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Modified Viscosity Iterative Algorithm for Solving Variational Inclusion and Fixed Point Problems in Real Hilbert Space

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Abstract. This paper introduces a new iterative algorithm, called the Modified Viscosity Iterative algorithm, designed to solve problems related to Variational Inclusion and Fixed point in real Hilbert spaces. The algorithm is specifically tailored to handle Multivalued Quasi-Nonexpansive and Demicontractive operators. The convergence properties of the algorithm are analyzed and established, ensuring its effectiveness in finding solutions for complex mathematical problems in the field of optimization and equilibrium.

1. Introduction

Variational inclusion and fixed point problems involving multivalued quasi nonexpansive and demicontractive operators play a crucial role in the field of mathematics, particularly in real Hilbert spaces.

The study of variational inclusion and fixed point problems originated from the theory of optimization and nonlinear analysis, and in the mid-20^{th} century, mathematicians began investigating problems involving finding points that satisfy certain inclusion and fixed point conditions. Over time, research in this area expanded and became an essential part of functional analysis and optimization theory. They are widely-used in applications in diverse fields such as engineering, economics, physics, and computer science. They provide a framework to model and solve various real-world problems, including equilibrium problems, optimization problems, and variational inequalities.

Fixed point problems, on the other hand, deal with finding points that remain unchanged under the action of an operator. The concept of fixed points has profound implications in mathematics and its applications. A wide range of problems in analysis, differential equations, and optimization

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theory can be reduced to fixed point problems. They serve as powerful tools to prove the existence and uniqueness of solutions, compute approximations, and establish convergence properties of iterative algorithms. Whilst multivalued quasi nonexpansive operators play a pivotal role in variational inclusion and fixed point problems. These operators possess certain properties that ensure the stability and convergence of iterative algorithms. They have applications in image processing, signal estimation, and constrained optimization, among others.

Moreover, fixed point theory for multivalued mappings has also contributed to the development of related areas of research, such as operator theory, topological degree theory, and convex analysis. By investigating the properties and behavior of fixed points in multivalued mappings, mathematicians have gained a deeper understanding of these fields and have been able to establish connections and develop new techniques.(See, [30], [10,11], [2], [7], [33], [9] and [26]).

Demicontractive operators, on the other hand, exhibit properties of both contractive and nonexpansive operators. They are broadly used in the study of variational inequalities and play a crucial role in convex analysis and optimization theory. They provide a bridge between nonlinear and linear problems, enabling the development of efficient numerical methods for solving variational problems arising in diverse areas.

Viscosity iterative algorithms have been extensively studied in recent years for finding common fixed points of single-valued nonexpansive mappings and solving variational inequality problems. These investigations have built upon the concepts of viscosity solutions introduced by various researchers. (see e.g [6], [25], [5], [29], [23], [19], [28]).

Throughout this paper, we denote H to be real Hilbert space with the inner product $\langle ., . \rangle$ induced by the norm $\|.\|$. Let $\mathbb{K}$, to be a nonempty, closed and convex subset of H.

An operator $\mathbf{A} : H \to H$ is said to be Lipschitz if there exists a constant $L > 0$ such that

$$\|\mathbf{A}x - \mathbf{A}y\| \leq L\|x - y\|, \forall x, y \in H \tag{1.1}$$

$\mathbf{A} : H \to H$ is said to be strongly positive if there exists a constant $k > 0$ such that

$$\langle \mathbf{A}x, x \rangle \geq k\|x\|^2, \quad \forall x \in H \tag{1.2}$$

$\mathbf{A} : H \to H$ is said to be $k-$strongly monotone if there exists a constant $k \in (0, 1)$ such that

$$\langle \mathbf{A}x - \mathbf{A}y, x - y \rangle_H \geq k\|x - y\|^2, \quad \forall x, y \in H \tag{1.3}$$

Definition 1.1. A multivalued mapping

(1) $T : D(T) \subseteq H \to CB(D)$ is called $L-$Lipschitzian if there exists $L > 0$, such that

$$H(Tx, Ty) \leq L\|x - y\|, \forall x, y \in D(T)$$

and T is contraction if $L \in (0, 1)$ and noneaxpansive if $L = 1$.

(2) T is called quasi-nonexpansive if $H(Tx, Tp) \leq \|x - p\|, \quad \forall x \in D(T), p \in Fix(T)$

(3) $T : D(T) \subseteq H \to CB(D)$ is said to be $k-$stritly pseudo-contractive, if there exists $k \in (0,1)$ such that for all $x, y \in D(T)$, the following holds;

$$\Big(H(Tx, Ty)\Big)^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2,$$

If $k = 1$, the map T is said to be pseudocontractive.

(4) [26] $T : D(T) \subseteq E \to 2^E$ is said to be demicontractive if $Fix(T) \neq \emptyset$ and for all $p \in Fix(T), x \in D(T)$ there exists $k \in (0,1)$ such that

$$\Big(H(Tx, Tp)\Big)^2 \leq \|x-p\|^2 + kd(x, Tx)^2.$$

If $k = 1$, the map T is said to be hemicontractive.

Let (X, d) be a metric space, $\mathbb{K}$ be a nonempty subset of X and $T : \mathbb{K} \to 2^{\mathbb{K}}$ be a multivalued mapping. An element $x \in \mathbb{K}$ is called a fixed point of T if $x \in Tx$. The fixed point set of T is denoted by $Fix(T) := \{x \in D(T) : x \in Tx\}$ where $D(T) := \{x \in X : Tx \neq \emptyset\}$. It is easy to see that single-valued mapping is a particular case of multivalued mapping.

Let D be a nonempty suset of a normed linear space E. The set D is called proximinal (see [13]) if for each $\psi \in E$, there exists $u \in D$ such that

$$d(x, u) := \inf\{\|x-y\| : y \in D\}, \forall x, y \in E \tag{1.4}$$

where $d(x, y) := \|x-y\|$ for all $x, y \in E$. Every closed, nonempty and convex set of real Hilbert space is proximinal. The family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets be donated as $CB(D)$, $K(D)$ and $P(D)$ respectively.

Let $\mathbf{A}, B \in CB(D)$. Then the Hausdorff metric in $\mathcal{H}$ is defined by

$$H(\mathbf{A}, B) = \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, \mathbf{A})\Big\}. \tag{1.5}$$

Let $\mathbf{A} : D(\mathbf{A}) \subset H \to 2^H$ be a multivalued operator. Then $\mathbf{A}$ is monotone if $(x, u), (y, v) \in D(\mathbf{A})$ such that

$$G(\mathbf{A}) := \{x, u) : x \in D(\mathbf{A}), u \in \mathbf{A}x\} \tag{1.6}$$

A monotone mapping $\mathbf{A} : H \to 2^H$ is said to be maximal if its graph $G(A)$ is not properly contained in the graph of any other monotone mapping.

A mapping $\mathbf{A} : H \to H$ is said to be $\alpha-$inverse strongly if there exits a constant $\alpha > 0$ such that

$$\langle \mathbf{A}x - \mathbf{A}y, x-y\rangle_H \geq \alpha\|\mathbf{A}x - \mathbf{A}y\|^2, \quad \forall x, y \in H \tag{1.7}$$

Remark 1.2. It can be seen that every $\alpha-$inverse strongly monotone mapping is $\dfrac{1}{\alpha}-$Lipschitz monotone.

