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# A New Study on Generalized Reverse Derivations of Semi-prime Ring

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Abstract. The aim of this paper is to extend the ideas from Generalized reverse derivation to Generalized $(\alpha, \beta)$-reverse derivations on Semi-prime ring. We prove that, if $0 \neq d$ be reverse derivation in $R$ and a Generalized $(\alpha, \beta)$-reverse derivation $g$, then $g$ is $\beta$-strong commutative preserved. Next we can prove that $R$ is commutative.

## 1. Introduction

The study of centralizing mapping of semi-prime rings given by Bell and Martindale [3]. Bell and Martindale [3] proved that $[d(u_1), u_1]_{\alpha,\beta} = 0 \ \forall \ u_1 \in B$, where $0 \neq d$ a derivation of $R$ and $R$ is semi-prime ring, then commutativity holds in $R$. Bell and Daif were studied the commutativity in prime and semi-prime rings that bind endomorphism or a derivation that preserves a $\beta$-strong commutativity on a non-zero ideal right in [2]. Further, Ali and Shah [1] extend some consequences for generalized derivation of Bell and Martindale [3]. Bresar established that, if $B \neq 0$ is left ideal in $R$ a prime ring, and two mappings $d_1$ and $d_2$ are $(\alpha, \beta)$-derivations in $R$ satisfies $(d_1\alpha(a) - \beta(a)d_2) \in Z(R)$, for each $a \in B$, so commutativity holds in $R$ [5]. Some properties are studied by Vukman in [12] and [4]. M. Samman and N. AL Yamani [8] studied reverse derivation on semi prime rings. They proved that the mapping $d : R \to R$ is central derivation iff it is reverse derivation and also that $d \neq 0$ a reverse derivation in semi-prime ring $R$, then the commutativity exists in $R$ resently Mukhtar Ahmad et.al[9]. Later, the idea of revers derivation and some properties

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of reverse derivation were studied by Bresar and Vukman [4]. The aim of this paper is extention the notion of generalized reverse derivation to generalized $(\alpha, \beta)$-reverse derivation resently Mukhtar Ahmad et.al[10]. A mapping $G : R \to R$ which associate with $(\alpha, \beta)$-reverse derivation $D$ is said to be a generalized $(\alpha, \beta)$-reverse derivation if, $G(u_1v_1) = G(v_1)\alpha(u_1) + \beta(v_1)D(u_1)$resently R. M. Kashif et.al[11].

## 2. Preliminaries

Throughout this paper,

**Definition 2.1.** Let $R$ is ring and it is considered as a semi-prime ring iff for any $u_1$; $u_1 \neq 0$ such that $u_1Ru_1 = 0$ implies $u_1 = 0$.

**Definition 2.2.** The additive mapping $d_1 : R \to R$ is known as $(\alpha, \beta)$-derivation, if $d_1(u_1v_1) = d_1(u_1)\alpha(v_1) + \beta(u_1)d_1(v_1)$ hold $\forall\ u_1,\ v_1\ \in\ R$, where $\alpha$ and $\beta$ are automorphism.

**Definition 2.3.** The mapping $d_1\ :\ R\ \to\ R$ is called a $(\alpha, \beta)$-reverse derivation if $d_1(u_1v_1) = d_1(v_1)\alpha(u_1) + \beta(v_1)d_1(u_1)$ holds $\forall\ u_1,\ v_1\ \in\ R$, where $\alpha$ and $\beta$ are automorphism.

**Definition 2.4.** An additive mapping $H\ :\ R\ \to\ R$ be a right (left) generalized $(\alpha, \beta)$-reverse derivation if there is a derivation $d$ from $R$ to $R$ such that $H(u_1v_1) = H(v_1)\alpha(u_1)+\beta(v_1)d(u_1)$ $(H(u_1v_1) = d(v_1)\alpha(u_1) + \beta(v_1)H(u_1)$ for all $u_1,\ v_1\ \in\ R$. H be a generalized reverse $(\alpha, \beta)$ of R associated with $(\alpha, \beta)$ derivation.

