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Eur. J. Math. Anal. 4 (2024) 1
doi: 10.28924/ada/ma.4.1

Frame Operators for Frames in Krein Spaces

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Abstract. In recent years, frames in Krein spaces and several generalizations have been extensively studied. In this paper, we propose an alternative way of looking at the notion of frames in Krein spaces and give a necessary and sufficient condition for a sequence in a Krein space to be a Bessel sequence. We observe that a subsequence of a frame in a Krein space need not be a frame. Also, two complementary subsequences are considered in which one of them is a frame for a Krein space. We obtain necessary and sufficient conditions under which the other one is also a frame for the Krein space.

1. Introduction

Hilbert space frames were originally introduced by Duffin and Schaeffer [7] to deal with some problems in non-harmonic Fourier analysis. The linear independence property for a (Hamel) basis, which allows every vector to be uniquely represented as a linear combination is very restrictive for practical problems. Frames allow each element in the space to be written as a linear combination of the elements in the frame, but linear independence is not required. Frames can be viewed as redundant bases which are generalization of Riesz bases. This redundancy property sometimes is extremely important in applications such as sampling theory [9], filter banks [3], signal and image processing [6] and so on.

Definition 1.1. *[4] Let $\mathcal{H}$ be a Hilbert space and I be a countable index set. A collection $\{f_n\}_{n\in I}$ in a Hilbert space $\mathcal{H}$ is said to be a frame for H if there exist $a, b > 0$ such that*

$$a\|f\|^2 \leq \sum_{n\in I} |\langle f, f_n\rangle|^2 \leq b\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Received: 19 Nov 2023.
2020 *Mathematics Subject Classification.* 42C15, 46C05, 46C20.
Key words and phrases. Krein space; Bessel sequence; frame sequence; frame operator.

We now look at the definition of frame which is equivalent to perceive as the map

$$\mathcal{H} \ni f \mapsto \sum_{n\in I} \langle f, f_n\rangle f_n \in \mathcal{H} \tag{1}$$

which is a well-defined bounded positive invertible operator.
The bounded linear operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ defined by

$$Sf = \sum_{n\in I} \langle f, f_n\rangle f_n, \quad f \in \mathcal{H},$$

is known as the frame operator associated to the frame $\{f_n\}_{n\in I}$. This operator S is bounded invertible, positive and self adjoint. It allows to reconstruct each vector in terms of the sequence $\{f_n\}_{n\in I}$ as follows:

$$f = \sum_{n\in I} \langle f, S^{-1}f_n\rangle f_n = \sum_{n\in I} \langle f, f_n\rangle S^{-1}f_n. \tag{2}$$

The formula (2) is known as reconstruction formula associated to $\{f_n\}_{n\in I}$ and if $S = I$, then the reconstruction formula resembles the Fourier series of f associated with the orthonormal sequence $\{f_n\}_{n\in I}$.

The concept of indefinite inner product was first found in a paper on quantum field theory by Dirac in 1942 [5]. Pontrjagin gave the mathematical interpretation of indefinite inner product. Giribet et al. have introduced and studied frames for Krein spaces [8]. Motivated by the equivalent definition of frame as given in (1), in this paper, we propose an alternative way of looking at the notion of frames in Krein spaces by decomposing the index set I in a natural way and obtain some new results on frames sequences.

The paper is organized as follows. Standard definition of Krein space is given in Section 2 along with some notations and examples which will be used in the sequel. In Section 3, we define the concept of Bessel sequence in Krein spaces and give a necessary and sufficient condition for a sequence to be a Bessel sequence in Krein spaces. In Section 4, we give the definition of frame for Krein space and study operators associated to the frame. In the last section, we study frame sequences in Krein spaces. In general, if $\{f_n\}_{n\in I}$ is a frame in a Krein space and $\{n_k\}$ is any infinite increasing sequence in I, then $\{f_{n_k}\}$ need not be a frame sequence. We provide some sufficient conditions under which subsequences become frame sequence for the Krein space.

2. Preliminaries

Let $\mathcal{K}$ be a complex vector space with a Hermitian sesquilinear form defined on it. Then we call $(\mathcal{K}, [.,.])$ an inner product space. An element $x \in \mathcal{K}$ is called neutral, positive, or negative if $[x, x] = 0$, $[x, x] > 0$, or $[x, x] < 0$ respectively. If $\mathcal{K}$ contains positive as well as negative elements, then it is called an indefinite inner product space, otherwise it is called a semi-definite inner product space. We refer [1, 2] for basics on indefinite inner product spaces.

