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Conditional Least Squares Estimation for Fractional Super Levy Processes in Nonlinear SPDEs

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Abstract. We consider infinite dimensional extension of affine models as super Levy processes satisfying a nonlinear SPDE. We obtain the asymptotics of the conditional least squares estimators. Finally we obtain the Berry-Esseen inequality.

1. Introduction and Preliminaries

Parameter estimation in finite dimensional diffusions is now classical. Bishwal [7] studied a new estimating function for discretely sampled diffusions. Bishwal [8] studied asymptotic theory of likelihood method and Bayesian method for drift estimation of finite dimensional stochastic differential equations. Bishwal [12] studied applications of Levy processes in stochastic volatility models in finance. Bishwal [13] studied parameter estimation for SPDEs driven by cylindrical stable processes. Bishwal [6] studied the Bernstein-von Mises theorem and spectral asymptotics of Bayes estimators for parabolic SPDEs when the number of Fourier coefficients becomes large. In this case, the measures generated by the process for different parameters are singular. Bishwal [11] studied Bernstein-von Mises theorem and small noise Bayesian asymptotics for parabolic stochastic partial differential equations. Bishwal [10] studied hypothesis testing for fractional stochastic partial differential equations with applications to neurophysiology and finance.
Consider the nonlinear SPDE

$$dX(t,x) = \frac{1}{2}\Delta X(t,x)dt + \sqrt{X(t,x)}dW(t,x) \tag{1.1}$$

where $W(t,x)$ a cylindrical Brownian motion. Konno and Shiga [26] studied the existence and weak uniqueness of the above equation as a martingale problem for the associated super-Brownian motion. The pathwise uniqueness of nonnegative solution still remains open. The main difficulty comes from the unbounded drift coefficient and non-Lipschitz diffusion coefficient. Wang *et al.* [39] studied a comparison theorem and showed that the solution of the nonlinear SPDE is distribution

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function valued. They also established pathwise uniqueness. As application they obtained well-posedness of martingale problems for two classes of measure-valued diffusions: interacting super-Brownian motions and interacting Fleming-Viot processes. He *et al.* [21] obtained pathwise unique solution to nonlinear SPDE with super Levy process, which is a combination of space-time Gaussian white noises and Poisson random measures which is a generalization of work of Xiong [40] where the result for a super-Brownian motion with binary branching mechanism was obtained. Using an extended Yamada-Watanabe argument, Xiong [40] established strong existence and uniqueness of the solution to the SPDE. Super-Brownian motion (SBM), also called the Dawson-Watanabe process introduced by Dawson and Watanabe is a measure valued process arising as the limit of empirical measure process of a branching particle system. SBM satisfies a martingale problem. When the state space is $\mathbb{R}$, SBM has a density w.r.t. Lebesgue measure and this density valued process $X(t, x)$ satisfies the above SPDE. When the space $\mathbb{R}$ is s single point, the SPDE becomes an SDE which is CIR diffusion $dX_t = \sqrt{X_t}dW_t$ whose uniqueness is established using the Yamada-Watanabe argument. Xiong and Yang (2019) studied existence and pathwise uniqueness to an SPDE with Hölder continuous coefficient driven by α-stable colored noise. The existence of the solution is shown by considering the weak limit of a sequence of SDE system which is obtained by replacing the Laplacian operator in the SPDE by its discrete version. The pathwise uniqueness is shown by using a backward doubly stochastic differential equation to take care of the Laplacian. In the case of $d = 1$, the pathwise uniqueness of a nonnegative solution to the corresponding equation was established by Yang and Zhou [42] for $1 < \alpha < \sqrt{5} - 1$ and pathwise uniqueness for $\sqrt{5} - 1 < \alpha < 2$ is still open.

The existence and pathwise uniqueness of solutions to the SDEs with non-Lipschitz coefficient driven by spectrally positive Levy processes were studied in Fu and Li [20].

Consider the SPDE with multiplicative noise:

$$du^\theta(t, x) = (A_0 + \theta A_1)u^\theta(t, x)dt + Mu^\theta(t, x)dZ(t, x), \;\; t \geq 0, \;\; x \in [0, 1] \tag{1.2}$$

where M is a known nonlinear operator.

Priola *et al.* [32] obtained exponential convergence to the invariant measure, in the total variation norm, for solutions to SDEs driven by α-stable noises in finite and infinite dimensions using two approaches: Lyapounov's function approach by Harris and Doeblin's coupling argument. In both approaches irreducibility and uniform strong Feller property play crucial role.

Equation (1.2) is called diagonalizable if A_0, A_1 and M have point spectrum and a common system of eigenfunction $\{h_j, j \geq 1\}$. Denote by ρ_k, ν_k and μ_k, the eigenvalues of the operators A_0, A_1 and M respectively. Then

$$u^\theta(t, x) = \sum_{j=1}^{\infty} u_{j,t} h_j. \tag{1.3}$$

We consider fractional stable CIR model as example.

Using fractional Levy process as the driving term, maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [9].

Fractional Levy Process (FLP) is defined as

$$M_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})} \int_{\mathbb{R}} [(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}] dL_s, \quad t \in \mathbb{R} \tag{1.4}$$

where $\{L_t, t \in \mathbb{R}\}$ is a Levy process on $\mathbb{R}$ with $E(L_1) = 0$, $E(L_1^2) < \infty$.

Here are some properties of the fractional Levy process:

1) the covariance of the process is given by

$$\mathrm{cov}(M_{H,t}, M_{H,s}) = \frac{E(L_1^2)}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]. \tag{1.5}$$

2) M_H is not a martingale. For a large class of Levy processes, M_H is neither a semimartingale. 3)M_H is Hölder continuous of any order β less than $H - \frac{1}{2}$. 4) M_H has stationary increments. 5) M_H is symmetric. 6) L is self-similar, but M_H is not self-similar. 7) M_H has infinite total variation on compacts.

Thus FLP is a generalization and a natural counterpart of FBM. Fractional stable motion is a special case of FLP.

