\varepsilon > 0$. Consider $g \in \mathcal{C}_c(G)$ and set $C_1 = \operatorname{supp}(g)$. Let us choose a compact neighborhood C_2 of the neutral element e. Set $C = C_1 \cup C_2 \cup (C_1C_2)$. We have for $s \in C_2$,

$$\begin{aligned} \|\Gamma^s_\omega g - g\|^p_{p,\omega} &= \int_C |\Gamma^s_\omega g(x) - g(x)|^p \omega(x)dx \\ &\leqslant \int_C |g(s^{-1}x)\omega(s^{-1}x) - g(x)\omega(x)|^p dx. \end{aligned}$$

The mapping $x \longmapsto (g\omega)(x)$ is uniformly continuous on G. Thus, there exists a neighborhood U of e which we may assume to be contained in C_2, such that

$$\forall s \in U,\ |(g\omega)(s^{-1}x) - (g\omega)(x)|^p < \frac{\varepsilon^p}{|C|}$$

where $|C|$ is the measure of the compact set C. Then, for $s \in U$, we have

$$\|\Gamma^s_\omega g - g\|^p_{p,\omega} \leqslant \int_C |(g\omega)(s^{-1}x) - (g\omega)(x)|^p dx < \frac{\varepsilon|C|}{|C|} = \varepsilon.$$

We will show the claim for $f \in L^p_\omega(G)$. Let K be a compact neighborhood of e. Since $\mathcal{C}_c(G)$ is dense in $L^p_\omega(G)$, then there exists $g \in \mathcal{C}_c(G)$ such that

$$\|f - g\|_{p,\omega} < \frac{\varepsilon}{3}.$$

There exists a compact neighborhood V of e which we may assume to be contained in K, such that $\|\Gamma_\omega^s g - g\|_{p,\omega} < \frac{\varepsilon}{3}$ for all $s \in V$.
Then, for $s \in V$, we have

$$\begin{aligned}
\|\Gamma_\omega^s f - f\|_{p,\omega} &\leqslant \|\Gamma_\omega^s f - \Gamma_\omega^s g\|_{p,\omega} + \|\Gamma_\omega^s g - g\|_{p,\omega} + \|f - g\|_{p,\omega} \\
&< \frac{1}{\omega(s)} \int_G |(f-g)(t)|^p \omega(st)dt + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\leqslant \frac{1}{\omega(s)} \int_G |(f-g)(t)|^p \omega(s)\omega(t)dt + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \int_G |(f-g)(t)|^p \omega(t)dt + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\|f-g\|_{p,\omega} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

□

Theorem 3.6. *Let G be a locally compact abelian group. Let $f \in L_\omega^p(G)$, $1 \leqslant p < \infty$. Let $\varepsilon > 0$. Then, there exists a positive function $g \in \mathcal{C}_c(G)$ such that $\|g\|_{1,\omega} = 1$ and $\|f *_\omega g - f\|_{p,\omega} \leqslant \varepsilon$.*

Proof. Let $f \in L_\omega^p(G)$ and $\varepsilon > 0$. According to Theorem 3.5, the mapping $s \longmapsto \Gamma_\omega^s f$ is continuous at the neutral element e of G. Then, there exists a compact neighborhood K of e such that

$$\|\Gamma_\omega^s f - f\|_{p,\omega} \leqslant \varepsilon,\ \forall s \in K.$$

Consider a positive function g such that $\text{supp}(g) \subset K$ and $\int_G g(y)\omega(y)dy = 1$ (that is $\|g\|_{1,\omega} = 1$). Then, $|(f *_\omega g)(x) - f(x)| \leqslant \int_G |\Gamma_\omega^s f(x) - f(x)| g(s)\omega(s)ds$. Using the Hölder's inequality with respect to the measure $g(s)\omega(s)ds$, one has

$$\begin{aligned}
|(f *_\omega g)(x) - f(x)| &\leqslant \left(\int_G |\Gamma_\omega^s f(x) - f(x)|^p g(s)\omega(s)ds\right)^{\frac{1}{p}} \left(\int_G g(s)\omega(s)ds\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_G |\Gamma_\omega^s f(x) - f(x)|^p g(s)\omega(s)ds\right)^{\frac{1}{p}},
\end{aligned}$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\begin{aligned}
\|f *_\omega g - f\|_{p,\omega}^p &= \int_G |(f *_\omega g)(x) - f(x)|^p \omega(x)dx \\
&\leqslant \iint_{G\times G} |\Gamma_\omega^s f(x) - f(x)|^p g(s)\omega(s)ds\omega(x)dx \\
&\leqslant \int_G \|\Gamma_\omega^s f - f\|_{p,\omega}^p g(s)\omega(s)ds \\
&= \|\Gamma_\omega^s f - f\|_{p,\omega}^p \int_G g(s)\omega(s)ds = \|\Gamma_\omega^s f - f\|_{p,\omega}^p \leqslant \varepsilon^p.
\end{aligned}$$