An indefinite inner product space $(\mathcal{K}, [., .])$ is decomposable if it can be written as an orthogonal direct sum of a neutral subspace $\mathcal{K}^0$, a positive definite subspace $\mathcal{K}^+$ and a negative definite subspace $\mathcal{K}^-$:

$$\mathcal{K} = \mathcal{K}^0[\dot{+}]\mathcal{K}^+[\dot{+}]\mathcal{K}^-. \tag{3}$$

Then (3) is known as a fundamental decomposition of $\mathcal{K}$.

An indefinite inner product space $(\mathcal{K}, [., .])$ is a Krein space if it can be written as an orthogonal direct sum of a positive definite subspace $\mathcal{K}^+$ and a negative definite subspace $\mathcal{K}^-$ such that $(\mathcal{K}^+, [., .])$ and $(\mathcal{K}^-, -[., .])$ are Hilbert spaces. Let a fundamental decomposition of a Krein space $\mathcal{K}$ be given by

$$\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^- \tag{4}$$

and $P^\pm$ be the orthogonal projections onto $\mathcal{K}^\pm$. The linear map

$$J = P^+ - P^-$$

is called the fundamental symmetry corresponding to (4). Then

$$(f, g)_J = [Jf, g]$$

is a positive definite inner product on $\mathcal{K}$, called J-inner product corresponding to the fundamental decomposition (4). We can write

$$(f, f)_J = [Jf, f] = [(2P^+ - I)f, f] = 2[P^+f, P^+f] - [f, f]. \tag{5}$$

The corresponding norm (called J-norm) is denoted by

$$\|f\|_J = (f, f)_J^{\frac{1}{2}} = [Jf, f]^{\frac{1}{2}}.$$

Example 2.1. *Consider $\mathcal{K} = \ell_2(\mathbb{N})$, the linear space of square-summable sequences, with*

$$[f, g] = \sum_{n=1}^{\infty}(-1)^n f_n\overline{g_n} \quad \text{for} \ \ f = (f_n)_{n=1}^\infty, \ g = (g_n)_{n=1}^\infty \in \mathcal{K}.$$

Let $\mathcal{K}^+ = \{(f_n)_{n=1}^\infty : f_n = 0 \text{ if } n \text{ is odd}\}$ and $\mathcal{K}^- = \{(f_n)_{n=1}^\infty : f_n = 0 \text{ if } n \text{ is even}\}$. Then $\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-$, where $\mathcal{K}^+$ and $\mathcal{K}^-$ are complete with respect to the induced norm and hence $\mathcal{K}$ is a Krein space.

Theorem 2.1. *[2] Let $\mathcal{K}$ be a Krein space. Then the following are equivalent:*

(1) *There exists a fundamental decomposition of $\mathcal{K}$.*
(2) *There exists a maximal uniformly positive ortho-complemented subspace.*
(3) *There exists a maximal uniformly negative ortho-complemented subspace.*
(4) *There exists a mapping J in K such that $J = J^* = J^{-1}$.*

Different fundamental decompositions induce different J-norms. Hence various norms can be defined on a Krein space by choosing different underlying fundamental decompositions.

Example 2.2. *Let $\mathcal{K}$ be a two-dimensional vector space with basis $\{e_1, e_2\}$ and an indefinite inner product defined by $[e_1, e_1] = 1, [e_2, e_2] = -1$ and $[e_1, e_2] = 0$. If we take $Y = span\{e_1\}$, then it is a maximal uniformly positive definite subspace and hence there exists a fundamental decomposition of $\mathcal{K}$ with $\mathcal{K}^+ = Y$ and $\mathcal{K}^- = span\{e_2\}$. Choosing $\mathcal{K}_n^+ = span\{(n, 1)\}$ and $\mathcal{K}_n^- = span\{(1, n)\}$ where $n > 1$, we get several fundamental decompositions. The corresponding fundamental symmetries J_n are given by*

$$J_n = \begin{pmatrix} \frac{n^2+1}{n^2-1} & \frac{-2n}{n^2-1} \\ \frac{2n}{n^2-1} & \frac{-(n^2+1)}{n^2-1} \end{pmatrix}.$$

Here we can see that the fundamental symmetries J_n satisfy $J_n^2 = I_n$, $[J_n f, g] = [f, J_n g]$ and $[J_n f, J_n g] = [f, g]$ for all $f, g \in \mathcal{K}$.