2. Conditional Least Squares Estimation

Let $\mathcal{H}$ be a real separable Hilbert space with inner product $\langle\cdot\rangle$ and norm $|\cdot|$. By $\mathcal{L}(\mathcal{H})$ we denote the Banach space of bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$ endowed with the operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{H})}$. We fix an orthonormal basis (e_n) in $\mathcal{H}$. Through the basis (e_n) we will often identify $\mathcal{H}$ in l^2. More generally, for a given sequence $\rho = (\rho_n)$ of real numbers we set

$$l_\rho^2 = \{(x_n) \in \mathbb{R}^\infty : \sum_{n\geq 1} x_n^2 \rho_n^2 < \infty\}.$$

where $\mathbb{R}^\infty = \mathbb{R}^{\mathbb{N}}$. The space l_ρ^2 becomes a separable Hilbert space with the inner product: $\langle x, y\rangle = \sum_{n\geq 1} x_n y_n \rho_n^2$ for $x = (x_n), y = (y_n) \in l_\rho^2$. Let us fix θ_0, the unknown true value of the parameter θ. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $Z(t, x)$ be a process on this space with values in the Schwarz space of distributions $D'(G)$ such that for $\phi, \psi \in C_0^\infty(G)$, $\|\phi\|_{L^2(G)}^{-1} \langle W(t,\cdot), \phi(\cdot)\rangle$ is a one dimensional stable process.

This process is usually referred to as the *cylindrical α-stable process* (C.S.P.), $\alpha \in (0, 2)$. We assume that there exists a complete orthonormal system $\{h_i\}_{i=1}^\infty$ in $L_2(G))$ such that for every $i = 1, 2, \ldots, h_i \in Z_0^{m,2}(G) \cap C^\infty(\overline{G})$ and

$$\Lambda_\theta h_i = \beta_i(\theta)h_i, \text{ and } \mathcal{L}_\theta h_i = \mu_i(\theta)h_i \text{ for all } \theta \in \Theta$$

where $\mathcal{L}_\theta$ is a closed self adjoint extension of A^θ, $\Lambda_\theta := (k(\theta)I - \mathcal{L}_\theta)^{1/2m}$, $k(\theta)$ is a constant and the spectrum of the operator Λ_θ consists of eigenvalues $\{\beta_i(\theta)\}_{i=1}^\infty$ of finite multiplicities and $\mu_i = -\beta_i^{2m} + k(\theta)$.

A Levy process (Z_t) with values in $\mathcal{H}$ is an $\mathcal{H}$-valued process defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ having stationary independent increments, cadlag trajectories such that $Z_0 = 0$, P-a.s. One has that

$$E[e^{i\langle Z_t, s\rangle}] = \exp(-t\psi(s)),\ \ s \in \mathcal{H}$$

where $\psi : \mathcal{H} \to \mathbb{C}$ is Sazonov continuous, negative definite function such that $\psi(0) = 0$. The function ψ is called the exponent of (Z_t).

The exponent ψ can be expressed by the infinite dimensional *Levy-Khintchine formula*

$$\psi(s) = \frac{1}{2}\langle Qs, s\rangle - i\langle a, s\rangle - \int_{\mathcal{H}} \left(e^{i\langle s,y\rangle} - 1 - \frac{i\langle s, y\rangle}{1 + |y|^2} \right) \nu(dy),\ \ s \in \mathcal{H}$$

where Q is the non-negative trace class operator on $\mathcal{H}$, $a \in \mathcal{H}$ and ν is the Levy measure or the jump intensity measure associated to (Z_t).

Cylindrical α-stable process (C.S.P.) is a Levy process taking values in the Hilbert space $\mathcal{H} = l^2_\rho$, with a properly chosen weight ρ.

Consider the linear SPDE

$$dX_t = \theta A X_t dt + dZ_t, \quad x \in \mathcal{H}$$

C.S.P. $Z(t)$ is a cylindrical α-stable process, $\alpha \in (0, 2)$ which can be expanded in the series

$$Z(t) = \sum_{i=1}^{\infty} \gamma_i Z_i(t) h_i,\ \ t \geq 0$$

where $\{Z_i(t)\}_{i=1}^{\infty}$ are independent, real valued, one dimensional, normalized, symmetric, α-stable processes and $(\gamma_i)_{i=1}^{\infty}$ is a given sequence of, possibly unbounded, positive numbers, and h_i is a fixed orthonormal basis in H. The latter series converges P-a.s. in $\mathcal{H}^{-\alpha}$ for $\alpha > d/2$. Indeed

$$\|Z(t)\|^2_{-\alpha} = \sum_{i=1}^{\infty} \gamma_i^2 Z_i^2(t) \|h_i\|^2_{-\alpha} = \sum_{i=1}^{\infty} Z_i^2(t) \beta_i^{-2\alpha}$$

and the later series converges P-a.s.

For any $j \in \mathbb{N}, t \geq 0$,

$$E[e^{iZ_j(t)h}] = e^{-t|h|^\alpha}.$$

Stable one-dimensional density: A one-dimensional, normalized, symmetric α-stable distribution μ_α, $\alpha \in (0, 2]$ has characteristic function

$$\hat{\mu}_\alpha(s) = e^{-|s|^\alpha}, s \in \mathbb{R}.$$

The density of μ_α with respect to Lebesgue measure will be denoted by p_α. This even function is known in closed form only if $\alpha = 1$ or 2. The precise asymptotic behavior of the density $p_\alpha, \alpha \in (0, 2)$ is as follows:

For any $\alpha \in (0, 2)$, there exists C_α such that

$$p_\alpha(x) \sim \frac{C_\alpha}{x^{\alpha+1}} \quad \text{as} \quad x \to \infty.$$

Stable measures on Hilbert space: A random variable ξ on $\mathcal{H}$ is called α-stable ($\alpha \in (0, 2]$) if for any n there exists a vector $a_n \in \mathcal{H}$ such that for any independent copies $\xi_1, \xi_2, \ldots, \xi_n$ of ξ, the random variable $n^{-1/\alpha}(\xi_1 + \xi_2, \ldots + \xi_n) - a_n$ has the same distribution as ξ. A Borel probability measure μ on H is said to be α-stable if it is the distribution of a stable random variable with vales in $\mathcal{H}$.

Consider the SPDE with multiplicative noise:

$$du^\theta(t,x) = (A_0 + \theta A_1)u^\theta(t,x)dt + Mu^\theta(t,x)dZ(t,x), \;\; t \geq 0, \;\; x \in [0,1] \tag{2.1}$$

where M is a known nonlinear operator and $Z(t,x)$ is a cylindrical subfractional Levy process.

Equation (2.1) is called diagonalizable if A_0, A_1 and M have point spectrum and a common system of eigenfunction $\{h_j, j \geq 1\}$. Denote by ρ_k, ν_k and μ_k, the eigenvalues of the operators A_0, A_1 and M respectively. Then $u^\theta(t,x) = \sum_{j=1}^{\infty} u_{j,t} h_j$.

