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On Geometric Constants for Discrete Morrey Spaces

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Abstract. In this paper we prove that the n-th Von Neumann-Jordan constant and the n-th James constant for discrete Morrey spaces ℓ_q^p where $1 \leq p < q < \infty$ are both equal to n. This result tells us that the discrete Morrey spaces are not uniformly non-ℓ^1, and hence they are not uniformly n-convex.

1. Introduction

Let $n \geq 2$ be a non-negative integer and $(X, \|\cdot\|)$ be a Banach space. The n-th *Von Neumann-Jordan constant* for X [6] is defined by

$$C_{NJ}^{(n)}(X) := \sup\left\{ \frac{\sum_{\pm} \|u_1 \pm u_2 \pm \cdots \pm u_n\|_X^2}{2^{n-1}\sum_{i=1}^{n} \|u_i\|_X} : u_i \neq 0, i = 1, 2, \ldots, n \right\}$$

and the n-th *James constant* for X [7] is defined by

$$C_J^{(n)}(X) := \sup\{\min \|u_1 \pm u_2 \pm \cdots \pm u_n\| : u_i \in S_X, i = 1, 2, \ldots, n\}.$$

Note that in the definition of $C_{NJ}^{(n)}(X)$, the sum $\sum_{\pm}$ is taken over all possible combinations of $\pm$ signs. Similarly, in the definition of $C_J^{(n)}(X)$, the minimum is taken over all possible combinations of $\pm$ signs, while the supremum is taken over all u_i's in the unit sphere $S_X := \{u \in X : \|u\| = 1\}$. These constants measure some sort of convexity of a Banach space.

We say that X is uniformly n-convex [2] if for every $\varepsilon \in (0, n]$ there exists a $\delta \in (0, 1)$ such that for every $u_1, u_2, \ldots, u_n \in S_X$ with $\|u_1 \pm u_2 \pm \cdots \pm u_n\| \geq \varepsilon$ for all combinations of $\pm$ signs except for $\|u_1 + u_2 + \cdots + u_n\|$, we have

$$\|u_1 + u_2 + \cdots + u_n\| \leq n(1 - \delta).$$

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Meanwhile, we say that X is uniformly non-ℓ_n^1 [1,5,8] if there exists a $\delta \in (0,1)$ such that for every $u_1, u_2, \dots, u_n \in S_X$ we have

$$\min \|u_1 \pm u_2 \pm \cdots \pm u_n\| \leq n(1-\delta).$$

Note that for $n = 2$, uniformly non-ℓ_n^1 spaces are known as uniformly nonsquare spaces, while for $n = 3$ they are known as uniformly non-octahedral spaces. One may verify that if X is uniformly n-convex, then X is uniformly non-ℓ_n^1 [2].

Now a few remarks about the two constants, and their associations with the uniformly non-ℓ_n^1 and uniformly n-convex properties.

- $1 \leq C_{NJ}^{(n)}(X) \leq n$ and $C_{NJ}^{(n)}(X) = 1$ if and only if X is a Hilbert space [6].
- $1 \leq C_J^{(n)}(X) \leq n$. If $\dim(X) = \infty$, then $\sqrt{n} \leq C_J^{(n)}(X) \leq n$. Moreover, if X is a Hilbert space, then $C_J^{(n)}(X) = \sqrt{n}$ [7].
- X is uniformly non-ℓ_n^1 if and only if $C_{NJ}^{(n)}(X) < n$ [6].
- X is uniformly non-ℓ_n^1 if and only if $C_J^{(n)}(X) < n$ [7].

The last two statements tell us that if $C_{NJ}^{(n)}(X) = n$ or $C_J^{(n)}(X) = n$, then X is not uniformly non-ℓ_n^1 and hence not uniformly n-convex.