Let $\mathbf{A} : H \to H$ be a single-valued nonlinear mapping and $\sqcap : H \to 2^H$ be a set-valued mapping. Then the variational inclusion problem is as follows: Find $x \in H$, such that

$$\omega \in \mathbf{M}(x) + \mathbf{A}(x) \tag{1.8}$$

where ω is the zero vector in H. We denote the solution of the problem (1.8) by $S(\mathbf{M}, \mathbf{A})$. If $\omega = \mathbf{A}$ then, problem (1.8) becomes the inclusion problem by Rockafellar [16].Further readings on Zeros of inclusion problem (See [17], [18], [19], [8], [12])

Let a set value mapping $\mathbf{M} : H \to 2^H$ be maximal monotone. We define a resolvent operator J_λ^{M} generated by $\sqcap$ *and* λ as follows

$$J_\lambda^{\mathrm{M}} = (I - \lambda \mathbf{M})^{-1}(x), \forall x \in H \tag{1.9}$$

where λ is a positive number. It is easily to see that the resolvent operator J_λ^{M} is single - valued nonexpensive and $1-$inverse strongly monotone , and moreover, a solution of the problem (1.8) is a fixed point of the operator $J_\lambda^{\mathrm{M}}(I - \lambda\mathbf{A})$, $\forall \lambda > 0$(See [4]).
Let $T : H \to P(H)$ be multivalued map and $P_T : H \to CB(H)$ be defined by

$$P_T(x) = \{y \in Tx : \|y - x\| = d(x, Tx)\} \tag{1.10}$$

See examples of a multivalued mapping T with $Fix(T) \neq \emptyset, Tp = \{q\}$ for all $q \in Tp$ which P_T is a demicontractive-type but not a $k-$strictly pseudocontractive-type mapping in Mendy et al [?]

To prove that a multivalued mapping T with $Fix(T) \neq \emptyset$ and $Tp = \{q\}$ for all $q \in Tp$ is a demicontractive-type but not a k-strictly pseudocontractive-type mapping, we need to demonstrate the following three steps:

Step 1.3. Show that T is demicontractive-type.

To prove that T is demicontractive-type, we need to show that for all $p \in Fix(T)$ and $x \in D(T)$, there exists $k \in (0, 1)$ such that

$$\Big(H(Tx, Tp)\Big)^2 \leq \|x - p\|^2 + kd(x, Tx)^2.$$

Since $Tp = \{q\}$ for all $q \in Tp$, we have $Tp = \{p\}$ for all $p \in Fix(T)$. Thus, for any $x \in D(T)$, $Tx = \{\phi\}$ for some $\phi \in Fix(T)$.

Now, consider the case when $x = \phi$. In this case, we have $H(Tx, Tp) = H(T\phi, Tp) = H(\{\phi\}, \{p\}) = 0$. Therefore, the inequality holds for any $k \in (0, 1)$.

Step 1.4. Show that T is not k-strictly pseudocontractive-type.

To prove that T is not k-strictly pseudocontractive-type, we need to show that there does not exist a constant $k \in (0, 1)$ such that

$$\Big(H(Tx, Tp)\Big)^2 \leq k\|x - p\|^2 + kd(x, Tx)^2$$

for all $p \in Fix(T)$ and $x \in D(T)$.

From step 1.3, we know that $H(Tx, Tp) = 0$ for any $p \in Fix(T)$ and $x \in D(T)$. Therefore, the inequality reduces to

$$0 \leq k\|x - p\|^2$$

for all $p \in Fix(T)$ and $x \in D(T)$. However, this inequality cannot hold for all $x \neq p$ since it implies $k \geq \frac{1}{\|x-p\|^2}$, which contradicts the requirement that $k \in (0, 1)$.

Step 1.5. Show that P_T is a demicontractive-type.

Let P_T denote the projection operator associated with multivalued mapping T. Since $Tp = \{q\}$ for all $q \in Tp$, P_T is the single-valued mapping that assigns each $p \in Fix(T)$ to itself.

Consider $p, x \in Fix(T)$, with $x \neq p$. Then $Tx = Tp$ and $\|x - p\| > 0$. Furthermore, $d(x, Tx) = d(p, Tp) = 0$ since $x, p \in Fix(T)$.

Using these values, let's rearrange the original inequality:

$$\begin{aligned}\Big(H(Tx, Tp)\Big)^2 &\leq \|x - p\|^2 + kd(x, Tx)^2\\ 0 &\leq \|x - p\|^2 + kd(x, Tx)^2.\end{aligned}$$

Since $\|x-p\| > 0$, the inequality can only hold if $k = 0$. However, $k \in (0, 1)$ by definition, so P_T cannot satisfy the inequality for any $k \in (0, 1)$. Hence, P_T is not a k-strictly pseudocontractive-type mapping.

A popular method for solving problem (1.8) is the well-known forward-backward splitting method introduced by Passty [14] and Lions and Mercier [27].The method is formulated as

$$x_{n+1} = (I - \lambda_n \mathbf{M})^{-1}(I - \lambda_n \mathbf{A})x, \quad \lambda_n > 0. \tag{1.11}$$

under the condition that $Dom(\mathbf{M}) \subset Dom(\mathbf{A})$. It was known in [31], that weak convergence of (1.11) requires quite restrictive assumptions on $\mathbf{A}$ and Π, such that the inverse of $\mathbf{A}$ is strongly monotone or Π is Lipschitz continuous and monotone and the operator $(\mathbf{A}+\mathbf{M})$ is strongly monotone on $Dom(B)$. Tseng in [20] and Gibali and Thong in [24] extended and improved results of G.H-G.Chen and R.T. Rockafellar [31].

Most recently, Sow [28] introduced and studied a new iterative algorithm and prove convergence theorems for variation inclusion problem (1.8) and fixed point problem involving multivalued demi-contractive and quasi-nonexpansive mappings in Hilbert spaces. They defined the sequence $\{\psi_n\}$

as follows

$$\begin{cases} \delta_n = J^{M}_{\lambda_n}(I - \lambda_n \mathbf{A})x_n, \\ \\ y_n = \theta_n\delta_n + (1-\theta_n)v_n, v_n \in T_{\delta_n}, \\ \\ z_n = \beta_n y_n + (1-\beta_n)u_n, u_n \in T_2 y_n, \\ \\ x_{n+1} = P_{\mathbb{K}}(\alpha_n \gamma f(x_n) + (1-\eta\alpha_n \mathbf{B})z_n) \end{cases} \tag{1.12}$$

and prove that under certain conditions, the sequence $\{x_n\}$ converges strongly to a unique fixed point that solved the variational inequality.

It is our purpose in this paper to construct a new iteration process, that modifies that of Sow [28] and prove that the corresponding sequence $\{x_n\}$ converges strongly to a common point of an inclusion problem and fixed point of a family of multivalued demicontractive and quasi-nonexpansive mappings in Hilbert spaces without any compactness. Our theorems generalize and extend that of Sow [28], and many other results in this directions.

2. Preliminaries

The following lemmas will play a crucial role in the sequel.

Let $\mathbb{K}$ be a nonempty, closed convex subset of H. The nearest point projection from H to $\mathbb{K}$ denoted by $P_{\mathbb{K}}$, assigns to each $\psi \in H$ the unique point of $\mathbb{K}$, $P_{\mathbb{K}}\psi$ such that

$$\|x - P_{\mathbb{K}}x\| \leq \|x - y\|,$$

for all $y \in \mathbb{K}$, and for every $x \in H$,

$$\langle x - P_{\mathbb{K}}x, y - P_{\mathbb{K}}x\rangle \leq 0, \quad \forall y \in \mathbb{K} \tag{2.1}$$

Lemma 2.1. *[27]. Let $\Pi : H \to 2^H$ be a maximal monotone mapping, and $\Lambda : H \to H$ be Lipschitz and continuous monotone mapping. Then $(\Pi + \Lambda) : H \to 2^H$ is a maximal monotone mapping.*

Lemma 2.2. *[28]. Let H be real Hilbert space and $\Lambda : H \to H$ be an $\alpha-$inverse strongly monotone mapping. Then, $(I - \theta\Lambda)$ is nonexpansive mapping for all $\psi, \pi \in H$ and $\theta \in [0, 2\alpha]$ such that*

$$\|(I - \theta\mathbf{A})x - (I - \theta\mathbf{A}y\|^2 \leq \|x - y\|^2 + \theta(\theta - 2\alpha)\|\mathbf{A}x - \mathbf{A}y\|^2 \tag{2.2}$$

Lemma 2.3. *[24]. Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} = (1 - b_n)a_n + \sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

$$i) \quad \sum_{n=0}^{\infty} b_n = \infty, \qquad ii) \quad \lim_{n\to\infty} \sup \frac{\sigma_n}{b_n} \leq 0.$$

Then $\lim_{n\to\infty} a_n = 0$

Lemma 2.4. *(Wang [1]). Let H be a real Hilbert space. Let $\mathbb{K}$ be a nonempty closed convex subset of H. $A : H \to H$ be $k-$ strongly monotone and $L-$ Lipschitzian operator with $k > 0$ and $L > 0$. Assume that $0 < \eta < \frac{2k}{L^2}$ and $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Then for each $t \in \Big(0, \min\Big(1, \frac{1}{\tau}\Big)\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (I - t\tau)\|x - y\|, \quad \forall x, y \in H \tag{2.3}$$

Lemma 2.5. *[29]. Let H be a real Hilbert space. Then for every $x, y \in H$, and every $\lambda \in (0, 1)$, the following holds:*

- **I):** $\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$
- **II:** $\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2$.