Consider SPDE model with multiplicative noise and mean reversion, where the j-th Fourier coefficient is the stable Cox-Ingersoll-Ross (SCIR) model:

$$du_{j,t} = (a - \theta u_{j,t})dt + \sigma u_{j,t-}^{1/\alpha} dZ_{j,t}, \;\; j \geq 1 \tag{2.2}$$

where a is the mean reverting level and θ is mean reverting speed. Recall that for $\alpha = 2$, for every $j \geq 1$, the process $Z_{j,t}$ is a standard Brownian motion, this is the famous Cox-Ingersoll-Ross (CIR) model used for modeling interest rate, which is also used a stochastic volatility process in Heston model. Note that there are Brownian CIR models with additive compound Poisson type jumps. When $1 < \alpha < 2$, $Z_{j,t}$ is stable process with Levy measure

$$\nu_\alpha(dz) = \frac{1_{\{z>0\}} dz}{\alpha\Gamma(-\alpha) z^{\alpha+1}}. \tag{2.3}$$

The discontinuous SCIR model captures the heavy tailed property in the sense of infinite variance. There is empirical evidence from high frequency data available in support of application of pure jump models in financial modeling.

The SCIR model has the unique stationary distribution μ with Laplace transform given by

$$L_\mu(\lambda) = \int_0^\infty e^{-\lambda x} \mu(dx) = \exp\left\{ - \int_0^\lambda \frac{\alpha a}{\alpha\theta + \sigma^\alpha z^{\alpha-1}} dz \right\}, \;\; \lambda \geq 0. \tag{2.4}$$

Now we focus on the fundamental semimartingale behind the CIR model. Define

$$\kappa_H := 2H\Gamma(3/2 - H)\Gamma(H + 1/2), k_H(t,s) := \kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H},$$
$$\eta_H := \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)}, v_t \equiv v_t^H := \eta_H^{-1} t^{2-2H}, \mathcal{M}_t^H := \int_0^t k_H(t,s) dM_s^H.$$

For using Girsanov theorem for Brownian motion, since a Radon-Nikodym derivative process is always a martingale, a central problem is how to construct an appropriate martingale which generates the same filtration, up to sets of measure zero, as the non-semimartingale called the *fundamental martingale*.

Extending Norros *et al.* (1999) it can be shown that $\mathcal{M}_t^H$ is a martingale, called the fundamental martingale whose quadratic variation $\langle \mathcal{M}^H \rangle_t$ is v_t^H. Moreover, the natural filtration of the martingale $\mathcal{M}^H$ coincides with the natural filtration of the FLP M^H since $M_t^H := \int_0^t K(t,s)d\mathcal{M}_s^H$ holds for $H \in (1/2, 1)$ where $K_H(t,s) := H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, \quad 0 \le s \le t$ and for $H = 1/2$, the convention $K_{1/2} \equiv 1$ is used.

Define $Q_i(t) := \frac{d}{dv_t}\int_0^t k_H(t,s)u_i(s)ds$. It is easy to see that $Q_i(t) = \frac{\eta_H}{2(2-2H)}\left\{t^{2H-1}Z_i(t) + \int_0^t r^{2H-1}dZ_i(s)\right\}$. Define the process $Z_i = (Z_i(t), t \in [0,T])$ by $Z_i(t) := \int_0^t k_H(t,s)du_i(s)$.

Extending Kleptsyna and Le Breton (2002), we have:
(i) Z_i is the fundamental semimartingale associated with the process u_i.
(ii) Z_i is a $(\mathcal{F}_t)$ -semimartingale with the decomposition $Z_i(t) = \mu_i(\theta)\int_0^t Q_i(s)dv_s + \beta_i^{-\nu}\mathcal{M}_t^H$.
(iii) u_i admits the representation $u_i(t) = \int_0^t K_H(t,s)dZ_i(s)$. (iv) The natural filtration $(\mathcal{Z}_i(t))$ of Z_i and $(\mathcal{U}_i(t))$ of u_i coincide.

We describe our observations now. Note that for equally spaced data (homoscedastic case) $v_{t_k} - v_{t_{k-1}} = \eta_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[k^{2-2H} - (k-1)^{2-2H}], \quad k = 1, 2, \cdots, n.$
For $H = 0.5$, $v_{t_k} - v_{t_{k-1}} = \eta_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[k^{2-2H} - (k-1)^{2-2H}] = \frac{T}{n}, \quad k = 1, 2, \ldots, n$. We have

$$\begin{aligned}
Q_i(t) &= \frac{d}{dv_t}\int_0^t k_H(t,s)u_i(s)ds = \kappa_H^{-1}\frac{d}{dv_t}\int_0^t s^{1/2-H}(t-s)^{1/2-H}u_i(s)ds \\
&= \kappa_H^{-1}\eta_H t^{2H-1}\frac{d}{dt}\int_0^t s^{1/2-H}(t-s)^{1/2-H}u_i(s)ds \\
&= \kappa_H^{-1}\eta_H t^{2H-1}\int_0^t \frac{d}{dt}s^{1/2-H}(t-s)^{1/2-H}u_i(s)ds \\
&= \kappa_H^{-1}\eta_H t^{2H-1}\int_0^t s^{1/2-H}(t-s)^{-1/2-H}u_i(s)ds.
\end{aligned}$$

The process Q_i depends continuously on u_i and therefore, the discrete observations of u_i does not allow one to obtain the discrete observations of Q_i. The process Q_i can be approximated by

$$q_i(n) = \kappa_H^{-1}\eta_H n^{2H-1}\sum_{j=0}^{n-1} j^{1/2-H}(n-j)^{-1/2-H}u_i(j).$$

It is easy to show that $q_i(n) \to Q_i(t)$ almost surely as $n \to \infty$, see Tudor and Viens (2007).

Define a new partition $0 \le r_1 < r_2 < r_3 < \cdots < r_{m_k} = t_k, \quad k = 1, 2, \cdots, n$. Define

$$q_i(t_k) = \kappa_H^{-1}\eta_H t_k^{2H-1}\sum_{j=1}^{m_k} r_j^{1/2-H}(r_{m_k} - r_j)^{-1/2-H}u_i(r_j)(r_j - r_{j-1}), \quad k = 1, 2, \cdots, n.$$

It is easy to show that $q_i(t_k) \to Q_i(t)$ almost surely as $m_k \to \infty$ for each $k = 1, 2, \cdots, n$. We use this approximate observation in the calculation of our estimators.