3. Frame Operator for Frames in Krein Spaces

Let $\mathcal{K}$ be a Krein space and let $\{f_n\}_{n\in I}$ be a sequence in $\mathcal{K}$. In relation to the sequence $\{f_n\}_{n\in I}$, the index set I is decomposed as $I_+ = \{n \in I : [f_n, f_n] \geq 0\}$ and $I_- = \{n \in I : [f_n, f_n] < 0\}$. It is easy to observe that $\ell_2(I)$ is the orthogonal direct sum of $\ell_2(I_+)$ and $\ell_2(I_-)$.

Definition 3.1. *A sequence $\{f_n\}_{n\in I}$ in a Krein space $\mathcal{K}$ is called a Bessel sequence if there exists a constant $B > 0$ such that*

$$\sum_{n\in I} |[f_n, f]|^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{K}. \tag{6}$$

The constant B in the inequality (6) is called a Bessel bound for $\{f_n\}_{n\in I}$.

Theorem 3.1. *Let $\{f_n\}_{n\in I}$ be a sequence in a Krein space $\mathcal{K}$. Then $\{f_n\}_{n\in I}$ is a Bessel sequence with a Bessel bound B if and only if the operators $T^+ : \ell_2(I_+) \longrightarrow \mathcal{K}^+$ defined by $T^+\{c_n\}_{n\in I_+} = \sum_{n\in I_+} c_n f_n$ and $T^- : \ell_2(I_-) \longrightarrow \mathcal{K}^-$ defined by $T^-\{c_n\}_{n\in I_-} = \sum_{n\in I_-} c_n f_n$ are well defined bounded operators and $\|T\| \leq \sqrt{B}$, where $T = T^+ + T^-$.*

Proof. Suppose first that $\{f_n\}_{n\in I}$ is a Bessel sequence with a Bessel bound B. Let $\{c_n\}_{n\in I_+} \in \ell_2(I_+)$ and $\{c_n\}_{n\in I_-} \in \ell_2(I_-)$. Let $\ell, m \in I_-$ such that $\ell > m$. Then

$$\begin{aligned} \left\|\sum_{n=1}^{\ell} c_n f_n - \sum_{n=1}^{m} c_n f_n\right\| &= \left\|\sum_{n=m+1}^{\ell} c_n f_n\right\| \\ &= \sup_{\|g\|=1} |[\sum_{n=m+1}^{\ell} c_n f_n, g]| \\ &\leq \sup_{\|g\|=1} \sum_{n=m+1}^{\ell} |[c_n f_n, g]| \end{aligned}$$

$$\leq \left(\sum_{n=m+1}^{\ell} |c_n|^2\right)^{\frac{1}{2}} \sup_{\|g\|=1} \left(\sum_{n=m+1}^{\ell} |[f_n, g]|^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{B}\left(\sum_{n=m+1}^{\ell} |c_n|^2\right)^{\frac{1}{2}}.$$

Since $\{c_n\}_{n\in I_-} \in \ell_2(I_-)$, $\{\sum_{n=1}^{\ell} |c_n|^2\}$ is a Cauchy sequence in $\mathbb{C}$. The above calculation shows that $\{\sum_{n=1}^{\ell} c_n f_n\}_{\ell\in I_-}$ is a Cauchy sequence in $\mathcal{K}^-$, and so it is convergent. Hence T^- is well defined. With similar arguments and by considering $\ell, m \in I_+$ with $\ell > m$ one may prove that $\{\sum_{n=1}^{\ell} c_n f_n\}_{\ell\in I_+}$ is a Cauchy sequence in $\mathcal{K}^+$ and so it is convergent. Thus T^- and T^+ are well defined and boundedness follows from the above calculation. Clearly T^- and T^+ are linear. Conversely, suppose that T^- and T^+ are well defined bounded linear operators with their adjoints $T^{-*} : \mathcal{K}^- \longrightarrow \ell_2(I_-)$ and $T^{+*} : \mathcal{K}^+ \longrightarrow \ell_2(I_+)$ defined by $T^{-*}f = \{-[f, f_n]\}_{n\in I_-}$ and $T^{+*}f = \{[f, f_n]\}_{n\in I_+}$ respectively. Since the adjoint of a bounded operator is bounded, $\|T^{-*}\| = \|T^-\|$ and $\|T^{+*}\| = \|T^+\|$. Also we have

$$\|T^{+*}f\|^2 \leq \|T^+\|^2\|f\|^2, \text{ for all } f \in \mathcal{K}^+$$

and

$$\|T^{-*}f\|^2 \leq \|T^-\|^2\|f\|^2, \text{ for all } f \in \mathcal{K}^-.$$

So $\|T^*f\|^2 \leq \|T\|^2\|f\|^2$, for all $f \in \mathcal{K}$. Hence $\{f_n\}_{n\in I}$ is a Bessel sequence. □