3. Main Results

In this section, we study the convergence properties of the iterative algorithm which is based on viscosity algorithm and forward - backward splitting Method. We now prove the following theorem.

Theorem 3.1. *Let H be a real Hilbert space and $\mathbb{K}$ be a nonempty, closed convex subset of H. Let $\mathbf{A} : \mathbb{K} \to H$ be an $\alpha-$inverse strongly monotone operator and let $\mathbf{B} : H \to H$ be an $k-$strongly monotone and $L-$Lipschitzian operator. Let $f : \mathbb{K} \to H$ be an $b-$Lipschitzian mapping and $\mathbb{M} : H \to 2^H$ be a maximal monotone mapping such that the domain of $\mathbb{M}$ is included in $\mathbb{K}$. Let $T_1, T_2 : \mathbb{K} \to CB(\mathbb{K})$ be a multivalued $\beta-$ demicontractive mapping and $T_3 : \mathbb{K} \to CB(\mathbb{K})$ be a multivalued quasi-nonexpansive mapping. Assume that $0 < \eta < \frac{2k}{L^2}, 0 < \gamma b < \tau$, where $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$, and $I - T_1, I - T_2$ and $I - T_3$ are demiclosed at origin, such that $\Omega := Fix(T_1) \cap Fix(T_2) \cap Fix(T_3) \cap S(\mathbb{M}, \mathbf{A}) \neq \emptyset$ and $T_1q = T_2q = T_3q = \{q\}, \forall q \in \Omega$. For given $x_0 \in \mathbb{K}$, let $\{x_n\}$ be generated by the algorithm:*

$$\begin{cases} \delta_n = J^{\mathbb{M}}_{\lambda_n}(I - \lambda_n\mathbf{A})x_n; \\ y_n = \theta_n\delta_n + (1 - \theta_n)v_n, \quad v_n \in T_1\delta_n; \\ z_n = \beta_n y_n + (1 - \beta_n)u_n, \quad u_n \in T_2 y_n; \\ t_n = \gamma_n z_n + (1 - \gamma_n)w_n, \quad w_n \in T_3 z_n; \\ x_{n+1} = P_{\mathbb{K}}(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n\mathbf{B})t_n) \end{cases} \tag{3.1}$$

where $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\}, \{\mu_n\}, \{\lambda_n\}$ and $\{\alpha_n\}$ are real sequence in $(0, 1)$ satisfying the following conditions

- **i):** $\lim_{n\to\infty} \alpha_n = 0 \qquad \sum_{n=0}^{\infty} \alpha_n < \infty, \qquad \lambda_n \in [a, b] \subset (0, \min\{1, 2\alpha\})$

ii): $\lim_{n\to\infty} \inf(1-\beta_n)(\beta_n-\beta) > 0$ *and* $\lim_{n\to\infty} \inf(1-\theta_n)(\theta_n-\beta) > 0, (\beta_n, \theta_n) \in (\beta, 1)$
iii): $\lim_{n\to\infty} \inf(1-\gamma_n)\gamma_n) > 0$

Then, the sequences defined in (3.1), that is $\{x_n\}$ and $\{\delta_n\}$ converge strongly to unique solution $x^* \in \Omega$, which also solve the following variational inequality:

$$\langle \eta \mathbf{B}x^* - \gamma f(x^*), x^* - q\rangle \le 0, \quad \forall q \in \Omega \tag{3.2}$$

Proof. From the choice of η and γ from [?], $(\eta\Phi - \gamma\psi)$ is strongly monotone, then the variational inequality (3.2) has a unique solution. We will first show that there is only one solution.

Lets assume that by contradiction that there exist two points $x^*, y^* \in \Omega$ which are two solution of the given inequality, and $x^* \neq y^*$, then we have

$$\langle \eta \mathbf{B}x^* - \gamma f(x^*), x^* - y^*\rangle \le 0 \tag{3.3}$$

and

$$\langle \eta \mathbf{B}y^* - \gamma f(y^*), y^* - x^*\rangle \le 0 \tag{3.4}$$

Therefore from (3.3) and (3.4), we have

$$\langle \eta \mathbf{B}y^* - \eta \mathbf{B}x^* + \gamma f(x^*) - \gamma f(y^*), y^* - x^*\rangle \le 0 \tag{3.5}$$

Now from the assumption that

$$\begin{aligned} \frac{L^2\eta}{2} > 0 \;\; &\Leftrightarrow \;\; \alpha - \frac{L^2\eta}{2} < \alpha \\ &\Leftrightarrow \;\; \eta\Big(\alpha - \frac{L^2\eta}{2}\Big) < \alpha\eta \\ &\Leftrightarrow \;\; \tau < \alpha\eta \end{aligned}$$

So that $0 < \gamma < \tau < \alpha\eta$

$$\begin{aligned} \langle \eta \mathbf{B}y^* - \eta \mathbf{B}x^* + \gamma f(x^*) - \gamma f(y^*), y^* - x^*\rangle &= \langle \eta \mathbf{B}y^* - \eta \mathbf{B}x^*, y^* - x^*\rangle \\ &- \langle \gamma f(y^*) - \gamma f(x^*), y^* - x^*\rangle \\ &= \langle \eta \mathbf{B}y^* - \eta \mathbf{B}x^*, y^* - x^*\rangle \\ &- \gamma\|f(y^*) - f(x^*)\|\|y^* - x^*\| \\ &\ge \alpha\eta\|x^* - y^*\|^2 - \gamma\rho\|x^* - y^*\|^2 \\ &= (\alpha\eta - \gamma\rho)\|x^* - y^*\|^2 \end{aligned}$$

and this is a contradiction to (3.5), and hence $x^* = y^*$, which is required. Again, we note that the operator $P_{\mathbb{K}}[I + (\alpha\gamma f - \eta\alpha\mathbf{B})]$ is a contradiction. Now for any fixed point $\alpha_0 \in (0, \min\left\{1, \frac{1}{\tau}\right\})$, and $\forall x, y \in H$, we have, by lemma (2.4), and letting $\Phi = [I + (\alpha_0\gamma f - \eta\alpha_0\mathbf{B})]x$ and $\Theta = [I + (\alpha_0\gamma f - \eta\alpha_0\mathbf{B})]y$, we have

$$\begin{aligned}\|P_{\mathbb{K}}\Phi - P_{\mathbb{K}}\Theta\| &\leq \|[I + (\alpha_0\gamma f - \eta\alpha_0\mathbf{B})]x - ([I + (\alpha_0\gamma f - \eta\alpha_0\mathbf{B})]y)\| \\ &\leq \alpha_0\gamma\|f(x) - f(y)\| + \|(I - \eta\alpha_0\mathbf{B})x - (I - \eta\alpha_0\mathbf{B})y\| \\ &\leq \alpha_0\gamma\rho\|x - y\| + (I - \alpha\tau)\|x - y\| \\ &\leq (I - \alpha_0(\tau - \gamma\rho))\|x - y\|\end{aligned}$$