Applying Itô's formula, for $t \ge r \ge 0$, we obtain

$$q_{j,t} = e^{-\theta(t-r)}q_{j,r} + a\int_r^t e^{-\theta(t-s)}ds + \sigma\int_r^t e^{-\theta(t-s)}q_{j,s-}^{1/\alpha}dZ_{j,s}, \ j \ge 1. \tag{2.5}$$

Let the process be observed at $\{kh, k = 0, 1, \ldots, n\}$ from a single realization $\{q_{j,t}, t \ge 0\}$ for fixed h. For simplicity, we take $h = 1$. This equation can be considered as a first order autoregressive

(AR(1)) equation

$$q_{j,k} = \rho + \gamma q_{j,k-1} + \epsilon_{j,k}, \quad j \geq 1 \tag{2.6}$$

where $\gamma = e^{-\theta}, \quad \rho = a\theta^{-1}(1-\gamma)$ and

$$\epsilon_{j,k} = \sigma \int_{k-1}^{k} e^{-\theta(k-s)} q_{j,s-}^{1/\alpha} dZ_{j,s}, \quad k \geq 1, \ j \geq 1. \tag{2.7}$$

For $B \in \mathcal{B}(R^+)$, let

$$S_{2,j,n}(B) = \sum_{k=1}^{n} q_{j,k-1}\epsilon_k I_B(|q_{j,k-1}\epsilon_{j,k}|), \quad S_{1,j,n}(B) = \sum_{k=1}^{n} q_{j,k-1}^2 I_B(q_{j,k-1}), \quad j \geq 1. \tag{2.8}$$

It is easy to see that

$$\epsilon_{j,k} = q_{j,k} - E(q_{j,k}|\mathcal{F}_{k-1}), \ k \geq 1, \ j \geq 1. \tag{2.9}$$

is a sequence of martingale differences for every fixed j.

Let $S_{1,j,n} := S_{1,j,n}(0,\infty), \quad S_{2,j,n} := S_{2,j,n}(0,\infty)$ and recall that $\gamma = e^{-\theta}$.

Then

$$\hat{\theta}_{j,n} - \theta = \frac{S_{2,j,n}}{S_{1,j,n}} \tag{2.10}$$

where $\hat{\theta}_n$ is the conditional least squares estimator (CLSE) which minimizes

$$\sum_{k=1}^{n} \epsilon_{j,k}^2 = \sum_{k=1}^{n} [q_{j,k} - E(q_{j,k}|\mathcal{F}_{k-1})]^2 = \sum_{k=1}^{n} [q_{j,k} - \rho - \gamma q_{j,k-1}]^2 \tag{2.11}$$

and are given by

$$\hat{\gamma}_{j,n} = \frac{\sum_{k=1}^{n} q_{j,k-1} \sum_{k=1}^{n} q_{j,k} - n\sum_{k=1}^{n} q_{j,k-1}q_{j,k}}{(\sum_{k=1}^{n} q_{j,k-1})^2 - n\sum_{k=1}^{n} q_{j,k-1}^2},$$

$$\hat{\rho}_{j,n} = \frac{1}{n}\sum_{k=1}^{n} q_{j,k} - \hat{\gamma}_n \frac{1}{n}\sum_{k=1}^{n} q_{j,k-1},$$

$$\hat{\theta}_{j,n} = -\log \hat{\gamma}_{j,n}, \quad \hat{a}_{j,n} = \frac{\hat{\rho}_n \hat{\theta}_n}{1-\hat{\gamma}_n}.$$

Let (S_1, S_2) have the characteristic function given by

$$E[\exp\{i\lambda_1 S_1 + i\lambda_2 S_2\}] := \exp\left\{-\frac{\sigma^\alpha}{\theta^2\Gamma(-\alpha)} \int_0^\infty E\left(1 - \exp\{i\lambda_1 y^2 + i\lambda_2 y^{(\alpha+1)/\alpha} V_{j,1}\}\right)\right.$$
$$\left. \times \ E\left(\exp\left\{\frac{ie^{-2\theta\lambda_1 y^2}}{1-e^{-2\theta}} + \frac{ie^{-\theta(\alpha+1)/\alpha}\lambda_2 y^{(\alpha+1)/\alpha} V_{j,2}}{(1-e^{\theta(\alpha+1)})^{1/\alpha}}\right\}\right) \frac{dy}{y^{\alpha+1}}\right\} \tag{2.12}$$

and

$$V_{j,k} := \sigma \int_{k-1}^{k} e^{-\theta(k-s)} e^{-\theta(s-k+1)/\alpha} dZ_{j,s}, \quad k = 1,2, \ j \geq 1 \tag{2.13}$$

which are i.i.d. with the same distribution as

$$\sigma \left(\frac{e^{-\theta}-1}{(\alpha-1)\theta}\right)^{1/\alpha} Z_{j,1}$$

which is regularly varying with index α. The limit distribution is normal only in the Gaussian case $\alpha = 2$.

Following Li and Ma [28] it can be shown that for every fixed j, if we have $1 < \alpha < (1+\sqrt{5})/2$, then we have as $n \to \infty$

$$(d_n^{-2} S_{1,j,n}, c_n^{-1} S_{2,j,n}) \xrightarrow{\mathcal{D}} (S_1, S_2) \;\text{ on }\; \mathbb{R}^2$$

where $d_n = n^{1/\alpha}$ and $c_n = n^{(\alpha+1)/\alpha^2} = d_n^{(\alpha+1)/\alpha}$.

For the stable SPDE model, we have the following result on the consistency and the limit distribution of the CLSE:

Theorem 2.1 If we have $1 < \alpha < (1+\sqrt{5})/2$, then for every fixed $j \geq 1$

a)

$$\hat{\theta}_{j,n} \to^{P} \theta \text{ as } \; n \to \infty.$$

b)

$$n^{(\alpha-1)/\alpha^2}(\hat{\theta}_{j,n} - \theta) \to^{\mathcal{D}} \left(\frac{\sigma^2}{\nu_j^2}\right)^{1/\alpha} \frac{S_2}{S_1} \text{ as } \; n \to \infty.$$

c) If in addition, $\lim_{j\to\infty} |\nu_j| = \infty$, then for every fixed $n \geq 1$,

$$\hat{\theta}_{j,n} \to^{P} \theta \;\text{ as }\; j \to \infty$$

and

$$|\nu_j| \, (\hat{\theta}_{j,n} - \theta) \to^{\mathcal{D}} \sigma \left(n^{-(\alpha-1)/\alpha^2}\right)^{1/\alpha} \frac{S_2}{S_1} \text{ as } \; j \to \infty.$$

where S_2 and S_1 are defined in (2.12).

Remarks

1) The limit distribution in the case $(1+\sqrt{5})/2 < \alpha < 2$ is still open.

2) The process (X_j) is exponentially ergodic and hence strongly mixing.