In this paper, we shall compute the value of the two constants for discrete Morrey spaces. Let $\omega := \mathbb{N} \cup \{0\}$ and $m = (m_1, m_2, \dots, m_d) \in \mathbb{Z}^d$. Define

$$S_{m,N} := \{k \in \mathbb{Z}^d : \|k - m\|_\infty \leq N\}$$

where $N \in \omega$ and $\|m\|_\infty = \max\{|m_i| : 1 \leq i \leq d\}$. Denote by $|S_{m,N}|$ the cardinality of $S_{m,N}$ for $m \in \mathbb{Z}^d$ and $N \in \omega$. Then we have $|S_{m,N}| = (2N+1)^d$.

Now let $1 \leq p \leq q < \infty$. Define $\ell_q^p = \ell_q^p(\mathbb{Z}^d)$ to be the discrete Morrey space as introduced in [3], which consists of all sequences $x : \mathbb{Z}^d \to \mathbb{R}$ with

$$\|x\|_{\ell_q^p} := \sup_{m \in \mathbb{Z}^d, N \in \omega} |S_{m,N}|^{\frac{1}{q} - \frac{1}{p}} \left(\sum_{k \in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} < \infty,$$

where $x := (x_k)$ with $k \in \mathbb{Z}^d$. One may observe that these discrete Morrey spaces are Banach spaces [3]. Note, in particular, that for $p = q$, we have $\ell_q^p = \ell^q$.

From [4] we already know that $C_{NJ}(\ell_q^p) = C_J(\ell_q^p) = 2$ for $1 \leq p < q < \infty$, which implies that ℓ_q^p are not uniformly nonsquares for those p's and q's. In this paper, we shall show that $C_{NJ}^{(n)}(\ell_q^p) = C_J^{(n)}(\ell_q^p) = n$ for $1 \leq p < q < \infty$, which leads us to the conclusion that ℓ_q^p are not uniformly non-ℓ_n^1 for those p's and q's, which is sharper than the existing result. (If X is not uniformly non-ℓ_n^1, then X is not uniformly non-ℓ_{n-1}^1, provided that $n \geq 3$.)

2. Main Results

The value of the n-th Von Neumann-Jordan constant and the n-th James constant for discrete Morrey spaces are stated in the following theorems. To understand the idea of the proof, we first present the result for $n = 3$.

Theorem 2.1. *For $1 \le p < q < \infty$, we have $C_{NJ}^{(3)}(\ell_q^p(\mathbb{Z}^d)) = C_J^{(3)}(\ell_q^p(\mathbb{Z}^d)) = 3$.*

Proof. To prove the theorem, it suffices for us to find $x^{(1)}, x^{(2)}, x^{(3)} \in \ell_q^p$ such that

$$\frac{\sum_{\pm} \|x^{(1)} \pm x^{(2)} \pm x^{(3)}\|_{\ell_q^p}^2}{2^2 \sum_{i=1}^{3} \|x^{(i)}\|_{\ell_q^p}} = 3$$

for the Von Neumann-Jordan constant, and

$$\min \|x^{(1)} \pm x^{(2)} \pm x^{(3)}\|_{\ell_q^p} = 3$$

for the James constant.

Case 1: $d = 1$. Let $j \in \mathbb{Z}$ be a nonnegative, even integer such that $j > 4^{\frac{q}{q-p}} - 1$, or equivalently

$$(j+1)^{\frac{1}{q}-\frac{1}{p}} < 4^{-\frac{1}{p}}.$$

Construct $x^{(1)}, x^{(2)}, x^{(3)} \in \ell_q^p(\mathbb{Z})$ as follows:

- $x^{(1)} = (x_k^{(1)})_{k\in\mathbb{Z}}$ is defined by

$$x_k^{(1)} = \begin{cases} 1, & k = 0, j, 2j, 3j, \\ 0, & \text{otherwise}; \end{cases}$$

- $x^{(2)} = (x_k^{(2)})_{k\in\mathbb{Z}}$ is defined by

$$x_k^{(2)} = \begin{cases} 1, & k = 0, j, \\ -1, & k = 2j, 3j, \\ 0, & \text{otherwise}; \end{cases}$$

- $x^{(3)} = (x_k^{(3)})_{k\in\mathbb{Z}}$ is defined by

$$x_k^{(3)} = \begin{cases} 1, & k = 0, 2j, \\ -1, & k = j, 3j, \\ 0, & \text{otherwise}. \end{cases}$$