Thus, by Banach contraction principle, the mapping $P_{\mathbb{K}}[I + (\alpha\gamma f - \eta\alpha\mathbf{B})]$ has a fixed point, say $\widehat{x} = P_{\mathbb{K}}[I + (\alpha\gamma f - \eta\alpha\mathbf{B})]$ and as such, from (2.1), it is similar in value to the variational inequality below

$$\langle \eta\mathbf{B}\widehat{x} - \gamma f(\widehat{x}), \widehat{x} - q\rangle \leq 0, \forall q \in \Omega$$

Now we continue with the proof of theorem (3.1)

Let $q \in \Omega$ with the fact that $J^{\prod}_{\lambda_n}$ is $1-$inverse strongly monotone,and from [?], we have the following

$$\|\delta_n - q\|^2 \leq \|x_n - q\|^2$$

Therefore from (3.6), we have

$$\|\delta_n - q\| \leq \|x_n - q\| \tag{3.6}$$

From lemma (2.5), with (3.1), and, for the fact that $T_1 q = \{q\}$, T_1 is $\beta-$demicontrative, we have

$$\begin{aligned}\|y_n - q\|^2 &= \|\theta_n(\delta_n - q) + (1 - \theta_n)(v_n - q)\|^2 \\ &= \theta_n\|\delta_n - q\|^2 + (1 - \theta_n)\|v_n - q\|^2 - (1 - \theta_n)\theta_n\|v_n - \delta_n\|^2 \\ &\leq \theta_n\|\delta_n - q\|^2 + (1 - \theta_n)H(T_1\delta_n, T_1 q)^2 - (1 - \theta_n)\theta_n\|v_n - \delta_n\|^2 \\ &\leq \theta_n\|\delta_n - q\|^2 + (1 - \theta_n)[\|\delta_n - q\|^2 + \beta d(\delta_n, T_1\delta_n)^2] - (1 - \theta_n)\theta_n\|v_n - \delta_n\|^2 \\ &\leq \|\delta_n - q\|^2 - (1 - \theta_n)(\theta_n - \beta)\|v_n - \delta_n\|^2\end{aligned} \tag{3.7}$$

Thus, we have

$$\|y_n - q\|^2 \leq \|\delta_n - q\|^2 - (1 - \theta_n)(\theta_n - \beta)\|v_n - \delta_n\|^2$$

Since $\theta_n \in (\beta, 1)$, we have

$$\|y_n - q\|^2 \leq \|\delta_n - q\|^2$$

Again from lemma (2.5), with (3.1), and, for the fact that $T_2 q = \{q\}$, T_2 is $\beta-$demicontrative, we have

$$\begin{aligned}
\|z_n - q\|^2 &= \|\beta_n(y_n - q) + (1-\beta_n)(u_n - q)\|^2 \\
&= \beta_n\|y_n - q\|^2 + (1-\beta_n)\|u_n - q\|^2 - (1-\beta_n)\beta_n\|u_n - y_n\|^2 \\
&\le \beta_n\|y_n - q\|^2 + (1-\beta_n)H(T_2y_n, T_2q)^2 - (1-\beta_n)\beta_n\|u_n - y_n\|^2 \\
&\le \beta_n\|y_n - q\|^2 + (1-\beta_n)[\|y_n - q\|^2 + \beta d(y_n, T_2y_n)^2] - (1-\beta_n)\beta_n\|u_n - y_n\|^2 \\
&\le \|y_n - q\|^2 - (1-\beta_n)(\beta_n - \beta)\|u_n - y_n\|^2 \qquad (3.8)
\end{aligned}$$

Thus, we have

$$\|z_n - q\|^2 \le \|y_n - q\|^2 - (1-\beta_n)(\beta_n - \beta)\|u_n - y_n\|^2$$

Since $\beta_n \in (\beta, 1)$, we have

$$\|z_n - q\|^2 \le \|y_n - q\|^2$$

Now, using the fact that $T_3q = q$, we have the following estimates

$$\begin{aligned}
\|t_n - q\| &= \|\gamma_n z_n + (1-\gamma_n)w_n - q\| \\
&\le \gamma_n\|z_n - q\| + (1-\gamma_n)\|w_n - q\| \\
&\le \gamma_n\|z_n - q\| + (1-\gamma_n)H(T_3z, T_3q) \\
&\le \gamma_n\|z_n - q\| + (1-\gamma_n)\|z_n - q\| \\
&\le \|z_n - q\|
\end{aligned} \qquad (3.9)$$

Hence, we can see that

$$\|t_n - q\| \le \|z_n - q\| \le \|y_n - q\| \le \|\delta_n - q\| \le \|x_n - q\| \qquad (3.10)$$

Using (3.1), inequality (3.10) and lemma (2.4)

$$\begin{aligned}
\|x_{n+1} - q\| &\le \|(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n\mathbf{B})t_n) - q\| \\
&\le \|\alpha_n\gamma(f(x_n) - f(q))\| + (1-\tau\alpha)\|t_n - q)\| + \alpha\|\gamma f(q) - \eta\alpha\mathbf{B}\| \\
&\le \alpha_n\gamma\|f(x_n) - f(q)\| + (1-\tau\alpha)\|x_n - q)\| + \alpha\|\gamma f(q) - \eta\alpha\mathbf{B}\| \\
&\le \alpha_n b\gamma\|x_n - q\| + (1-\tau\alpha)\|x_n - q)\| + \alpha\|\gamma f(q) - \eta\alpha\mathbf{B}\| \\
&\le (1 - \alpha(\tau - b\gamma))\|x_n - q)\| + \alpha\|\gamma f(q) - \eta\alpha\mathbf{B}\| \\
&\le \max\left\{\|x_n - q\|, \frac{\|\gamma f(q) - \eta\mathbf{B}q\|}{\tau - b\gamma}\right\}.
\end{aligned}$$

Therefore, by induction, it is easy to see that

$$\|x_{n+1} - q\| \le \max\left\{\|x_0 - q\|, \frac{\|\gamma f(q) - \eta\mathbf{B}q\|}{\tau - b\gamma}\right\}, \quad \forall n \ge 1$$

Hence $\{x_n\}$, $\{f(x_n)\}$ and $\{\mathbf{B}x_n\}$ are bounded.

Secondly, we now have the following estimates. From (3.1) and lemma (2.5), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \|\alpha_n(\gamma f(x_n)-\eta\mathbf{B}q)+(I-\eta\alpha_n\mathbf{B})(t_n-q)\|^2 \\
&\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}q\|^2+(1-\tau\alpha_n)^2\|t_n-q\|^2 \\
&+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|t_n-q\| \\
&\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}q\|^2+(1-\tau\alpha_n)^2\|z_n-q\|^2 \\
&+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|t_n-q\| \\
&\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}q\|^2+(1-\tau\alpha_n)^2\|y_n-q\|^2 \\
&- (1-\tau\alpha_n)^2(1-\beta_n)(\beta_n-\beta)\|u_n-y_n\|^2 \\
&+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|t_n-q\| \\
&\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}q\|^2+(1-\tau\alpha_n)^2\|\delta_n-q\|^2 \\
&- (1-\tau\alpha_n)^2(1-\theta_n)(\theta_n-\beta)\|v_n-\delta_n\|^2 \\
&- (1-\tau\alpha_n)^2(1-\beta_n)(\beta_n-\beta)\|u_n-y_n\|^2 \\
&+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|t_n-q\| \\
&\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}q\|^2+(1-\tau\alpha_n)^2\|x_n-q\|^2 \\
&- (1-\tau\alpha_n)^2(1-\theta_n)(\theta_n-\beta)\|v_n-\delta_n\|^2 \\
&- (1-\tau\alpha_n)^2(1-\beta_n)(\beta_n-\beta)\|u_n-y_n\|^2 \\
&+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|x_n-q\| \\
&\leq \|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}q\|^2-\alpha_n(2\tau-\tau^2\alpha_n)\|x_n-q\|^2 \\
&- (1-\tau\alpha_n)^2(1-\theta_n)(\theta_n-\beta)\|v_n-\delta_n\|^2 \\
&- (1-\tau\alpha_n)^2(1-\beta_n)(\beta_n-\beta)\|u_n-y_n\|^2 \\
&+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|x_n-q\|
\end{aligned}$$

Therefore

$$\begin{aligned}
(1-\tau\alpha_n)^2\Big[(1-\theta_n)(\theta_n-\beta)\|v_n-\delta_n\|^2 &+ (1-\beta_n)(\beta_n-\beta)\|u_n-y_n\|^2\Big] \\
&\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2-\alpha_n(2\tau-\tau^2\alpha_n)\|x_n-q\|^2 \\
&+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|x_n-q\| \\
&+ \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}q\|^2
\end{aligned}$$

Due to the boundedness of $\{f(x_n)\}$ and $\{x_n\}$, and for some constant $M > 0$, we have

$$\begin{aligned}(1-\tau\alpha_n)^2\Big[(1-\theta_n)(\theta_n-\beta)\|v_n-\delta_n\|^2 &+ (1-\beta_n)(\beta_n-\beta)\|u_n-y_n\|^2\Big] \\ &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2+\alpha_n M \end{aligned} \tag{3.11}$$

We now show that $x_n \to x$. We then consider two cases.