3) For the Gaussian case ($\alpha = 2$), the limit results are based on ergodic theory and martingale convergence theorem. For the non-Gaussian case ($1 < \alpha < 2$), limit results are obtained by the theory of regular variation and convergence of point processes.

4) Let $0 < \alpha < 2$ and let Z_t be a one dimensional α-stable process with Levy measure $\nu(dz)$. Then as $n \to \infty$, $nP(n^{-1/\alpha} Z_t \in \cdot) \to^{v} t\nu(\cdot)$.

We consider the *Stable Cox-Ingersoll-Ross Model* as an example. Xiong and Yang [41] studied existence and strong uniqueness of the following SPDE:

$$du_k(t) = (\theta\nu_k + \rho_k) u_k(t) dt + \sigma_k (u_k(t))^{1/\alpha} dZ_k(t), \; k \geq 1.$$

The existence of the solution in the case of space-time white noise is shown by considering the weak limit of a sequence of SDE systems which is obtained by replacing the Laplacian operator in the SPDE by its discrete version. The weak uniqueness follows from the uniqueness of solution to the martingale problem for the associated super-Brownian motion. In the case of α-stable noise the existence and pathwise uniqueness of the solution is studied in Xiong and Yang [41].

3. Interacting Particle Systems

First consider the fractional Cox-Ingersoll-Ross (FCIR) model

$$dY_t = a(b - Y_t)dt + \sigma\sqrt{Y_t}dW_t^H \tag{3.1}$$

where W_t^H is a fractional Brownian motion with Hurst parameter $H > 1/2$.

Then by Proposition 5.7 of Buchmann and Kluppelberg [15], we have

$$Y_t = f(X_t) \tag{3.2}$$

where

$$dX_t = a(b - X_t)dt + dW_t^H, \quad X_0 = f^{-1}(Y_0), \quad t \in [0, T] \tag{3.3}$$

and $f(x) = \text{sgn}(x)\sigma^2 x^2/4$.

Let $b = 0$, $\sigma = 1$ and $a > 0$. Then X_t is described by the Ornstein-Uhlenbeck SDE

$$dX_t = -aX_t dt + dW_t^H, \; X_0 = f^{-1}(Y_0). \tag{3.4}$$

For $H = 0.5$, let us consider maximum likelihood estimator (MLE) for the simple mean-field model

$$dX_j(t) = \alpha X_j(t)dt - \beta(X_j(t) - \bar{X}_n(t))dt + dW_j(t), \quad X_j(0) = x_j(0), \quad j = 1, 2, \cdots, n \tag{3.5}$$

where $\bar{X}_n(t)) = n^{-1}\sum_{j=1}^n X_j(t)$, $\beta \neq \alpha$, and $\alpha \neq 0$. The middle term on the right side of (3.5) can be viewed as an interaction among the subsystems which create a tendency for the subsystems to relax towards the center of gravity of the ensemble. Thus the system provides a simple example of a cooperative interaction. Mean-field type models have applications in physics, biology and economics, see Dawson [19]. The case $\beta = 0$ corresponds to sampling independent replications of Ornstein-Uhlenbeck processes on $[0, T]$. Our parameter here is $\theta = (\alpha, \beta)$.

Suppose $\frac{1}{n}\sum_{j=1}^n x_j(0) \to \nu_0$ almost surely and $\frac{1}{n}\sum_{j=1}^n x_j^2(0) \to \gamma_0^2 + \nu_0^2$ almost surely as $n \to \infty$. Then the estimator $\hat{\theta}^n \to^P \theta$ as $n \to \infty$ and $\sqrt{n}(\hat{\theta}^n - \theta) \to^{\mathcal{D}} \mathcal{N}(0, I^{-1}(T))$ as $n \to \infty$

where

$$I(T) = \begin{pmatrix} A(T) & -B(T) \\ -B(T) & B(T) \end{pmatrix}$$

with

$$A(T) := \frac{\nu_0^2}{2\alpha}(e^{2\alpha T} - 1) + B(T), \quad B(T) := \frac{e^{2(\alpha-\beta)T} - 1}{4(\alpha-\beta)^2} - \frac{T}{2(\alpha-\beta)} + \frac{\gamma_0^2(e^{2(\alpha-\beta)T} - 1)}{2(\alpha-\beta)}.$$

The case $\beta = 0$ corresponds to sampling independent replications of the same process given below:

$$dX_j(t) = \alpha X_j(t)dt + dW_j(t), \quad j = 1, 2, \cdots, n \tag{3.6}$$

In the classical case when $\beta = 0$, the MLE is given by

$$\widehat{\alpha}^n = \frac{\sum_{j=1}^n \int_0^T X_j(t)dX_j(t)}{\sum_{j=1}^n \int_0^T (X_j(t))^2 dt}.$$

Sampling n independent Ornstein-Uhlenbeck processes on $[0, T]$ and letting $n \to \infty$ give weak consistency and asymptotic normality of the MLE: $\widehat{\alpha}^n \to^P \alpha$ and $\sqrt{n}(\widehat{\alpha}^n - \alpha) \to^{\mathcal{D}} \mathcal{N}(0, \frac{2\alpha}{\nu_0^2(e^{2\alpha T}-1)})$ as $n \to \infty$. See also Bishwal (2010) for independent sampling case.

For $H \geq 0.5$, let us consider maximum likelihood estimator (MLE) for the fractional mean-field model

$$dX_j(t) = \alpha X_j(t)dt - \beta(X_j(t) - \bar{X}_n(t))dt + dW_j^H(t), \quad X_j(0) = x_j(0), \quad j = 1, 2, \cdots, n \tag{3.7}$$

where $\bar{X}_n(t)) = n^{-1} \sum_{j=1}^n X_j(t)$, $\beta \neq \alpha$, and $\alpha \neq 0$.

The case $\beta = 0$ corresponds to sampling independent replications of the same process given below:

$$dX_j(t) = \alpha X_j(t)dt + dW_j^H(t), \quad j = 1, 2, \cdots, n \tag{3.8}$$

First consider the FCIR model

$$dY_j(t) = a(b - Y_j(t))dt + \sigma\sqrt{Y_j(t)}dW_j^H(t), \quad j = 1, 2, \cdots, n \tag{3.9}$$

where $W_j^H(t)$ is a fractional Brownian motion with Hurst parameter $H > 1/2$.

Then by Proposition 5.7 of Buchmann and Kluppelberg [15], we have

$$Y_j(t) = S(X_j(t)) \tag{3.10}$$

where

$$dX_j(t) = a(b - X_j(t))dt + dW_j^H(t), \quad X_j(0) = S^{-1}(Y_j(0)), \quad t \in [0, T], \quad j = 1, 2, \cdots, n \tag{3.11}$$

and $S(x) = \text{sgn}(x)\sigma^2 x^2/4$. Here S is the state space transform.