The three sequences are in the unit sphere of $\ell_q^p(\mathbb{Z})$. Indeed, for the first sequence, we have

$$\begin{aligned}\|x^{(1)}\|_{\ell_q^p} &= \sup_{m\in\mathbb{Z}, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k^{(1)}|^p\right)^{\frac{1}{p}} \\ &= \sup_{m\in\mathbb{Z}\cap[0,3j], N\in\mathbb{Z}\cap[0,3j/2]} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k^{(1)}|^p\right)^{\frac{1}{p}} \\ &= \max\{1, (j+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{1}{p}}, (2j+1)^{\frac{1}{q}-\frac{1}{p}} 3^{\frac{1}{p}}, (3j+1)^{\frac{1}{q}-\frac{1}{p}} 4^{\frac{1}{p}}\}.\end{aligned}$$

Since $(3j+1)^{\frac{1}{q}-\frac{1}{p}} < (2j+1)^{\frac{1}{q}-\frac{1}{p}} < (j+1)^{\frac{1}{q}-\frac{1}{p}} < 4^{-\frac{1}{p}}$, we get $\|x^{(1)}\|_{\ell_q^p} = 1$. Similarly, one may observe that $\|x^{(2)}\|_{\ell_q^p} = \|x^{(3)}\|_{\ell_q^p} = 1$.

Next, we observe that

$$x_k^{(1)} + x_k^{(2)} + x_k^{(3)} = \begin{cases} 3, & k = 0, \\ 1, & k = j, 2j, \\ -1, & k = 3j, \\ 0, & \text{otherwise}; \end{cases}$$

$$x_k^{(1)} + x_k^{(2)} - x_k^{(3)} = \begin{cases} 3, & k = j, \\ 1, & k = 0, 3j, \\ -1, & k = 2j, \\ 0, & \text{otherwise}; \end{cases}$$

$$x_k^{(1)} - x_k^{(2)} + x_k^{(3)} = \begin{cases} 3, & k = 2j, \\ 1, & k = 0, 3j, \\ -1, & k = j, \\ 0, & \text{otherwise}; \end{cases}$$

$$x_k^{(1)} - x_k^{(2)} - x_k^{(3)} = \begin{cases} 3, & k = 3j, \\ 1, & k = j, 2j, \\ -1, & k = 0, \\ 0, & \text{otherwise}. \end{cases}$$

We first compute that

$$\|x^{(1)} + x^{(2)} + x^{(3)}\|_{\ell_q^p} = \max\{3, (j+1)^{\frac{1}{q}-\frac{1}{p}}(3^p+1)^{\frac{1}{p}}, (2j+1)^{\frac{1}{q}-\frac{1}{p}}(3^p+2)^{\frac{1}{p}}, (3j+1)^{\frac{1}{q}-\frac{1}{p}}(3^p+3)^{\frac{1}{p}}\}.$$

Notice that

- $(j+1)^{\frac{1}{q}-\frac{1}{p}}(3^p+1)^{\frac{1}{p}} < \left(\frac{3^p+1^p}{4}\right)^{\frac{1}{p}} < (3^p)^{\frac{1}{p}} = 3.$
- $(2j+1)^{\frac{1}{q}-\frac{1}{p}}(3^p+2)^{\frac{1}{p}} < (j+1)^{\frac{1}{q}-\frac{1}{p}}(3^p+2)^{\frac{1}{p}} < \left(\frac{3^p+2}{4}\right)^{\frac{1}{p}} < 3.$

- $(3j+1)^{\frac{1}{q}-\frac{1}{p}}(3^p+3)^{\frac{1}{p}} < (j+1)^{\frac{1}{q}-\frac{1}{p}}(3^p+3)^{\frac{1}{p}} < \left(\frac{3^p+3}{4}\right)^{\frac{1}{p}} < 3.$

Hence, we obtain $\|x^{(1)}+x^{(2)}+x^{(3)}\|_{\ell_q^p} = 3$.