**Definition 2.5.** Some identities holds for every $u_1, v_1, w_1 \in R$ $[u_1, v_1w_1] = v_1[u_1, w_1]+[u_1, v_1]w_1$
$[u_1v_1, w_1] = [u_1, w_1]v_1 + u_1[v_1, w_1]$
$[u_1v_1, w_1]_{\alpha,\beta} = u_1[v_1, w_1]_{\alpha,\beta} + [u_1, \beta(w_1)]v_1 = u_1[v_1, \alpha(w_1)] + [u_1, w_1]_{\alpha,\beta}\, v_1$
$[u_1, v_1w_1]_{\alpha,\beta} = \beta(v_1)[u_1, w_1]_{\alpha,\beta} + [u_1, v_1]_{\alpha,\beta}\, \alpha(w_1)$

**Definition 2.6.** The derivation H would be commuting, if $0 = [v_1, H(u_1)]$, $\forall\ u_1, v_1\ \in\ R$.

**Definition 2.7.** The strong commutativity preserving is defined as $[g(u_1), g(v_1)] = [u_1, v_1]$ for all $u_1, v_1 \in R$, where $g : R \to R$ is a mapping on $R$.

**Lemma 2.8.** Let $u_1 \neq 0$ in $Z$(center of ring), if $u_1, v_1 \in Z$ , then $v_1 \in Z$.

**Lemma 2.9.** Let $g : R \to R$ be an additive map and on a left ideal $B$ of $R$, $g$ is centralizing, then $g(u_1) \in R\ \forall\ \ u_1 \in B \cup Z$.

**Lemma 2.10.** Let $0 \neq B$ be an ideal of a semi-prime ring $R$. If the set $[B, B]$ centralizes $Z$ in $R$, then $B$ centralizes $Z$.

### 2.1. Point-Wise Operation.

**Theorem 2.11.** Suppose $0 \neq d$ from $R$ to $R$ a derivation in a semi-prime ring $R$. Let generalized $(\alpha, \beta)$-reverse derivation $g$ on a left ideal $B \neq 0$ of $R$. Then $g$ satisfies $[g(w_1), g(v_1)] = \beta([w_1, v_1])$ for all $v_1, w_1 \in B$ (that is, $g$ is $\beta$-strong commutativity preserved), when $g$ is a homomorphism on $B$.

**Proof.** Since $g$ is generalized $(\alpha, \beta)$-reverse derivation and homomorphism on $B$, such that $g(u_1v_1) = g(u_1)g(v_1)\, \forall\, u_1, v_1 \in B$.

This implies

$$g(u_1v_1) = g(u_1)g(v_1) = g(v_1)\alpha(u_1) + \beta(v_1)d(u_1),\ for\, all\, u_1, v_1 \in B. \quad (1)$$

We replace $v_1$ by $v_1w_1$ where $w_1 \in B$, in equation (1), we obtain

$g(u_1)g(v_1w_1) = g(v_1w_1)\alpha(u_1) + \beta(v_1w_1)d(u_1)$

this gives

$$g(u_1)g(v_1w_1) = g(u_1v_1w_1) = g(v_1)g(w_1)\alpha(u_1) + \beta(v_1w_1)d(u_1),\quad for\, all\, u_1, v_1 \in B. \quad (2)$$

As $g$ is homomorphism, so we get

$g(u_1)g(v_1w_1) = g(u_1)g(v_1)g(w_1) = g(u_1v_1)g(w_1)$

this equalized to

$g(u_1v_1)g(w_1) = (g(v_1)\alpha(u_1) + \beta(v_1)d(u_1))g(w_1)$

this relates to

$g(u_1v_1)g(w_1) = g(v_1)\alpha(u_1)g(w_1) + \beta(v_1)d(u_1)g(w_1)$

By the equation (2), we get

$$g(u_1v_1)g(w_1) = g(v_1)g(w_1)\alpha(u_1) + \beta(v_1)d(u_1)g(w_1),\ for\, all\, u_1, v_1 \in B. \quad (3)$$

From equation (2) and equation (3), we obtain

$\beta(v_1)d(u_1)g(w_1) = \beta(v_1)d(u_1)\beta(w_1)$

this implies

$$\beta(v_1)d(u_1)(g(w_1) - \beta(w_1)) = 0,\ for\, all\, u_1, v_1 \in B. \quad (4)$$