Let $b = 0$, $\sigma = 1$ and $a > 0$. Then $X_j(t)$ is described by the Ornstein-Uhlenbeck SDEs

$$dX_j(t) = -aX_j(t)dt + dW_j^H(t), \ X_j(0) = S^{-1}(Y_j(0)), \quad j = 1, 2, \cdots, n \tag{3.12}$$

Consider the model of n interacting particles of fractional diffusions satisfying the Itô stochastic differential equations

$$dX_j(t) = \sum_{l=1}^p \theta_l \mu_{jl}(X(t)) + \sigma_j(X(t))dW_j^H(t), \quad j = 1, 2, \cdots, n \tag{3.13}$$

where $X(t) = (X_1(t), X_2(t), \cdots, X_n(t))'$ and $(W_j^H(t); t \geq 0)$, $j = 1, 2, \cdots, n$ are independent fractional Wiener processes. Here $\theta_l(\cdot) \in L^2([0, T], dt), l = 1, \ldots, p$ are unknown functions to be estimated based on observation of the process X in the time interval $[0, T]$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_p)$ and $\mu_j(x) = (\mu_{j1}(x), \mu_{j2}(x), \ldots, \mu_{jp}(x))'$. The processes $X_j(t), j = 1, 2, \cdots, n$ are observed on $[0, T]$.

The functions $\mu_j, \sigma_j; \ \ j = 1, 2, \cdots, n$ are assumed to be known such that the system has a unique solution.

We need the following assumption and results to prove the main results.

(A0) Suppose that $b_{jl} := \mu_{jl}(s)\sigma_j^{-1}(s); j = 1, 2, \cdots, n; l = 1, 2 \ldots, p$ are measurable and adapted processes satisfying

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^t b_{jl}(s)b_{jm}(s)ds \to c_{lm}(t) \text{ a.s.} \quad \text{as } n \to \infty$$

$l, m = 1, 2 \ldots, p$ where $c_{lm}(t)$ are finite and continuous nonrandom functions of $t \in [0, T]$. The limiting matrix $I(t) = (c_{lm}(t))_{l,m=1,2\ldots,p}$ is positive definite, $\delta' I(t)\delta$ is increasing for all $\delta \in \mathbb{R}^p$ and $I(0) = 0$.

In the exchangeable case, (A0) follows from McKean-Vlasov Law of Large Numbers. In particular, (A0) will be satisfied when $\mu_{jl}(X) = \mu_l X_j$ and $\sigma_j(X) = \sigma(X_j)$ which corresponds to the independent replicated sampling on $[0, T]$. See Oelschlager [29].

We also need the following version of Rebolledo's Central Limit Theorem for Martingales, see Rebolledo [34]:

Theorem 3.1 *Let $M_n, n \in \mathbb{Z}_+$ be a sequence of locally square integrable martingales with $M_n(0) = 0$. Suppose the following condition holds: $\sum_{s\le t} E\{|\Delta M_n(s)|^2 I(|\Delta M_n(s)| > \epsilon)\} \to 0$ for all $t \in [0, T]$, $\epsilon > 0$; and $\langle M_n\rangle(t) \to c(t)$ a.s. for all $t \in [0, T]$, where $c(t)$ is a continuous increasing function with $c(0) = 0$. Then $M_n \to^{\mathcal{D}} M$, a continuous Gaussian martingale with zero mean and covariance function $K(s, t) = c(s \wedge t), s, t \in [0, T]$ where $\Delta M_s = M_s - M_{s-}$ denotes the jump of M at the point s.*

The model is given by

$$dX_j(t) = \sum_{l=1}^{p}\theta_l\mu_{jl}(X(t)) + \sigma_j(X(t))dW_j^H(t), \quad j = 1, 2, \cdots, n \tag{3.14}$$

where $X(t) = (X_1(t), X_2(t), \cdots, X_n(t))'$ and $(W_j^H(t); t \ge 0)$, $j = 1, 2, \cdots, n$ are independent fractional Wiener processes. Here $\theta = (\theta_1, \theta_2, \ldots, \theta_p)$ is the unknown parameter. The functions $\mu_{jl}, \sigma_j, j = 1, \ldots, n; l = 1, \ldots, p$ are assumed to be known such that there exists a unique solution $X(t)$ to the above SDE.

Our aim is to estimate the parameter θ based on n particles $q_1(\cdot), q_2(\cdot), \cdots, q_n(\cdot)$ of $q(t)$ on $[0, T]$. We denote this data by $q^{n,T}$.

The Radon-Nikodym derivative (likelihood) is given by

$$\begin{aligned}\Lambda_n^\theta(q^{n,T}) := \frac{dP_\theta}{dP_0}(q^{n,T}) = \exp\Big\{&\textstyle\sum_{l=1}^{p}\theta_l\sum_{j=1}^{n}\int_0^T\mu_{jl}(q(t))\sigma_j^{-2}(q(t))dq_j(t)\\ &\textstyle-\frac{1}{2}\sum_{l=1}^{p}\sum_{m=1}^{p}\theta_l\theta_m\sum_{j=1}^{n}\int_0^T\mu_{jl}(q(t))\sigma_j^{-2}(q(t))\mu_{jm}(q(t))dt\Big\}.\end{aligned} \tag{3.15}$$

4. Approximate Maximum Likelihood Estimation

The approximate maximum likelihood estimator is defined as

$$\hat{\theta}^n = \arg\max_{\theta} \Lambda_n^{\theta}(q^{n,T}).$$

Extending Kasonga [24], using McKean-Vlasov Law of Large Numbers and Rebolledo's Central Limit Theorem for Martingales, we obtain the consistency and asymptotic normality of the approximate maximum likelihood estimator $\hat{\theta}^n$ which is given below:

Theorem 4.1 *Under (A0), we have a)* $\hat{\theta}^n \to^P \theta$ *as* $n \to \infty$.

b) $\sqrt{n}(\hat{\theta}^n - \theta) \to^{\mathcal{D}} \mathcal{N}(0, I^{-1}(T))$ *as* $n \to \infty$ *where* $I(T)$ *is the Fisher information.*

5. Berry-Esseen Inequality

In this section we consider the case $H = 0.5$ and $\alpha = 2$, i.e., the standard Brownian motion case.

$$dX_j(t) = f_j(\theta, X(t)) + \sigma_j(X(t))dW_j(t), X_j(0) = X_j^0, \quad j = 1, 2, \cdots, n$$

We assume the following conditions for $j = 1, 2, \cdots, n$:

(A1) $|f_j(\theta, x)| \le a_j(\theta)(1 + |x|), \qquad |f_j(\theta, x) - f_j(\theta, y)| \le a_j(\theta)|x - y|$.