Similarly, we have

$$\|x^{(1)} \pm x^{(2)} \pm x^{(3)}\|_{\ell_q^p} = \sup_{m\in\mathbb{Z}\cap[0,3j], N\in\mathbb{Z}\cap[0,3j/2]} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k^{(1)} \pm x_k^{(2)} \pm x_k^{(3)}|^p\right)^{\frac{1}{p}} = 3$$

for every combination of $\pm$ signs.

Consequently, $\frac{\sum_{\pm}\|x^{(1)}\pm x^{(2)}\pm x^{(3)}\|^2_{\ell_q^p}}{2^2\sum_{i=1}^3\|x^{(i)}\|_{\ell_q^p}} = 3$ and $\min\|x^{(1)} \pm x^{(2)} \pm x^{(3)}\|_{\ell_q^p} = 3$, so we come to the conclusion that

$$C_{NJ}^{(3)}(\ell_q^p(\mathbb{Z})) = C_J^{(3)}(\ell_q^p(\mathbb{Z})) = 3.$$

Case 2: $d > 1$. Let $j \in \mathbb{Z}$ be a nonnegative, even integer such that $j > 4^{\frac{q}{d(q-p)}} - 1$, which is equivalent to

$$(j+1)^{d(\frac{1}{q}-\frac{1}{p})} < 4^{-\frac{1}{p}}.$$

We then construct $x^{(1)}, x^{(2)}, x^{(3)} \in \ell_q^p(\mathbb{Z}^d)$ as follows:

- $x^{(1)} = (x_k^{(1)})_{k\in\mathbb{Z}^d}$ is defined by

$$x_k^{(1)} = \begin{cases} 1, & k = (0,0,\ldots,0), (j,0,\ldots,0), (2j,0,\ldots,0), (3j,0,\ldots,0), \\ 0, & \text{otherwise}; \end{cases}$$

- $x^{(2)} = (x_k^{(2)})_{k\in\mathbb{Z}^d}$ is defined by

$$x_k^{(2)} = \begin{cases} 1, & k = (0,0,\ldots,0), (j,0,\ldots,0), \\ -1, & k = (2j,0,\ldots,0), (3j,0,\ldots,0), \\ 0, & \text{otherwise}; \end{cases}$$

- $x^{(3)} = (x_k^{(3)})_{k\in\mathbb{Z}^d}$ is defined by

$$x_k^{(3)} = \begin{cases} 1, & k = (0,0,\ldots,0), (2j,0,\ldots,0), \\ -1, & k = (j,0,\ldots,0), (3j,0,\ldots,0), \\ 0, & \text{otherwise}. \end{cases}$$

As in the case where $d = 1$, one may observe that

$$\begin{aligned}\|x^{(1)}\|_{\ell_q^p} &= \sup_{m\in\mathbb{Z}^d, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k^{(1)}|^p\right)^{\frac{1}{p}} \\ &= \max\{1, (j+1)^{d(\frac{1}{q}-\frac{1}{p})}2^{\frac{1}{p}}, (2j+1)^{d(\frac{1}{q}-\frac{1}{p})}3^{\frac{1}{p}}, (3j+1)^{d(\frac{1}{q}-\frac{1}{p})}4^{\frac{1}{p}}\} \\ &= 1.\end{aligned}$$

We also get $\|x^{(2)}\|_{\ell_q^p} = \|x^{(3)}\|_{\ell_q^p} = 1$. Moreover, through similar observation as in the 1-dimensional case, we have

$$\|x^{(1)} \pm x^{(2)} \pm x^{(3)}\|_{\ell_q^p} = 3$$

for every possible combinations of $\pm$ signs. It thus follows that

$$C_J^{(3)}(\ell_q^p(\mathbb{Z}^d)) = \sup\{\min \|x_1 \pm x_2 \pm x_3\|_{\ell_q^p} : x_1, x_2, x_3 \in S_{\ell_q^p}\} = 3$$

and

$$C_{NJ}^{(3)}(\ell_q^p(\mathbb{Z}^d)) = \sup\left\{\frac{\sum_{\pm} \|x_1 \pm x_2 \pm x_3\|_{\ell_q^p}^2}{2^2 \sum_{i=1}^3 \|x_i\|_{\ell_q^p}} : x_i \neq 0, i = 1, 2, 3\right\} = 3.$$

□

We now state the general result for $n \geq 3$. (The proof is also valid for $n = 2$, which amounts to the work of [3].)