Corollary 3.2. *Let $\{f_n\}_{n\in I}$ be a sequence in a Krein space $\mathcal{K}$ such that both $\sum_{n\in I_+} c_n f_n$ and $\sum_{n\in I_-} c_n f_n$ are convergent for all $\{c_n\}_{n\in I_-} \in \ell_2(I_-)$ and $\{c_n\}_{n\in I_+} \in \ell_2(I_+)$. Then $\{f_n\}_{n\in I}$ is a Bessel sequence.*

The condition (6) remains unchanged regardless how the elements of $\{f_n\}_{n\in I}$ are numbered.

Corollary 3.3. *Let $\{f_n\}_{n\in I}$ be a Bessel sequence in a Krein space $\mathcal{K}$. Then $\sum_{n\in I_+} c_n f_n$ and $\sum_{n\in I_-} c_n f_n$ converge unconditionally for all $\{c_n\}_{n\in I_+} \in \ell_2(I_+)$ and $\{c_n\}_{n\in I_-} \in \ell_2(I_-)$ respectively.*

The following example illustrates that how the norm of a single element actually depends upon the choice of fundamental decomposition. If a frame is defined relative to fundamental decomposition, then frame bounds will vary arbitrarily when difference fundamental decompositions are considered.

Example 3.2. *Consider the two dimensional Minkowski space $\mathcal{K} = \mathbb{R}^2$ with the inner product $[f, g] = f_1g_1 - f_2g_2$ where $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathbb{R}^2$. Consider the fundamental decompositions with $\mathcal{K}_n^+ = span\{(\frac{n+1}{n}, \frac{n-1}{n})\}$ and $\mathcal{K}_n^- = span\{(\frac{n-1}{n}, \frac{n+1}{n})\}$ where $n > 1$. Then we get*

$$\|f\|_{J_n}^2 = \frac{1}{4}[(2n + 2/n)(f_1^2 + f_2^2) + 4f_1f_2(1/n - n)].$$

Let $f = (1, 1)$ and $g = (1, 0)$. Then $\|f\|_{J_n}^2 = \frac{2}{n}$ and $\|g\|_{J_n}^2 = \frac{1}{2}(n + \frac{1}{n})$.

Let $\{f_n\}_{n\in I}$ be a Bessel sequence in $\mathcal{K}$. Then both $\{f_n\}_{n\in I_+} \subset \mathcal{K}^+$ and $\{f_n\}_{n\in I_-} \subset \mathcal{K}^-$ are Bessel sequences. Define $T^+ : \ell_2(I_+) \longrightarrow \mathcal{K}^+$ and $T^- : \ell_2(I_-) \longrightarrow \mathcal{K}^-$ by

$$T^+\{c_n\} = \sum_{n\in I_+} c_n f_n$$

and

$$T^-\{c_n\} = \sum_{n\in I_-} c_n f_n$$

respectively. Then T^+ and T^- are both bounded linear operators and are called synthesis operators. Define $T^{+*} : \mathcal{K}^+ \longrightarrow \ell_2(I_+)$ and $T^{-*} : \mathcal{K}^- \longrightarrow \ell_2(I_-)$ by

$$T^{+*}f = \{[f, f_n]\}_{n\in I_+}$$

and

$$T^{-*}f = \{[f, f_n]\}_{n\in I_-}$$

are called analysis operators. Thus $S^+ = T^+T^{+*} : \mathcal{K}^+ \longrightarrow \mathcal{K}^+$ given by $S^+f = T^+T^{+*}f = \sum_{n\in I_+}[f, f_n]f_n$ and $S^- = T^-T^{-*} : \mathcal{K}^- \longrightarrow \mathcal{K}^-$ given by $S^-f = T^-T^{-*}f = \sum_{n\in I_-}[f, f_n]f_n$. Therefore the frame operator $S = S^+ + S^- : \mathcal{K} \longrightarrow \mathcal{K}$ is defined by

$$Sf = (S^+ + S^-)f = \sum_{n\in I_+}[f, f_n]f_n + \sum_{n\in I_-}[f, f_n]f_n.$$

Esmeral et al. [5] have given a defintion of frame which involves fundamental symmetry of the Krein space $\mathcal{K}$. As shown in the Example 3.2, for sufficiently large values on n, J-norms of elements of $\mathcal{K}$ can be too small or too large. Thus we propose the following definition for frame in Krein spaces.