Thus, $\|f *_\omega g - f\|_{p,\omega} \leqslant \varepsilon$. □

Theorem 3.7. *Let G be a locally compact abelian group. If $T \in \mathcal{M}_\omega^{1,p}(G)$, then $\|Tf\|_p \leqslant \|T\|\|f\|_1$. In other words, $T : L_\omega^1(G) \longrightarrow L_\omega^p(G)$ is a bounded operator.*

Proof. Let $\varepsilon > 0$. Via Theorem 3.6, there exists a positive function g in $\mathcal{C}_c(G)$ such that $\int_G g(t)\omega(t)dt = 1$ and $\|g *_\omega Tf - Tf\|_p \leqslant \varepsilon$ because $\|\cdot\|_p \leqslant \|\cdot\|_{p,\omega}$. We have,

$$\|g *_\omega Tf - Tf\|_p \geqslant \|Tf\|_p - \|g *_\omega Tf\|_p.$$

Therefore,

$$\begin{aligned}\|Tf\|_p &\leqslant \|g *_\omega Tf\|_p + \varepsilon \\ &= \|Tg *_\omega f\|_p + \varepsilon \\ &\leqslant \|Tg\|_p\|f\|_1 + \varepsilon \\ &\leqslant \|Tg\|_{p,\omega}\|f\|_1 + \varepsilon \\ &\leqslant \|T\|\|\|_{1,\omega}\|f\|_1 + \varepsilon \\ &= \|T\|\|f\|_1 + \varepsilon.\end{aligned}$$

Since the latter inequality is true for arbitrary $\varepsilon > 0$, then we obtain $\|Tf\|_p \leqslant \|T\|\|f\|_1$. □

For a function f in $L^p_\omega(G)$, we define the convolution operator T_f by

$$T_f g = f *_\omega g.$$

Theorem 3.8. *Let G be a locally compact abelian group. Let $1 < p < \infty$. Let f be a function in $L^p_\omega(G)$. Then, $\|T_f\| = \|f\|_{p,\omega}$.*

Proof. Let $f \in L^p_\omega(G)$ and let $\varepsilon > 0$. From Theorem 3.6, there exits a positive function g such that $\int_G g(t)\omega(t)dt = 1$ and $\|f *_\omega g - f\|_{p,\omega} \leqslant \varepsilon$. Then,

$$\begin{aligned}\|f\|_{p,\omega} &\leqslant \varepsilon + \|f *_\omega g\|_{p,\omega} = \varepsilon + \|T_f g\|_{p,\omega} \\ &\leqslant \varepsilon + \|T_f\|\|g\|_{1,\omega} = \varepsilon + \|T_f\|.\end{aligned}$$

Thus $\|f\|_{p,\omega} \leqslant \|T_f\|$.

Let us prove the inverse inequality. Let $g \in L^1_\omega(G)$. Applying the Hölder's inequality with respect to the measure $g(y)\omega(y)dy$, one has

$$\begin{aligned}\|f * g\|^p_{p,\omega} &= \int_G |g *_\omega f|^p \omega(x)dx \\ &= \int_G \left|\int_G g(y)\Gamma^y_\omega f(x)\omega(y)\right|^p \omega(x)dx \\ &\leqslant \int_G \left[\int_G |\Gamma^y_\omega f(x)|^p |g(y)|\omega(y)dy\right]\left[\int_G |g(y)|\omega(y)dy\right]^{\frac{p}{q}} \omega(x)dx \\ &\leqslant \int_G (|f|^p *_\omega |g|)\,\omega(x)dx \left[\int_G |g(y)|\omega(y)dy\right]^{\frac{p}{q}} \\ &\leqslant \||f|^p *_\omega |g|\|_{1,\omega}\|g\|^{\frac{p}{q}}_{1,\omega} \leqslant \|f\|^p_{p,\omega}\|g\|_{1,\omega}\|g\|^{\frac{p}{q}}_{1,\omega} = \|f\|^p_{p,\omega}\|g\|^p_{1,\omega}.\end{aligned}$$

Then, $\|T_f g\|_{p,\omega}^p \leqslant \|f\|_{p,\omega}^p \|g\|_{1,\omega}^p$. Thus, $\|T_f\| \leqslant \|f\|_{p,\omega}$. □

As a consequence of Theorem 3.3 and Theorem 3.8, we have the following result.