Case 1: Assuming that the sequence $\{\|x_n-q\|\}$ is monotonically decreasing. Then $\{\|x_n-q\|\}$ must be a convergent sequence. Therefore, we have

$$\lim_{n\to\infty}[\|x_n-q\|^2-\|x_{n+1}-q\|^2]=0, \tag{3.12}$$

This implies that from (3.11), that

$$\lim_{n\to\infty}(1-\theta_n)(\theta_n-\beta)\|v_n-\delta_n\|^2=0 \tag{3.13}$$

and

$$\lim_{n\to\infty}(1-\beta_n)(\beta_n-\beta)\|u_n-y_n\|^2=0 \tag{3.14}$$

Since $\lim_{n\to\infty}\inf(1-\theta_n)(\theta_n-\beta)>0$ and $\lim_{n\to\infty}\inf(1-\beta_n)(\beta_n-\beta)>0$, with the fact that $v_n\in T_1\delta_n$ and $u_n\in T_2y_n$, it follows that

$$\lim_{n\to\infty}d(\delta_n,T_1\delta_n)=0 \tag{3.15}$$

and

$$\lim_{n\to\infty}d(y_n,T_2y_n)=0 \tag{3.16}$$

observing that

$$\begin{aligned}\|y_n-\delta_n\| &= \|\theta_n\delta_n+(1-\theta_n)v_n-\delta_n\| \\ &= \|\theta_n\delta_n+(1-\theta_n)v_n-\delta_n+\theta_n\delta_n-\theta_n\delta_n\| \\ &= (1-\theta_n)\|v_n-\delta_n\| \\ &\leq \|v_n-\delta_n\|\end{aligned} \tag{3.17}$$

Taking the limits and from (3.13), we can see that

$$\lim_{n\to\infty}\|y_n-\delta_n\|=0$$

$$\begin{aligned}\|z_n - y_n\| &= \|\beta_n y_n + (1-\beta_n)u_n - y_n\|\\ &= \|\beta_n y_n + (1-\beta_n)u_n - y_n + \beta_n y_n - \beta_n y_n\|\\ &= (1-\beta_n)\|u_n - y_n\|\\ &\leq \|u_n - y_n\|\end{aligned} \tag{3.18}$$

Again, from (3.14), we can see that

$$\lim_{n\to\infty}\|z_n - y_n\| = 0$$

$$\begin{aligned}\|z_n - \delta_n\| &= \|z_n - y_n + y_n - \delta_n\|\\ &\leq \|z_n - y_n\| + \|y_n - \delta_n\|\end{aligned}$$

Hence $\lim\limits_{n\to\infty}\|z_n - \delta_n\| = 0$

Now from lemma (2.2), lemma (2.4) and (3.1), we have the following

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \|\alpha_n(\gamma f(x_n) - \eta\mathbf{B}q) + (I - \eta\alpha_n\mathbf{B})(t_n - q)\|^2\\ &\leq \alpha_n^2\|\gamma f(x_n) - \eta\mathbf{B}\|^2 + (1-\tau\alpha)^2\|t_n - q\|^2\\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n) - \eta\mathbf{B}q\|\|t_n - q\|\\ &\leq \alpha_n^2\|\gamma f(x_n) - \eta\mathbf{B}\|^2 + (1-\tau\alpha_n)^2\|\delta_n - q\|^2\\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n) - \eta\mathbf{B}q\|\|t_n - q\|\\ &\leq \alpha_n^2\|\gamma f(x_n) - \eta\mathbf{B}\|^2 + (1-\tau\alpha_n)^2\|J_{\lambda_n}^{\mathbb{M}}(1-\lambda\mathbf{A})x_n - J_{\lambda_n}^{\mathbb{M}}(1-\lambda\mathbf{A})q\|^2\\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n) - \eta\mathbf{B}q\|\|t_n - q\|\\ &\leq \alpha_n^2\|\gamma f(x_n) - \eta\mathbf{B}\|^2 + (1-\tau\alpha_n)^2\Big[\|x_n - q\|^2 + a(b-2\alpha)\|\mathbf{A}x_n - \mathbf{A}q\|^2\Big]\\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n) - \eta\mathbf{B}q\|\|x_n - q\|\\ &\leq \alpha_n^2\|\gamma f(x_n) - \eta\mathbf{B}\|^2 + \|x_n - q\|^2 - \alpha_n(2\tau - \tau^2\alpha_n)\|x_n - q\|^2\\ &- (1-\tau\alpha)^2 a(2\alpha - b)\|\mathbf{A}x_n - \mathbf{A}q\|^2\\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n) - \eta\mathbf{B}q\|\|x_n - q\|\end{aligned} \tag{3.19}$$

Therefore, from (3.19),and with a constant $D > 0$, we have

$$(1-\tau\alpha)^2 a(2\alpha - b)\|\mathbf{A}x_n - \mathbf{A}q\|^2 \leq \|x_n - q)\|^2 - \|x_{n+1} - q)\|^2 + \alpha_n D$$

Since, $\lim_{n\to\infty} \alpha_n = 0$, and from the inequality (3.12), with the fact that $\{x_n\}$ is bounded, we have

$$\lim_{n\to\infty} \|\mathbf{A}x_n - \mathbf{A}q\|^2 = 0 \tag{3.20}$$

Since $J_{\lambda_n}^{\mathbb{M}}$ is 1−inverse strongly monotone,and $\|t_n - q\| \le \|\delta_n - q\|$, we have the following

$$\begin{aligned}
\|t_n - q\|^2 &= \|J_{\lambda_n}^{(M)}(I - \lambda_n\mathbf{A})x_n - J_{\lambda_n}^{\mathbb{M}}(I - \lambda_n\mathbf{A})q\|^2 \\
&\le \langle t_n - q, (I - \lambda_n\mathbf{A})x_n - (I - \lambda_n\mathbf{A})q\rangle \\
&= \frac{1}{2}\Big[\|(I - \lambda_n\mathbf{A})x_n - (I - \lambda_n\mathbf{A})q\|^2 \\
&+ \|t_n - q\|^2 - \|(I - \lambda_n\mathbf{A})x_n - (I - \lambda_n\mathbf{A})q - (t_n - q)\|^2\Big] \\
&\le \frac{1}{2}\Big[\|x_n - q\|^2 + \|t_n - q\|^2 - \|x_n - t_n\|^2 \\
&+ 2\lambda_n\langle t_n - q, \mathbf{A}x_n - \mathbf{A}q\rangle - \lambda_n^2\|\mathbf{A}x_n - \mathbf{A})q)\|^2\Big] \\
&\le \|x_n - q\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle x_n - q, \mathbf{A}x_n - \mathbf{A}q\rangle - \lambda_n^2\|\mathbf{A}x_n - \mathbf{A}q\|^2
\end{aligned}$$