Definition 3.3. *Let $\{f_n\}_{n\in I}$ be a Bessel sequence in $\mathcal{K}$. The sequence $\{f_n\}_{n\in I}\}$ is said to be a frame if the frame operator $S = S^+ + S^- : \mathcal{K} \longrightarrow \mathcal{K}$ defined by*

$$Sf = (S^+ + S^-)f = \sum_{n\in I_+}[f, f_n]f_n + \sum_{n\in I_-}[f, f_n]f_n$$

is a bounded positive invertible operator.

The following is the frame decomposition theorem for the Krein space $\mathcal{K}$, which states that if $\{f_n\}_{n\in I}$ is a frame for a Krein space $\mathcal{K}$, then every element can be written as a linear combination of frame elements.

Theorem 3.4. *Let $\{f_n\}_{n\in I}$ be a frame for a Krein space $\mathcal{K}$ with the frame operator S. Then*

$$f = \sum_{n\in I_+}[f, {S^+}^{-1}f_n]f_n + \sum_{n\in I_-}[f, {S^-}^{-1}f_n]f_n, \text{ for all } f \in \mathcal{K}, \tag{7}$$

and both the series converges unconditionally for all $f \in \mathcal{K}$.

Proof. Since the operator S is self adjoint and invertible, we have

$$\begin{aligned} f &= SS^{-1}f \\ &= \sum_{n\in I}[S^{-1}f, f_n]f_n \\ &= \sum_{n\in I_+}[S^{+^{-1}}f, f_n]f_n + \sum_{n\in I_-}[S^{-^{-1}}f, f_n]f_n \\ &= \sum_{n\in I_+}[f, S^{+^{-1}}f_n]f_n + \sum_{n\in I_-}[f, S^{-^{-1}}f_n]f_n \end{aligned}$$

Since $\{f_n\}_{n\in I}$ is a frame with $[f, S^{+^{-1}}f_n] \in \ell_2(I_+)$ and $[f, S^{-^{-1}}f_n]f_n \in \ell_2(I_-)$, the unconditional convergence follows from Corollary 3.3. □

4. Frame Sequences in Krein Spaces

We begin this section with the following definitions for frame sequence

Definition 4.1. *Let $\mathcal{K}$ be a Krein space. A sequence $\{f_n\}_{n\in I} \in \mathcal{K}$ is called a*

(a) *frame sequence if it is a frame for $[f_n] = \overline{span}\{f_n \ : n \in I\}$.*
(b) *exact if removal of an arbitrary f_n render the collection $\{f_n\}$ no longer a frame for the Krein space $\mathcal{K}$.*
(c) *near exact if it can be made exact by removing finitely many elements from it.*

Example 4.2. *Let $\{f_n\}_{n\in I}$ be a sequence of unit orthonormal vectors in Krein space $\mathcal{K}$ and $\{n_k\}$ be any infinite increasing subset of I. Then $\{f_{n_k}\}$ is a frame sequence.*

Let $\{f_n\}_{n\in I}$ be a frame for a Krein space $\mathcal{K}$ and $\{n_k\}$ be any infinite increasing sequence in I. Then $\{f_{n_k}\}$ need not be a frame sequence.

Example 4.3. *Let $\{f_n\}_{n\in I}$ be a sequence of unit orthonormal vectors in a Krein space $\mathcal{K}$. Define a sequence $\{h_n\}_{n\in I} \in \mathcal{K}$ by $h_n = \frac{1}{\sqrt{n}}f_n, n \in I$. Let $n_k = n_{k-1} + (k-1), k \in I$ and $n_0 = \pm 1$. Then $\{n_k\}$ is an infinite increasing sequence in I. Define another sequence $\{g_n\}_{n\in I} \in \mathcal{K}$ by $g_1 = h_1, g_{n_k} = g_{n_k+1} = g_{n_k+2} = \cdots = g_{n_k+1} - 1 = h_k, k \geq 2$. Then $\{g_n\}_{n\in I}$ is a tight frame for $\mathcal{K}$. But note that $\{g_{n_k}\} = \{h_k\}$ is not a frame sequence.*

The following theorem gives a necessary and sufficient condition for a existence of a subsequence to be a frame for a Krein space $\mathcal{K}$.