(A2) $|f_j(\theta, x) - f_j(\phi, y)| \le b_j(x)|\theta - \phi|$ for all $\theta, \phi \in \Theta, x, y \in \mathbb{R}$ where $\sup_{\theta\in\Theta} |a_j(\theta)| = a < \infty, E|b_j(X_j^0)|^r < \infty$ for any integer r.

(A3) The diffusion process X is stationary and ergodic with invariant measure ν, i.e., for any g_j with $E[g_j(\cdot)] < \infty$, $\frac{1}{n}\sum_{j=1}^n \sum_{i=1}^m g_j(X_{t_i}) \to E_\nu[g(X_0)]$ a.s. as $n \to \infty$ and $h \to 0$.

(A4) $\sup_{t\ge 0} E|X_j(t)|^r < \infty$ for all $r \ge 0$.

(A5) $E|f_j(\theta, X_j^0) - f_j(\theta_0, X_j^0)|^2 = 0$ iff $\theta = \theta_0$.

(A6) f_j is twice continuously differentiable function in x for all θ.

(A7) $f_j(\cdot, x)$ and all its derivatives are three times continuously differentiable with respect to θ for all $x \in \mathbb{R}$. Moreover, these derivatives upto third order with respect to θ are of polynomial growth in x uniformly in θ.

The Fisher information is given by $0 < I(\theta) := \int_{-\infty}^{\infty} (f_j'(\theta, x))^2 d\nu(x) < \infty$ and for any $\delta > 0$, or any compact $\bar{\Theta} \subset \Theta$,

$$\inf_{\theta_0\in\bar{\Theta}} \sup_{|\theta-\theta_0|>\delta} E_{\theta_0}|f_j'(\theta, X_0) - f_j'(\theta_0, X_j(0))|^2 > 0.$$

(A8) The Malliavin covariance of the process is nondegenerate.

Let $f_j = \mu_j/\sigma_j, \ j = 1, 2, \cdots, n$. The model is given by

$$dX_j(t) = \sum_{l=1}^{p} \theta_l \mu_{jl}(X(t)) + \sigma_j(X(t))dW_j(t), \quad j = 1, 2, \cdots, n.$$

The Radon-Nikodym derivative (likelihood) is given by

$$L_n^\theta(X^{n,T}) := \frac{dP_\theta}{dP_0}(X^{n,T}) = \exp\Big\{\sum_{l=1}^p \theta_l \sum_{j=1}^n \int_0^T \mu_{jl}(X(t))\sigma_j^{-2}(X(t))dX_j(t)$$
$$-\frac{1}{2}\sum_{l=1}^p\sum_{m=1}^p \theta_l\theta_m \sum_{j=1}^n \int_0^T \mu_{jl}(X(t))\sigma_j^{-2}(X(t))\mu_{jm}(X(t))dt\Big\}.$$

We observe the process $\{X_t\}$ at times $0 = t_0 < t_1 < \cdots t_m = T$ with $t_i - t_{i-1} = \frac{T}{m} = h, i = 1, 2\cdots, n$. We assume equispaced sampling for simplicity with T being fixed, $m \to \infty$ and $n \to \infty$.

The dataset is n particles $X_1(\cdot), X_2(\cdot), \cdots, X_n(\cdot)$ of $X(t)$ on $[0, T]$. The approximate log-likelihood based on observations $X_j(t_1), X_j(t_2), \ldots, X_j(t_n),\ j = 1, 2, \cdots, n$ with $t_i = iT/m = ih$ is defined as

$$K_{n,m}(\theta) = \sum_{l=1}^p \theta_l \sum_{j=1}^n \sum_{i=1}^m \mu_{jl}(X(t_{i-1}))\sigma_j^{-2}(X(t_{i-1}))(X_j(t_i) - X_j(t_{i-1})$$
$$-\frac{1}{2}\sum_{l=1}^p\sum_{k=1}^p \theta_l\theta_k \sum_{j=1}^n\sum_{i=1}^m \mu_{jl}(X(t_{i-1}))\sigma_j^{-2}(X(t_{i-1})))\mu_{jk}(X(t_{i-1}))(t_i - t_{i-1}).$$

We start with some preliminary lemmas. The first lemma is from Michel and Pfanzagl (1971) which will be needed to prove our main results.

Lemma 5.1 Let ξ, ζ and η be any three random variables on a probability space $(\Omega, \mathcal{F}, P)$ with $P(\eta > 0) = 1$. Then, for any $\epsilon > 0$, we have

$$(a)\ \sup_{x\in\mathbb{R}} |P\{\xi + \zeta \le x\} - \Phi(x)| \le \sup_{x\in\mathbb{R}} |P\{\xi \le x\} - \Phi(x)| + P(|\zeta| > \epsilon) + \epsilon,$$

$$(b)\ \sup_{x\in\mathbb{R}} |P\{\frac{\xi}{\eta} \le x\} - \Phi(x)| \le \sup_{x\in\mathbb{R}} |P\{\xi \le x\} - \Phi(x)| + P\{|\eta - 1| > \epsilon\} + \epsilon.$$

The strong rate of convergence of particle approximations of McKean-Vlasov SDEs with Lipschitz coefficients is $O(n^{-1/2})$ where n is the number of particles. This rate is driven by the statistical error. The bias is of the order $O(n^{-1})$. Talay and Tubaro [37] showed that for smooth coefficients the the weak error is $O(h)$. Bencheikh and Jourdain [2] showed that weak error between a SDE with nonlinear in the sense of McKean given by moments and its approximation by the Euler discretization with time step h of a system of n interacting particles is $O(n^{-1} + h)$.

From Talay and Tubaro [37] and Bencheikh and Jourdain [2], we have

Lemma 5.2 Let $f_j = \mu_j/\sigma_j$. Then

$$\sup_{t\in\Pi} |E[f_j(X_t^n)] - E[f_j(X_t^{n,m})]| \le C\frac{T}{m},\ j \ge 1.$$

The following lemma follows from Yoshida [43, 44].

Lemma 5.3 Let $I_n(\theta) := \frac{1}{nI(\theta_0)}\sum_{j=1}^n \int_0^T \mu_j^2(\theta, X_t)dt$. Then under the conditions (A1)-(A8),

$$\sup_{\theta\in\Theta} E[I_n(\theta) - 1]^2 \le Cn^{-1}.$$

The following lemma follows from Theorem 1 in Yoshida [44].