This gives us

$$\|t_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle t_n - q, \mathbf{A}x_n - \mathbf{A}q\rangle - \lambda_n^2\|\mathbf{A}x_n - \mathbf{A}q\|^2 \tag{3.21}$$

Therefore

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \|\alpha_n(\gamma f(x_n) - \eta\mathbf{B}q) + (I - \eta\alpha_n\mathbf{B})(t_n - q)\|^2 \\
&\le \alpha_n^2\|\gamma f(x_n) - \eta\mathbf{B}\|^2 + (1 - \tau\alpha_n)^2\|t_n - q\|^2 \\
&+ 2\alpha_n(1 - \tau\alpha_n)\|\gamma f(x_n) - \eta\mathbf{B}q\|\|t_n - q\| \\
&\le \alpha_n^2\|\gamma f(x_n) - \eta\mathbf{B}\|^2 + (1 - \tau\alpha_n)^2\Big[\|x_n - q\|^2 - \|x_n - t_n\|^2 \\
&+ 2\lambda_n\langle t_n - q, \mathbf{A}x_n - \mathbf{A}q\rangle - \lambda_n^2\|\mathbf{A}x_n - \mathbf{A}q\|^2\Big] \\
&+ 2\alpha_n(1 - \tau\alpha_n)\|\gamma f(x_n) - \eta\mathbf{B}q\|\|x_n - q\| \\
&\le \alpha_n^2\|\gamma f(x_n) - \eta\mathbf{B}\|^2 + (1 - \tau\alpha_n)^2\|x_n - q\|^2 - (1 - \tau\alpha_n)^2\|x_n - t_n\|^2 \\
&+ 2\lambda_n(1 - \tau\alpha_n)^2\langle t_n - q, \mathbf{A}x_n - \mathbf{A}q\rangle - \lambda_n^2(1 - \tau\alpha_n)^2\|\mathbf{A}x_n - \mathbf{A}q\|^2 \\
&+ 2\alpha_n(1 - \tau\alpha_n)\|\gamma f(x_n) - \eta\mathbf{B}q\|\|x_n - q\| \qquad (3.22)
\end{aligned}$$

Thus, from (3.25), we have

$$\begin{aligned}(1-\tau\alpha_n)^2\|x_n-t_n\|^2 &\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\ &- \alpha_n\tau(2-\tau\alpha_n)\|x_n-q\|^2+2\lambda_n(1-\tau\alpha_n)^2\langle t_n-q,\mathbf{A}x_n-\mathbf{A}q\rangle \\ &- \lambda_n^2(1-\tau\alpha_n)^2\|\mathbf{A}x_n-\mathbf{A}q\|^2+2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|x_n-q\|\end{aligned} \tag{3.23}$$

Therefor, since $\alpha_n \to 0$ as $n \to \infty$ with inequalities (3.12) and (3.20), we have

$$\lim_{n\to\infty}\|x_n-t_n\|=0$$

From (3.10) and lemma (2.5) with the fact that T_3 is quasi-nonexpansive, we have the following estimate

$$\begin{aligned}\|t_n-q\|^2 &= \|\gamma_n z_n+(1-\gamma_n)w_n-q\|^2 \\ &= \gamma_n\|z_n-q\|^2+(1-\gamma_n)\|w_n-q\|^2-(1-\gamma_n)\gamma_n\|w_n-z_n\|^2 \\ &= \gamma_n\|z_n-q\|^2+(1-\gamma_n)H(T_3z_n,T_3q)^2-(1-\gamma_n)\gamma_n\|w_n-z_n\|^2 \\ &= \gamma_n\|z_n-q\|^2+(1-\gamma_n)\|z_n-q\|^2-(1-\gamma_n)\gamma_n\|w_n-z_n\|^2 \\ &\leq \|x_n-q\|^2-(1-\gamma_n)\gamma_n\|w_n-z_n\|^2\end{aligned} \tag{3.24}$$

Therefore

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \|\alpha_n(\gamma f(x_n)-\eta\mathbf{B}q)+(I-\eta\alpha_n\mathbf{B})(t_n-q)\|^2 \\ &\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}\|^2+(1-\tau\alpha_n)^2\|t_n-q\|^2 \\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|t_n-q\| \\ &\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}\|^2+(1-\tau\alpha_n)^2\Big[\|x_n-q\|^2 \\ &- (1-\gamma_n)\gamma_n\|w_n-z_n\|^2\Big]+2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|t_n-q\| \\ &\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}\|^2+(1-\tau\alpha_n)^2\|x_n-q\|^2 \\ &- (1-\tau\alpha_n)^2(1-\gamma_n)\gamma_n\|w_n-z_n\|^2+2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|t_n-q\|\end{aligned}$$

Hence, we have the following

$$\begin{aligned}(1-\tau\alpha_n)^2(1-\gamma_n)\gamma_n\|w_n-z_n\|^2 &\leq \alpha_n^2\|\gamma f(x_n)-\eta\mathbf{B}\|^2-\tau\alpha_n(2-\tau\alpha_n)\|x_n-q\|^2 \\ &+ \|x_n-q\|^2-\|x_{n+1}-q\|^2 \\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(x_n)-\eta\mathbf{B}q\|\|t_n-q\|\end{aligned}$$

Therefor, since $\alpha_n \to 0$ as $n \to \infty$ with inequalities (3.12) and (3.20), we have

$$(1-\tau\alpha_n)^2(1-\gamma_n)\gamma_n\|w_n - z_n\|^2 \leq 0$$

From this we have

$$\lim_{n\to\infty}(1-\gamma_n)\gamma_n\|w_n - z_n\|^2 = 0 \tag{3.25}$$

Since $\lim_{n\to\infty}\inf((1-\gamma_n)\gamma_n) > 0$

$$\lim_{n\to\infty}\|w_n - z_n\| = 0 \tag{3.26}$$

Again, with $w \in T_3 z_n$

$$\lim_{n\to\infty} d(z_n, T_3 z_n) = 0 \tag{3.27}$$

Moreover, since H is reflexive and $\{x_n\}$ is bounded, we then prove that $\lim_{n\to+\infty}\sup\langle\eta\mathbf{B}x^* - \gamma f(x^*), x^* - x_n\rangle \leq 0$. We let the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ to converge weakly to x^{**} in $\mathbb{K}$, and

$$\lim_{n\to+\infty}\langle\eta\mathbf{B}x^* - \gamma f(x^*), x^* - x_n\rangle = \lim_{n\to+\infty}\langle\eta\mathbf{B}x^* - \gamma f(x^*), x^* - x_{n_i}\rangle$$

Again, since $I-T_1$, $I-T_2$ and $I-T_3$ satisfies the demiclosed principle and from (3.32), (3.16) and (3.27), we obtain $x^{**} \in Fix(T_1)\cap Fix(T_2)\cap Fix(T_3)$. We now show that $x^{**} \in S(\mathbb{M}, \mathbf{A})$. Since $\mathbf{A}$ is $\alpha-$inverse strongly monotone, and Lipschitz continuous mapping. Then From lemma (2.1), it follows that $(\mathbb{M}+\mathbf{A})$ is maximal monotone.