Theorem 4.1. *Let $\{f_n\}_{n\in I}$ be a frame for a Krein space $\mathcal{K}$ and let $\{m_k\}$ and $\{n_k\}$ be two infinite increasing sequences in I with $\{m_k^+\}\cup\{n_k^+\} = I_+$ and $\{m_k^-\}\cup\{n_k^-\} = I_-$. If $\{f_{m_k}\}_{m_k\in I}$ is a frame, then $\{f_{n_k}\}_{n_k\in I}$ is a frame if and only if there exists a bounded linear operator $T : \ell_2(I) \longrightarrow \ell_2(I)$ such that $T = T^+ + T^-$, where $T^+ : \ell_2(I_+) \longrightarrow \ell_2(I_+)$ defined by $T^+\{[f_{n_k}^+, f^+]\} = \{[f_{m_k}^+, f^+]\}, f^+ \in \mathcal{K}^+$ and $T^- : \ell_2(I_-) \longrightarrow \ell_2(I_-)$ is defined by $T^-\{[f_{n_k}^-, f^-]\} = \{[f_{m_k}^-, f^-]\}, f^- \in \mathcal{K}^-$.*

Proof. Suppose that $\{f_{m_k}\}_{m_k\in I}$ is a frame with lower frame bounds A and $A^{'}$. Then

$$\begin{aligned}\sum_{m_k\in I_+}|[f_{m_k},f^+]|^2 &= \sum_{n_k\in I_+}\|[T^+\{[f^+_{n_k},f^+]\}\|\\ &\le \|T^+\|\sum_{n_k\in I_+}|[f_{n_k},f^+]|^2.\end{aligned}$$

So, we have

$$\begin{aligned}\sum_{n_k\in I_+}|[f_{n_k},f^+]|^2 &\ge \frac{\sum_{m_k\in I_+}|[f_{m_k},f^+]|^2}{\|T^+\|}\\ &\ge \frac{A}{\|T\|}\|f^+\|^2.\end{aligned}$$

Similarly, we have

$$\sum_{n_k\in I_-}|[f_{n_k},f^-]|^2 \ge \frac{A^{'}}{\|T\|}\|f^-\|^2.$$

Hence $\{f_{n_k}\}_{n_k\in I}$ is a frame for the Krein space $\mathcal{K}$.

Conversely, suppose that $\{f_{n_k}\}_{n_k\in I}$ is a frame for the Krein space $\mathcal{K}$. Then there exist operators ${T_1}^+:\ell_2(I_+)\longrightarrow\mathcal{K}_1$ given by ${T_1}^+\{[f^+_{n_k},f^+]\}\longrightarrow f^+$ and $T_1^{*+}:\mathcal{K}_1\longrightarrow\ell_2(I_+)$ given by $T_1^{*+}f^+=\{[f^+_{n_k},f^+]\}$ and similarly there exist operators ${T_1}^-:\ell_2(I_-)\longrightarrow\mathcal{K}_2$ given by ${T_1}^-\{[f^-_{n_k},f^-]\}=f^-$ and ${T_1^-}^*:\mathcal{K}_2\longrightarrow\ell_2(I_-)$ given by ${T_1^-}^*f^-=\{[f^-_{n_k},f^-]\}$. Also, since $\{f_{m_k}\}_{m_k\in I}$ is a frame for the Krein space $\mathcal{K}$, there exist operators $T_2^+:\ell_2(I_+)\longrightarrow\mathcal{K}_1$ given by ${T_2}^+\{[f^+_{m_k},f^+]\}=f^+$ and $T_2^{+*}:\mathcal{K}_1\longrightarrow\ell_2(I_+)$ given by $T_2^{+*}f^+=\{[f^+_{m_k},f^+]\}$.