Corollary 3.9. *Let G be a locally compact abelian group. Let $1 < p < \infty$. Then, the multipliers space $\mathcal{M}_\omega^{1,p}(G)$ and the Beurling space $L_\omega^p(G)$ are isometricaly identified by the mapping $T : f \longmapsto T_f$.*

Conclusion

In this paper, we obtain a characterization of multipliers for the pair (L_ω^1, L_ω^p) using the Fourier transform related to a Beurling weight. We also obtain the identification of the space of such multipliers with the Beurling space L_ω^p when $1 < p < \infty$. It would be interesting in the future to consider the case of the pair (L_ω^p, L_ω^q) in this framework of the weight dependent convolution.

Competing Interests

The authors declare that no competing interests exist.

References

[1] R. Akylzhanov, M. Ruzhansky, L_p-L_q multipliers on locally compact groups, J. Funct. Anal. 278 (3) (2020) 108324. https://doi.org/10.1016/j.jfa.2019.108324

[2] A. Bourouihiya, Beurling weighted spaces, product-convolution operators and the tensor product of frames, PhD thesis, University of Maryland,USA, 2006. http://www.norbertwiener.umd.edu/Research/lectures/2006/ABourouihiya_Thesis.pdf

[3] G.I. Gaudry, Multipliers of weighted Lebesgue and measure spaces, Proc. London Math. Soc. 19 (3) (1969) 327-340. https://doi.org/10.1112/plms/s3-19.2.327

[4] A. Issa, Y. Mensah, Multipliers on weighted group algebras, Gulf J. Math. 8(2) (2020) 35-45. https://doi.org/10.56947/gjom.v8i2.283

[5] A. Issa, Y. Mensah, On Beurling spaces provided with a weight dependent convolution, Funct. Anal. 1 (2022) ID 6. https://fac-seams.org/journal/index.php/fa/issue/view/1

[6] A. Issa, Y. Mensah, Multipliers for Gelfand pairs, Palest. J. Math. 10(1) (2021) 151-159. https://pjm.ppu.edu/paper/801-multipliers-gelfand-pairs

[7] A. El Kinani, A. Roukbi, A. Benazzouz, Structure d'algèbre de Banach sur l'espace à poids $L_\omega^p(G)$, Le Mathematiche. vol. LXIV Fasc. I (2009) 179-193. https://lematematiche.dmi.unict.it/index.php/lematematiche/article/view/253

[8] Yu. Kuznetsova, C. Molitor-Braun, Harmonic analysis of weighted L^p-algebras, Expo. Math. 30 (2012) 124-153. https://doi.org/10.1016/j.exmath.2012.01.002

[9] R. Larsen, An introduction to the theory of multipliers, Springer-Verlag, 1971.

[10] A. Mahmoodi, A new convolution on Beurling algebras, Math. Sci. 3 (9) (2009) 63-82. https://www.sid.ir/paper/322483/en

[11] S. Naik, K. Rajbangshi, Operators and multipliers on weighted Bergman spaces, Afr. Mat. 30 (2019) 269-277. https://doi.org/10.1007/s13370-018-0645-6

[12] S. Öztop, Multipliers on some weighted L^p-spaces, Internat. J. Math. and Math. Sci. 23 (9) (2000) 651-656. https://doi.org/10.1155/S016117120000096X

[13] T. W. Palmer, Banah algebras and the general theory of ∗-algebras, Volume 1, Algebras and Banach Algebras, Cambridge University Press, 1994.

[14] R. Radha, N. S. Kumar, Weyl transform and Weyl multipliers associated with locally compact abelian groups, J. Pseudo-Differ. Oper. Appl. 9 (2018) 229-245. https://doi.org/10.1007/s11868-017-0213-0

[15] H. Reiter, J. D. Stegeman, Classical harmonic analysis and locally compact groups, London Math. Soc., Monographs, 22, Clarendon Press, Oxford, 2009.

[16] H. I. Royden, P. M. Fitzpatrick, Real Analysis, (Fourth Edition), China Machine Press, 2010.

[17] W. Rudin, Fourier analysis on groups, New York, Interscience Publishers, Inc., 1962.

[18] V. S. Shulman, I. G. Todorov, L. Turowska, Sets of multiplicity and closable multipliers on group algebras, J. Funct. Anal. 268 (6) (2015) 1454-1508. https://doi.org/10.1016/j.jfa.2014.11.019

[19] T. Strohmer, Pseudodifferential operators and Banach algebras in mobile communications, Appl. Comput. Harmon. Anal. 20 (2006) 237-249. https://doi.org/10.1016/j.acha.2005.06.003