Theorem 2.2. *For $1 \leq p < q < \infty$, we have $C_{NJ}^{(n)}(\ell_q^p(\mathbb{Z}^d)) = C_J^{(n)}(\ell_q^p(\mathbb{Z}^d)) = n$.*

Proof. As for $n = 3$, we shall consider the case where $d = 1$ first, and then the case where $d > 1$ later.

Case 1: $d = 1$. Let $j \in \mathbb{Z}$ be a nonnegative, even integer such that $j > 2^{(n-1)(\frac{q}{q-p})} - 1$, which is equivalent to

$$(j+1)^{\frac{1}{q}-\frac{1}{p}} < 2^{-\frac{(n-1)}{p}}.$$

We construct $x^{(i)} \in \ell_q^p \in \mathbb{Z}$ for $i = 1, 2, \ldots, n$ as follows:

- $x^{(1)} = (x_k^{(1)})_{k\in\mathbb{Z}}$ is defined by

$$x_k^{(1)} = \begin{cases} 1, & k \in S_1^{(1)}, \\ 0, & \text{otherwise}, \end{cases}$$

  where

$$S_1^{(1)} = \{0, j, 2j, 3j, \ldots, (2^{n-1} - 1)j\};$$

- $x^{(i)} = (x_k^{(i)})_{k\in\mathbb{Z}}$ for $2 \leq i \leq n$ is defined by

$$x_k^{(i)} = \begin{cases} 1, & k \in S_1^{(i)}, \\ -1, & k \in S_{-1}^{(i)}, \\ 0, & \text{otherwise}, \end{cases}$$

  with the following rules: Write $P = \{0, j, 2j, \ldots, (2^{n-1} - 1)j\}$ as

$$P = P_1^{(i)} \cup P_2^{(i)} \cup \cdots \cup P_{2^{i-1}}^{(i)}$$

where $P_1^{(i)}$ consists of the first $\frac{2^{n-1}}{2^{i-1}}$ terms of P, $P_2^{(i)}$ consists of the next $\frac{2^{n-1}}{2^{i-1}}$ terms of P, and so on. Then $S_1^{(i)}$ and $S_{-1}^{(i)}$ are given by

$$S_1^{(i)} = P_1^{(i)} \cup P_3^{(i)} \cup \cdots \cup P_{2^{i-1}-1}^{(i)},$$
$$S_{-1}^{(i)} = P_2^{(i)} \cup P_4^{(i)} \cup \cdots \cup P_{2^{i-1}}^{(i)}.$$

For example, for $i = 2$, $x^{(2)} = (x_k^{(2)})_{k\in\mathbb{Z}}$ is defined by

$$x_k^{(2)} = \begin{cases} 1, & k \in S_1^{(2)}, \\ -1, & k \in S_{-1}^{(2)}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$S_1^{(2)} = \left\{0, j, 2j, 3j, \ldots, \left(\frac{2^{n-1}}{2} - 1\right)j\right\}$$
$$S_{-1}^{(2)} = \left\{\left(\frac{2^{n-1}}{2}\right)j, \left(\frac{2^{n-1}}{2} + 1\right)j, \ldots, (2^{n-1} - 1)j\right\};$$

Note that the largest absolute value of the terms of $x^{(i)}$ in the above construction will be equal to 1 for each $i = 1, \ldots, n$. Next, since the number of possible combinations of $\pm$ signs in $x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)}$ is 2^{n-1}, the above construction will give us $1 + 1 + \cdots + 1 = n$ as the largest absolute value of $x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)}$ for every combination of $\pm$ signs. This means that, if $x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)} = (x_k)_{k\in\mathbb{Z}}$, then $\max_{k\in\mathbb{Z}} |x_k| = n$.