Similarly, there exist operators ${T_2}^-:\ell_2(I_-)\longrightarrow\mathcal{K}_2$ given by ${T_2}^-\{[f^-_{m_k},f^-]\}=f^-$ and $T_2^{-*}:\mathcal{K}_2\longrightarrow\ell_2(I_-)$ given by ${T_2}^{-*}f^-=\{[f^-_{m_k},f^-]\}$. Then $T^+={T_2^+}^*{T_1}^+:\ell_2(I_+)\longrightarrow\ell_2(I_+)$ is a bounded linear operator such that $T^+\{[f^+_{n_k},f^+]\}=\{[f^+_{m_k},f^+]\},f^+\in\mathcal{K}^+$ and $T^-={T_2^-}^*{T_1}^-:\ell_2(I_-)\longrightarrow\ell_2(I_-)$ is a bounded linear operator such that $T^-\{[f^-_{n_k},f^-]\}=\{[f^-_{m_k},f^-]\},f^-\in\mathcal{K}^-$. □

Next, we give a sufficient condition for two subsequences of a frame for $\mathcal{K}$ to be a frame sequence.

Theorem 4.2. *Let $\{f_n\}_{n\in I}$ be a frame for a Krein space $\mathcal{K}$. Let $\{m_k\}$ and $\{n_k\}$ be two infinite increasing sequences in I with $\{m_k^+\}\cup\{n_k^+\}=I_+$ and $\{m_k^-\}\cup\{n_k^-\}=I_-$. Let $\mathcal{K}_1=[f^+_{m_k}]\cap[f^+_{n_k}]$. If $\mathcal{K}_1$ is a finite dimensional space, then $\{f^+_{m_k}\}$ and $\{f^+_{n_k}\}$ are frame sequences for $\mathcal{K}^+$. Further, if $\mathcal{K}_2=[f^-_{m_k}]\cap[f^-_{n_k}]$ is finite dimensional, then $\{f^-_{m_k}\}$ and $\{f^-_{n_k}\}$ are frame sequences for $\mathcal{K}^-$.*

Proof. Let $\{\ell_k^+\}$ be a finite subsequence of $\{n_k^+\}$ such that $\mathcal{K}_1=[f_{\ell_k}]_{\ell_k\in I_+}$. Since $\mathcal{K}_1$ is finite dimensional, $\{f^+_{\ell_k}\}$ is a frame for $\mathcal{K}_1$. Let $A^{'}$ and $B^{'}$ be the frame bounds for $\{f^+_{\ell_k}\}$. Consider $\{f_{n_k}\}_{n_k\in I_+}$, let $f^+\in\{f_{n_k}\}_{n_k\in I_+}$ be any element. Now, if $f^+[\perp]\mathcal{K}_1$, then

$$\begin{aligned}\sum_{n\in I_+}[f^+,f_n]^2 &= \sum_{n_k\in I_+}[f^+,f_{n_k}]^2\\ &\ge A\|f^+\|^2.\end{aligned}$$

Also, if $f^+ \in \mathcal{K}_1$, then

$$\begin{aligned}\sum_{n_k \in I_+} [f^+, f_{n_k}]^2 &\geq \sum_{\ell_k \in I_+} [f^+, f_{l_k}]^2 \\ &\geq A^{'} \|f^+\|^2.\end{aligned}$$

Otherwise, we have

$$\begin{aligned} f^+ &= \sum \alpha_k f_{n_k} \\ &= \sum \alpha_i f_n + \sum \alpha_j f_j, \quad i \in \{n_k\} \backslash \{\ell_k\}, j \in \{\ell_k\} \\ &= (f^+)^{'} + (f^+)^{''}, \text{where } (f^+)^{'} [\perp] \mathcal{K}_1 \text{ and } (f^+)^{''} \in \mathcal{K}_1. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n_k \in I_+} [f^+, f_{n_k}]^2 &= \sum [f^+, f_n]^2 + \sum [f^+, f_j]^2, \quad i \in \{n_k\} \backslash \{\ell_k\}, j \in \{\ell_k\} \\ &= \sum [(f^+)^{'} + (f^+)^{''}, f_n]^2 + \sum [(f^+)^{'} + (f^+)^{''}, f_j]^2 \\ &= \sum [(f^+)^{'}, f_n]^2 + \sum [(f^+)^{''}, f_j]^2 \\ &\geq A \|(f^+)^{'}\|^2 + A^{'} \|(f^+)^{''}\| \\ &\geq \min\left\{\frac{A}{2}, \frac{A^{'}}{2}\right\} \|f^+\|^2. \end{aligned}$$

Hence $\{f_{n_k}\}_{n_k \in I_+}$ is a frame sequence for $\mathcal{K}^+$. Similarly we can show that $\{f_{m_k}\}_{m_k \in I_+}$ is a frame sequence for $\mathcal{K}^+$. Also, in a similar way, one can prove that $\{f_{n_k}\}_{n_k \in I_-}$ and $\{f_{m_k}\}_{m_k \in I_-}$ are frame sequences for $\mathcal{K}^-$. □