Lemma 5.4 Let $M_n := \frac{1}{\sqrt{nI(\theta_0)}} \sum_{j=1}^{n} \int_0^T \mu_j(\theta_0, X_t) dW_t$. Then under the conditions (A1)-(A8),

$$\sup_{x \in \mathbb{R}} |P_{\theta_0} \{M_n \le x\} - \Phi(x)| \le C n^{-1/2}.$$

In this section, our main result is the following theorem.

Theorem 5.5 Under the conditions (A1)–(A8), we have

$$\sup_{x\in\mathbb{R}} \left| P_{\theta_0} \left\{ \sqrt{nI(\theta_0)}(\theta_{m,n} - \theta_0) \le x \right\} - \Phi(x) \right| = O\left(n^{-1/2} \bigvee \frac{T}{m} \right).$$

Proof By Taylor expansion, we have

$$K'_{m,n}(\theta_{m,n}) = K'_{m,n}(\theta_0) + (\theta_{m,n} - \theta_0) K''_{m,n}(\bar{\theta}_{m,n})$$

where $\left|\bar{\theta}_{m,n} - \theta\right| \le |\theta_{m,n} - \theta_0|$. Since $K'_{m,n}(\theta_{m,n}) = 0$, hence we have

$$\sqrt{nI(\theta_0)}(\theta_{m,n} - \theta_0) = -\frac{\frac{1}{\sqrt{nI(\theta_0)}} K'_{m,n}(\theta_0)}{\frac{1}{nI(\theta_0)} K''_{m,n}(\bar{\theta}_{m,n})} = -\frac{\frac{1}{\sqrt{nI(\theta_0)}} \sum_{j=1}^{n} \sum_{i=1}^{m} \mu'_j(\theta_0, X_{t_{i-1}}) \Delta W_i}{\frac{1}{nI(\theta_0)} \sum_{j=1}^{n} \sum_{i=1}^{m} \mu''_j(\bar{\theta}_{m,n}, X_{t_{i-1}}) \Delta t_i} =: \frac{U_{m,n}}{V_{m,n}}$$

Note that

$$V_{m,n} = \frac{1}{nI(\theta_0)} \sum_{j=1}^{n} \sum_{i=1}^{m} \mu''_j(\bar{\theta}_{m,n}, X_{t_{i-1}}) \Delta t_i = \frac{1}{nI(\theta_0)} \sum_{j=1}^{n} \sum_{i=1}^{m} \mu'_j(\bar{\theta}_{m,n}, X_{t_{i-1}})^2 \Delta t_i.$$

Let $\lim V_{m,n} = V_n$ in L_2 as $\frac{T}{m} \to 0$. Similar to Lemma 5.3, it can be shown that $E(V_n - 1)^2 \le Cn^{-1}$ (see also Pardoux and Veretennikov (2001) and Yoshida (2011)). It can be shown that $E(V_{m,n} - V_n)^2 \le C\frac{T}{m}$ (see Altmeyer and Chorowski (2018)). Hence

$$E(V_{m,n} - 1)^2 = E[(V_{m,n} - V_n) + (V_n - 1)]^2 \le C(n^{-1} \bigvee \frac{T}{m}).$$

Further by Lemma 5.1 (b), we have

$$\begin{aligned}
& \sup_{x\in\mathbb{R}} \left| P_\theta \left\{ \sqrt{nI(\theta)}(\theta_{m,n} - \theta) \le x \right\} - \Phi(x) \right| \\
= & \sup_{x\in\mathbb{R}} \left| P_\theta \left\{ \frac{U_{m,n}}{V_{m,n}} \le x \right\} - \Phi(x) \right| \\
= & \sup_{x\in\mathbb{R}} |P_\theta \{U_{m,n} \le x\} - \Phi(x)| + P_\theta \{|V_{m,n} - 1| \ge \epsilon\} + \epsilon \\
\le & C(n^{-1/2} \bigvee \frac{T^2}{m}) + \epsilon^{-2} C(n^{-1} \bigvee \frac{T}{m}) + \epsilon.
\end{aligned}$$

since by Lemma 5.1 (a), Lemma 5.2 and Lemma 5.4, we have

$$\begin{aligned} & \sup_{x\in\mathbb{R}} \left| P_\theta \{U_{m,n} \le x\} - \Phi(x) \right| \\ \le & \sup_{x\in\mathbb{R}} \left| P_\theta \{M_n \le x\} - \Phi(x) \right| + P_\theta \{|U_{m,n} - M_n| \ge \epsilon\} + \epsilon \\ \le & \; Cn^{-1/2} + \epsilon^{-2} E \left| U_{m,n} - M_n \right|^2 + \epsilon \\ \le & \; Cn^{-1/2} + \epsilon^{-2} C \frac{T}{m} + \epsilon. \end{aligned}$$

Choosing $\epsilon = n^{-1/2}$, we have the result. □

Remarks We considered fractional Levy process driving term in this paper whose increments are stationary. Using fractional Levy process as the driving term which include jumps, maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [9]. Recently, sub-fractional Brownian (sub-FBM) motion which is a centered Gaussian process with covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s,t > 0$$

for $0 < H < 1$ introduced by Bojdecki, Gorostiza and Talarczyk [14] has received some attention recently in finite dimensional models. The interesting feature of this process is that this process has some of the main properties of FBM, but the increments of the process are nonstationary, more weakly correlated on non-overlapping time intervals than that of FBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. It would be interesting to see extension of this paper to sub-FBM case. We generalize sub-fBM to Sub-fractional Levy process (sub-FLP).

Sub-fractional Levy process (SFLP) is defined as

$$S_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})} \int_{\mathbb{R}} [(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}] dM_s, \quad t \in \mathbb{R}$$

where $M_t, t \in \mathbb{R}$ is a Levy process on $\mathbb{R}$ with $E(M_1) = 0$, $E(M_1^2) < \infty$ and without Brownian component. SFLP has the following properties:

1) The covariance of the process is given by

$$\mathrm{Cov}(S_{H,t}, S_{H,s}) = s^{2H} + t^{2H} + \frac{E[L(1)^2]}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

2) S_H is not a martingale. For a large class of Levy processes, S_H is neither a semimartingale nor a Markov process. 3) S_H is Hölder continuous of any order β less than $H - \frac{1}{2}$. 4) S_H has nonstationary increments. 5) S_H is symmetric. 6) S_H is self similar. 7) S_H has infinite total variation on compacts.

It would be interesting to investigate estimation in SPDE driven by subfractional Levy processes which incorporate both jumps and long memory apart from nonstationarity.

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