Let us now compute the norms. For $x^{(1)}$, we have

$$\begin{aligned} \|x^{(1)}\|_{\ell_q^p} &= \sup_{m\in\mathbb{Z}, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k^{(1)}|^p\right)^{\frac{1}{p}} \\ &= \sup_{m\in\mathbb{Z}\cap[0,(2^{n-1}-1)j], N\in\mathbb{Z}\cap[0,(2^{n-1}-1)j/2]} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k^{(1)}|^p\right)^{\frac{1}{p}} \\ &= \max\{1, (j+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{1}{p}}, (2j+1)^{\frac{1}{q}-\frac{1}{p}} 3^{\frac{1}{p}}, \ldots, ((2^{n-1}-1)j+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{n-1}{p}}\}. \end{aligned}$$

For each $r = 1, 2, \ldots, 2^{n-1} - 1$, we have $(rj+1)^{\frac{1}{q}-\frac{1}{p}} \le (j+1)^{\frac{1}{q}-\frac{1}{p}}$ and $(r+1)^{\frac{1}{p}} \le 2^{\frac{n-1}{p}}$, so that

$$(rj+1)^{\frac{1}{q}-\frac{1}{p}} (r+1)^{\frac{1}{p}} \le (j+1)^{\frac{1}{q}-\frac{1}{p}} 2^{\frac{n-1}{p}} < 2^{-\frac{n-1}{p}} 2^{\frac{n-1}{p}} = 1.$$

Hence we obtain $\|x^{(1)}\|_{\ell_q^p} = 1$. Similarly, one may verify that

$$\|x^{(2)}\|_{\ell_q^p} = \|x^{(3)}\|_{\ell_q^p} = \cdots = \|x^{(n)}\|_{\ell_q^p} = 1.$$

Next, we shall compute the norms of $x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)}$. Write $x^{(1)} + x^{(2)} + \cdots + x^{(n)} = (x_k)_{k\in\mathbb{Z}}$ where

$$x_k := \begin{cases} a_1, & k = 0, \\ a_2, & k = j, \\ a_3, & k = 2j, \\ \vdots & \\ a_{2^{n-1}}, & k = (2^{n-1}-1)j, \\ 0, & \text{otherwise}, \end{cases}$$

with $a_1 = n$ and $|a_i| < n$ for $i = 2, 3, \ldots, (2^{n-1})j$. Accordingly, we have

$$\begin{aligned} \|x^{(1)} + x^{(2)} + \cdots + x^{(n)}\|_{\ell_q^p} &= \sup_{m\in\mathbb{Z}, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \Big(\sum_{k\in S_{m,N}} |x_k|^p \Big)^{\frac{1}{p}} \\ &= \sup_{m\in\mathbb{Z}\cap[0,(2^{n-1}-1)j], N\in\mathbb{Z}\cap[0,(2^{n-1}-1)j/2]} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \Big(\sum_{k\in S_{m,N}} |x_k|^p \Big)^{\frac{1}{p}} \\ &= \max\Big\{ n, (j+1)^{\frac{1}{q}-\frac{1}{p}} (n^p + a_2^p)^{\frac{1}{p}}, (2j+1)^{\frac{1}{q}-\frac{1}{p}} (n^p + a_2^p + a_3^p)^{\frac{1}{p}}, \\ &\qquad \ldots, ((2^{n-1}-1)j+1)^{\frac{1}{q}-\frac{1}{p}} \Big(n^p + \sum_{i=2}^{2^{n-1}} a_i^p \Big)^{\frac{1}{p}} \Big\}. \end{aligned}$$