Let $(\nu, g) \in G(\mathbb{M}+\mathbf{A})$, that is $g - \mathbf{A}\nu \in \mathbb{M}(\nu)$. Since $\delta_{n_i} = J^{\mathbb{M}}_{\lambda_{n_i}}(x_{n_i}) - \lambda_{n_i}\mathbf{A}x_{n_i})$, we have $x_{n_i} - \lambda_{n_i}x_{n_i} \in (I + \lambda_{n_i}\mathbb{M})\delta_{n_i}$, that is $\frac{1}{\lambda_{n_i}}(x_{n_i} - \delta_{n_i} - \lambda_{n_i}\mathbf{A}x_{n_i}) \in \mathbb{M}(\delta_{n_i})$. By maximal monotonocity of $(\mathbb{M}+\mathbf{A})$, gives

$$\langle\nu - \delta_{n_i}, g - \mathbf{A}\nu - \frac{1}{\lambda_{n_i}}(x_{n_i} - \delta_{n_i} - \lambda_{n_i}\mathbf{A}x_{n_i}) \geq 0$$

and, therefore

$$\begin{aligned}
\langle\nu - \delta_{n_i}, g\rangle &\geq \langle\nu - \delta_{n_i}, \mathbf{A}\nu - \frac{1}{\lambda_{n_i}}(x_{n_i} - \delta_{n_i} - \lambda_{n_i}\mathbf{A}x_{n_i}))\rangle \\
&= \langle\nu - \delta_{n_i}, \mathbf{A}\nu - \mathbf{A}\delta_{n_i} + \mathbf{A}\delta_{n_i} + \frac{1}{\lambda_{n_i}}(x_{n_i} - \delta_{n_i} - \lambda_{n_i}\mathbf{A}x_{n_i}))\rangle \\
&\geq \langle\nu - \delta_{n_i}, \mathbf{A}\nu - \mathbf{A}x_{n_i}\rangle + \langle\nu - \delta_{n_i}, \frac{1}{\lambda_{n_i}}(x_{n_i} - \delta_{n_i}\rangle
\end{aligned}$$

It then follows from $\|\delta_n - x_n\| \to 0, \|\mathbf{A}\delta_n - \mathbf{A}x_n\| \to 0$ *and* $\delta_{n_i} \to x^{**}$ weakly that $\lim_{n\to\infty}\langle\nu - \delta_{n_i}, g\rangle = \langle\nu - x^{**}, g\rangle$ and hence $x^{**} \in S(\Pi, \mathbf{A})$. Therefore, $x^{**} \in \Omega$,

On the other hand, for the fact that x^* solves the variational inequality (3.30).

$$\begin{aligned}\lim_{n\to+\infty}\sup\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_n\rangle &= \lim_{n\to+\infty}\sup\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_{n_i}\rangle\\ &= \langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x^{**}\rangle\leq 0\end{aligned} \tag{3.28}$$

Lastly, we now prove that $\lim_{n\to\infty}\|x_n-x^*\|=0$, that is $x_n\to x^*$ *as* $n\to\infty$.

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \|\alpha_n\gamma f(x_n)+(I-\eta\alpha_n\mathbf{B})t_n-x^*\|^2\\ &\leq \|\alpha_n(\gamma f(x_n)-\gamma f(x^*))+(I-\eta\alpha_n\mathbf{B})t_n-x^*\|^2\\ &+ 2\alpha_n\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_{n+1}\rangle\\ &\leq \Big[\alpha_n\gamma\|f(x_n)-f(x^*)\|+\|(I-\eta\alpha_n\mathbf{B})(t_n-x^*)\|\Big]^2\\ &+ 2\alpha_n\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_{n+1}\rangle\\ &\leq \Big[\alpha_n\gamma b\|x_n-x^*\|+(1-\tau\alpha_n)\|x_n-x^*\|\Big]^2\\ &+ 2\alpha_n\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_{n+1}\rangle\\ &\leq \Big[1-\alpha_n(\tau-\gamma b)\Big]^2\|x_n-x^*\|^2+2\alpha_n\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_{n+1}\rangle\\ &\leq \Big[1-\alpha_n(\tau-\gamma b)\Big]\|x_n-x^*\|^2+2\alpha_n\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_{n+1}\rangle\end{aligned}$$

Thus, from lemma (2.3), it follows that $\psi_n\to\psi^*$ *as* $n\to\infty$, where $b_n=\alpha_n(\tau-\gamma b)$, $a_n=\|x_n-x^*\|^2$ and $\sigma_n=2\alpha_n\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_{n+1}\rangle$

Case 2: Suppose that the sequence $\Big\{\|x_n-x^*\|\Big\}$ is monotonically increasing. Set $\mathbf{W}_n:=\|x_n-x^*\|^2$ and $\tau:=\mathbf{N}\to\mathbf{N}$ be a mapping for all $n\geq n_0$ (for some n_0 sufficient large), by $\tau_n:=\max\{k\in\mathbf{N}:k\leq n,\mathbf{W}_k\leq\mathbf{W}_{k+1}\}$. Then, τ is a nondecreasing sequence, such that $\tau_n\to\infty$ as $n\to\infty$ and $\mathbf{W}_{\tau(n)}\leq\mathbf{W}_{\tau(n)+1}\}$ for all $n\geq n_0$. Now, from (3.11), we have

$$\begin{aligned}(1-\alpha_{\tau(n)}\tau)\Big[(1-\theta_n)(\theta_n-\beta)\|v_{\tau(n)}-\delta_{\tau(n)}\|^2 &+ (1-\beta_n)(\beta_n-\beta)\|u_{\tau(n)}-y_{\tau(n)}\|^2\Big]\\ &\leq 2\alpha_{\tau(n)}M\end{aligned} \tag{3.29}$$

$$\lim_{n\to+\infty}\tau(1-\alpha_{\tau(n)})\Big[(1-\theta_{\tau(n)})(\theta_{\tau(n)}-\beta)\|v_{\tau(n)}-\delta_{\tau(n)}\|^2+(1-\beta_{\tau(n)})(\beta_{\tau(n)}-\beta)\|u_{\tau(n)}-y_{\tau(n)}\|^2\Big]=0$$

Since $(\beta_{\tau(n)},\theta_{\tau(n)})\in(\beta,1)$ and $\lim_{n\to\infty}\inf\gamma_{\tau(n)}(1-\gamma_{\tau(n)})>0$, we have

$$\lim_{n\to\infty}\|u_{\tau(n)}-y_{\tau(n)}\|=0 \qquad \textit{and} \qquad \lim_{n\to\infty}\|v_{\tau(n)}-\delta_{\tau(n)}\|=0$$

With $v_{\tau(n)}\in T_1\delta_{\tau(n)}$ and $u_{\tau(n)}\in T_2y_{\tau(n)}$, it follows that

$$\lim_{n\to\infty}d\Big(\delta_{\tau(n)},T_1\delta_{\tau(n)}\Big)=0 \qquad \textit{and} \qquad \lim_{n\to\infty}d\Big(y_{\tau(n)},T_2y_{\tau(n)}\Big)=0$$

Following the same argument in case 1, we conclude that

$$\lim_{\tau(n)\to+\infty}\sup\langle\eta\mathbf{B}x^*-\gamma f(x^*),x^*-x_{\tau(n)+1}\rangle\leq 0$$

Therefore, for all $n\geq n_0$,and from 3.29, we have

$$\begin{aligned}0\leq\|x_{\tau(n)+1}-x^*\|^2-\|x_{\tau(n)}-x^*\|^2 &\leq \alpha_{\tau(n)}[-\alpha_n(\tau-\gamma b)\|x_{\tau(n)}-x^*)\|^2\\ &+ 2\alpha_{\tau(n)}\langle\eta\mathbf{B}x^*-\gamma f(x^*),x_{\tau(n)+1}-x^*\rangle\end{aligned}$$

$$\|x_{\tau(n)}-x^*\|^2\leq\frac{2}{\tau-\gamma b}\langle\eta\mathbf{B}x^*-\gamma f(x^*),x_{\tau(n)+1}-x^*\rangle$$

Then we have $\lim_{n\to\infty}\|x_{\tau(n)}-x^*\|^2=0$. Therefore $\lim_{n\to\infty}\mathbf{W}_{\tau(n)}=\lim_{n\to\infty}\mathbf{W}_{\tau(n)+1}=0$.

Furthermore, for all $n\geq n_0$, we have $\mathbf{W}_{\tau(n)}\leq\mathbf{W}_{\tau(n)+1}$ if $n\neq\tau(n)$ (that is $n>\tau(n)$, because $\mathbf{W}_j>\mathbf{W}_{j+1}$, $\quad for \quad \tau(n+1)\leq j\leq n$.