Corollary 4.3. *Let $\{f_n\}_{n \in I}$ be a frame for a Krein space $\mathcal{K}$. Let $\{m_k\}$ and $\{n_k\}$ be two infinite increasing sequences in I with $\{m_k^+\} \cup \{n_k^+\} = I_+$ and $\{m_k^-\} \cup \{n_k^-\} = I_-$. Let $\{f_{m_k}\}_{m_k \in I^+}$ and $\{f_{n_k}\}_{n_k \in I^+}$ be frames for $[f_{m_k}]_{m_k \in I^+}$ and $[f_{n_k}]_{n_k \in I^+}$ respectively and let $\{f_{m_k}\}_{m_k \in I^-}$ and $\{f_{n_k}\}_{n_k \in I^-}$ be frames for $[f_{m_k}]_{m_k \in I^-}$ and $[f_{n_k}]_{n_k \in I^-}$ respectively. If $\{g_i^+\} = \{f_{m_k}\}_{m_k \in I^+} \cup \{f_{n_k}\}_{n_k \in I^+}$ and $\{g_i^-\} = \{f_{m_k}\}_{m_k \in I^-} \cup \{f_{n_k}\}_{n_k \in I^-}$, then $\{g_i^+\}$ and $\{g_i^-\}$ are frame sequences.*

Proof. The proof of the corollary follows from the Theorem 4.2 and the fact that $\{f_{m_k}\}_{m_k \in I^+}$ and $\{f_{n_k}\}_{n_k \in I^+}$ are frames for the $[f_{m_k}]_{m_k \in I^+}$ and $[f_{n_k}]_{n_k \in I^+}$ respectively. □

Finally, we give a sufficient condition for the exactness of frames in a Krein space $\mathcal{K}$.

Theorem 4.4. *Let $\{f_n\}_{n \in I}$ be a frame for a Krein space $(\mathcal{K}, [.,.])$ with bounds A , $A^{'}$ and B, $B^{'}$ such that $f_n \neq 0$, for all $n \in I$. If for every infinite increasing sequence $\{n_k\} \in I_+$ and $\{m_k\} \in I_-$, $\{f_{n_k}\}_{n_k \in I_+}$ and $\{f_{m_k}\}_{m_k \in I_-}$ are frame sequences with bounds A, B and $A^{'}$, $B^{'}$ respectively, then $\{f_n\}_{n \in I}$ is an exact frame.*

Proof. Suppose on the contrary that $\{f_n\}_{n\in I}$ is not an exact frame. Then, there exists $m \in I$ such that $f_m \in [f_n], \quad i \neq m$. Let $\{n_k\}$ be an increasing sequence in I, given by $n_k = k, k = 1, 2, 3, \ldots, m-1$ and $n_k = k+1, k = m, m+1, \ldots$. Since $\{f_{n_k}\}_{n_k \in I_+}$ is a frame for $\mathcal{K}^+$ and $\{f_{n_k}\}_{n_k \in I_-}$ is a frame for $\mathcal{K}^-$ with bounds A, B and $A^{'}$, $B^{'}$ respectively, we have

$$A\|f\|^2 \leq \sum_{\substack{n\neq m \\ n\in I_+}} [f, f_n]^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{K}^+ \tag{8}$$

and

$$A^{'}\|f\|^2 \leq \sum_{\substack{n\neq m \\ n\in I_-}} |[f, f_n]|^2 \leq B^{'}\|f\|^2, \text{ for all } f \in \mathcal{K}^-. \tag{9}$$

Since $\{f_n\}_{n\in I}$ is a frame for the Krein space $(\mathcal{K}, [.,.])$, by (8), we have $[f, f_m] = 0$ for all $f \in \mathcal{K}^+$. In particular, $[f_m, f_m] = 0$. This gives $f_m = 0$. Also, by (8), $|[f^-, f_m]| = 0$, for all $f \in \mathcal{K}^-$. In particular $|[f_m, f_m]| = 0$. This implies that $f_m = 0$ which is a contradiction. Hence $\{f_{n_k}\}_{n_k\in I}$ is an exact frame. □

Acknowledgements

The present work of the first author is partially supported by University Grants Commission (UGC), Government of India. The present work of the second author is partially supported by Science and Engineering Research Board (SERB), Government of India (Reference Number : TAR/2022/000219).

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