Since $(rj+1)^{\frac{1}{q}-\frac{1}{p}} \leq (j+1)^{\frac{1}{q}-\frac{1}{p}}$ for each $r = 1, 2, \ldots, 2^{n-1}-1$, we obtain

$$\begin{aligned} (rj+1)^{\frac{1}{q}-\frac{1}{p}} \Big(n^p + \sum_{i=2}^{r+1} a_i^p \Big)^{\frac{1}{p}} &\leq (j+1)^{\frac{1}{q}-\frac{1}{p}} \Big(n^p + \sum_{i=2}^{r+1} a_i^p \Big)^{\frac{1}{p}} \\ &< 2^{-\frac{(n-1)}{p}} \Big(n^p + \sum_{i=2}^{r+1} a_i^p \Big)^{\frac{1}{p}} \\ &< 2^{-\frac{(n-1)}{p}} (\underbrace{n^p + n^p + \cdots + n^p}_{r+1 \text{ times}})^{\frac{1}{p}} \\ &= 2^{-\frac{(n-1)}{p}} (r+1)^{\frac{1}{p}} (n^p)^{\frac{1}{p}} \\ &\leq 2^{-\frac{(n-1)}{p}} 2^{\frac{(n-1)}{p}} n \\ &= n. \end{aligned}$$

It thus follows that

$$\|x^{(1)} + x^{(2)} + \cdots + x^{(n)}\|_{\ell_q^p} = n.$$

As we have remarked earlier, the largest absolute value of $x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)}$ is equal to n for every combination of $\pm$ signs. Moreover, it is clear that for $k \notin \{0, 2j, \ldots, (2^{n-1}-1)j\}$, the

k-th term of $x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)}$ is equal to 0. Hence, we obtain

$$\|x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)}\|_{\ell_q^p} = \sup_{m\in\mathbb{Z}, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k^{(1)} \pm x_k^{(2)} \pm \cdots \pm x_k^{(n)}|^p \right)^{\frac{1}{p}}$$

$$= \sup_{m\in\mathbb{Z}\cap[0,(2^{n-1}-1)j], N\in\mathbb{Z}\cap[0,(2^{n-1}-1)j/2]} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k^{(1)} \pm x_k^{(2)} \pm \cdots \pm x_k^{(n)}|^p \right)^{\frac{1}{p}} = n.$$

Consequently, we get

$$\frac{\sum_{\pm} \|x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)}\|^2_{\ell_q^p}}{2^{n-1}\sum_{i=1}^n \|x_i\|_{\ell_q^p}} = \frac{2^{n-1}n^2}{2^{n-1}n} = n$$

and

$$\min \|x^{(1)} \pm x^{(2)} \pm \cdots \pm x^{(n)}\|_{\ell_q^p} = n,$$

whence

$$C_{NJ}^{(n)}(\ell_q^p(\mathbb{Z})) = C_J^{(n)}(\ell_q^p(\mathbb{Z})) = n.$$

Case 2: $d > 1$. Here we choose $j \in \mathbb{Z}$ to be a nonnegative, even integer such that $j > 2^{(\frac{n-1}{d})(\frac{q}{q-p})} - 1$ or, equivalently,

$$(j+1)^{d(\frac{1}{q}-\frac{1}{p})} < 2^{-\frac{(n-1)}{p}}.$$

Then, using the sequences

$$x^{(i)} = (x_{k_1}^{(i)})_{k_1\in\mathbb{Z}} \in \ell_q^p(\mathbb{Z}), \quad i = 1, \ldots, n,$$

in the case where $d = 1$, we now define $x^{(i)} := (x_k^{(i)})_{k\in\mathbb{Z}^d} \in \ell_q^p(\mathbb{Z}^d)$ for $i = 1, \ldots, n$, where

$$x_k^{(i)} = \begin{cases} x_{k_1}^{(i)}, & k = (k_1, 0, 0, \ldots, 0), \\ 0, & \text{otherwise}. \end{cases}$$

We shall then obtain

$$C_{NJ}^{(n)}(\ell_q^p(\mathbb{Z}^d)) = C_J^{(n)}(\ell_q^p(\mathbb{Z}^d)) = n,$$

as desired. □

Corollary 2.2.1. *For $1 \leq p < q < \infty$, the space ℓ_q^p is not uniformly non-ℓ_n^1.*

Corollary 2.2.2. *For $1 \leq p < q < \infty$, the space ℓ_q^p is not uniformly n-convex.*

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