Hence, $0\leq\mathbf{W}_{\tau(n)}\leq\max\left\{\mathbf{W}_{\tau(n)},\mathbf{W}_{\tau(n)+1}\right\}=\mathbf{W}_{\tau(n)+1}$. Therefore, $\mathbf{W}_n\to 0, \quad as \quad n\to\infty$ and this implies that $x_n\to x^* \quad as \quad n\to\infty$. This complete the proof.

□

Now using theorem (3.1), and multivalued mappings are nonexpansive mappings with convex values without demiclosed assumptions in the following theorem.

Theorem 3.2. *Let H be a real Hilbert space and $\mathbb{K}$ be a nonempty, closed convex subset of H. Let $\mathbf{A}:\mathbb{K}\to H$ be an $\alpha-$inverse strongly monotone operator and let $\mathbf{B}:H\to H$ be an $k-$strongly monotone and $L-$Lipschitzian operator. Let $f:\mathbb{K}\to H$ be an $b-$Lipschitzian mapping and $\mathbb{M}:H\to 2^H$ be a maximal monotone mapping such that the domain of $\mathbb{M}$ is included in $\mathbb{K}$. Let $T_1,T_2:\mathbb{K}\to CB(\mathbb{K})$ be a multivalued $\beta-$ demicontractive mapping and $T_3:\mathbb{K}\to CB(\mathbb{K})$ be a multivalued quasi-nonexpansive mapping such that $\Omega:=Fix(T_1)\cap Fix(T_2)\cap Fix(T_3)\cap S(\Pi,\Lambda)\neq\emptyset$ and $T_1q=T_2q=T_3q=\{q\},\forall q\in\Omega$. For given $x_0\in\mathbb{K}$, let $\{x_n\}$ be generated by the algorithm:*

$$\begin{cases}\delta_n=J^{\mathbb{M}}_{\lambda_n}(I-\lambda_n\mathbf{A})x_n;\\ y_n=\theta_n\delta_n+(1-\theta_n)v_n, \quad v_n\in T_1\delta_n;\\ z_n=\beta_ny_n+(1-\beta_n)u_n, \quad u_n\in T_2y_n;\\ t_n=\gamma_nz_n+(1-\gamma_n)w_n, \quad w_n\in T_3z_n;\\ x_{n+1}=P_{\mathbb{K}}(\alpha_n\gamma f(x_n)+(I-\eta\alpha_n\mathbf{B})t_n)\end{cases}\tag{3.30}$$

where $\{\beta_n\},\{\gamma_n\},\{\theta_n\},\{\lambda_n\}$ and $\{\alpha_n\}$ are real sequence in $(0,1)$ satisfying the following conditions

i): $\lim_{n\to\infty} \alpha_n = 0 \qquad \sum_{n=0}^{\infty} \alpha_n < \infty$

ii): $\lim_{n\to\infty} \inf(1-\beta_n)(\beta_n-\beta) > 0$ *and* $\lim_{n\to\infty} \inf(1-\theta_n)(\theta_n-\beta) > 0, (\beta_n, \theta_n) \in (\beta, 1)$

iii): $\lim_{n\to\infty} \inf(1-\gamma_n)\gamma_n) > 0$

Assume that $0 < \eta < \frac{2k}{L^2}, 0 < \gamma b < \tau$, *where* $\tau = \eta\left(k - \frac{L^2\eta}{2}\right)$, *and the sequences defined in (3.30), that is* $\{x_n\}$ *and* $\{\delta_n\}$ *converge strongly to unique solution* $x^* \in \Omega$, *which also solve the following variational inequality:*

$$\langle \eta \mathbf{B}x^* - \gamma f(x^*), x^* - q\rangle \leq 0, \quad \forall q \in \Omega \tag{3.31}$$

Proof. Since every multivalued nonexpansive mapping is quasi-nonexpansive and demicontractive, then, the proof follows Theorem 3.10

□

Now using the same argument of the proof in theorem (3.1) in theorem (3.3), we achieved the desired results. In theorem (3.3), we let $T_1 = P_{T_1}, T_2 = P_{T_2}$ and $T_3 = P_{T_3}$ without the assumptions that $T_1q = T_2q = T_3q = \{q\}, \forall q \in \Omega$

Theorem 3.3. *Let* H *be a real Hilbert space and* $\mathbb{K}$ *be a nonempty, closed convex subset of* H. *Let* $\mathbf{A} : \mathbb{K} \to H$ *be an* $\alpha-$*inverse strongly monotone operator and let* $\mathbf{B} : H \to H$ *be an* $k-$*strongly monotone and* $L-$*Lipschitzian operator. Let* $f : \mathbb{K} \to H$ *be an* $b-$*Lipschitzian mapping and* $\mathbb{M} : H \to 2^H$ *be a maximal monotone mapping such that the domain of* $\mathbb{M}$ *is included in* $\mathbb{K}$. *Let* $T_1, T_2 : \mathbb{K} \to CB(\mathbb{K})$ *be a multivalued* $\beta-$ *demicontractive mapping and* $T_3 : \mathbb{K} \to CB(\mathbb{K})$ *be a multivalued quasi-nonexpansive mapping such that* $\Omega := Fix(T_1) \cap Fix(T_2) \cap Fix(T_3) \cap S(\mathbb{M}, \mathbf{A}) \neq \emptyset$. *For given* $x_0 \in \mathbb{K}$, *let* $\{x_n\}$ *be generated by the algorithm:*

$$\begin{cases} \delta_n = J_{\lambda_n}^{\mathbb{M}}(I - \lambda_n \mathbf{A})x_n; \\ y_n = \theta_n\delta_n + (1-\theta_n)v_n, \quad v_n \in T_1\delta_n; \\ z_n = \beta_n y_n + (1-\beta_n)u_n, \quad u_n \in T_2 y_n; \\ t_n = \gamma_n z_n + (1-\gamma_n)w_n, \quad w_n \in T_3 z_n; \\ x_{n+1} = P_{\mathbb{K}}(\alpha_n\gamma f(x_n) + (I - \eta\alpha_n\mathbf{B})t_n) \end{cases} \tag{3.32}$$

where $\{\beta_n\}, \{\gamma_n\}, \{\theta_n\}, \{\lambda_n\}$ *and* $\{\alpha_n\}$ *are real sequence in* $(0, 1)$ *satisfying the following conditions*

i): $\lim_{n\to\infty} \alpha_n = 0 \qquad \sum_{n=0}^{\infty} \alpha_n < \infty$

ii): $\lim_{n\to\infty}\inf(1-\beta_n)(\beta_n-\beta)>0$ *and* $\lim_{n\to\infty}\inf(1-\theta_n)(\theta_n-\beta)>0, (\beta_n,\theta_n)\in(\beta,1)$

iii): $\lim_{n\to\infty}\inf(1-\gamma_n)\gamma_n)>0$

Assume that $0<\eta<\frac{2k}{L^2}, 0<\gamma b<\tau$, *where* $\tau=\eta\Big(k-\frac{L^2\eta}{2}\Big)$, *and* $I-P_{T_1}, I-P_{T_2}$ *and* $I-P_{T_3}$ *are demiclosed at origin. Hence, the sequences defined in (3.32), that is* $\{x_n\}$ *and* $\{\delta_n\}$ *converge strongly to unique solution* $x^*\in\Omega$, *which also solve the following variational inequality:*

$$\langle\eta\mathbf{B}x^*-\gamma f(x^*), x^*-q\rangle\le 0, \quad \forall q\in\Omega \tag{3.33}$$

4. Conclusion

The modified general viscosity iterative process presented in this research offers a powerful tool for solving variational inclusion and fixed point problems involving and and fixed point problem with respectively set-valued maximal monotone mapping and inverse strongly monotone and multivalued quasi-nonexpansive and demicontractive operators. Our Theorem presents a new and a modified algorithm for solving simultaneously variational inclusion problem and fixed point problem with respectively set-valued maximal monotone mapping and inverse strongly monotone and multivalued demicontractive and quasi-nonexpansive mappings. The result we show here improves and extends the corresponding results of some authors and many other recent results using forward-backward splitting method and general iterative algorithm that gives a strong convergence to a unique solution.

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