Put $w_1 = [w_1, v_1]$ in equation (4), we have

$\beta(v_1)d(u_1)(g([w_1, v_1]) - \beta([w_1, v_1])) = 0,$

we arrives to

$d(u_1)\beta(v_1)(g([w_1, v_1]) - \beta([w_1, v_1])) = 0.$

By replacing $\beta(v_1)$ by $(g([w_1, v_1]) - \beta([w_1, v_1]))\alpha(r)d(u_1)$, we obtain

$d(u_1)(g([w_1, v_1]) - \beta([w_1, v_1]))\alpha(r)d(u_1)(g([w_1, v_1]) - \beta([w_1, v_1])) = 0,$

it gives

$d(u_1)(g([w_1, v_1]) - \beta([w_1, v_1]))Rd(u_1)(g([w_1, v_1]) - \beta([w_1, v_1])) = 0.$

As $R$ semi-prime, so we obtain

$d(u_1)(g([w_1, v_1]) - \beta([w_1, v_1])) = 0,$

since $d \neq 0$, we have

$g([w_1, v_1]) - \beta([w_1, v_1]) = 0,$

we get

$g([w_1, v_1]) = \beta([w_1, v_1])$

As $g$ is homomorphism, so we have
$[g(w_1), g(v_1)] = \beta([w_1, v_1])$
So $g$ is $\beta$-strong commutative preserved on $B$.

**Theorem 2.12.** Let $g$ on a left ideal $B \neq 0$ of $R$, is generalized $(\alpha, \beta)$-reverse derivation. If $g$ is homomorphism on $B$, then on $B$, $g$ is commuting.

**Proof.** By theorem 2.11, $g$ is $\beta$-strong commutative preserved, then $\forall\ u_1, v_1 \in B$, we get

$$\beta([u_1, v_1]) = [g(u_1), g(v_1)] \tag{5}$$

Replace $v_1 = v_1 u_1$ in equation (5) we have
$\beta([u_1, v_1 u_1]) = [g(u_1), g(v_1 u_1)]$
$\beta([u_1, v_1])\beta(u_1) = [g(u_1), g(v_1)]g(u_1)$
By equation (5), we get
$\beta([u_1, v_1])\beta(u_1) = \beta([u_1, v_1])g(u_1)$
this implies

$$\beta([u_1, v_1])(g(u_1) - \beta(u_1)) = 0. \tag{6}$$

Now put $v_1 = u_1 v_1$ in equation (5) we have
$\beta([u_1, u_1 v_1]) = [g(u_1), g(u_1 v_1)]$
$\beta(u_1)\beta([u_1, v_1]) = g(u_1)[g(u_1), g(v_1)]$
By equation (5), we have
$\beta(u_1)\beta([u_1, v_1]) = g(u_1)\beta([u_1, v_1])$
this implies

$$(g(u_1) - \beta(u_1))\beta([u_1, v_1]) = 0. \tag{7}$$

Put $v_1 = r_1 v_1$ in equation (6), we have
$\beta([u_1, r_1 v_1])(g(u_1) - \beta(u_1)) = 0,$
this implies
$\beta([u_1, r_1])\beta(v_1)(g(u_1) - \beta(u_1)) = 0,$
we get
$\beta([u_1, r_1])B(g(u_1) - \beta(u_1)) = 0,$
also that
$\beta([u_1, r_1])RB(g(u_1) - \beta(u_1)) = 0,$
By semi-primeness of $R$, there exist a family $w = \{P_\theta/\theta \in \bigwedge\}$ of prime ideals such that $\bigcap P_\theta = 0$.If $w$ has a member $P$ and $u_1 \in B$, then last relation, we get, $B(g(u_1) - \beta(u_1))$ not in $P$ or $[\beta(u_1), R] \subseteq P$. If $\exists\ v_1 \in B$ such that $[\beta(u_1), R]$ not in $P$. It implies $B(g(v_1) - \beta(v_1)) \subseteq P$. Let $w_1 \in B$ is arbitrary such that $[\beta(v_1 + w_1), R] \subseteq P$. This means that $[\beta(w_1), R]$ not in $P$ and hence $(g(w_1) - \beta(w_1)) \subseteq P$. In other ways $[\beta(v_1 + w_1), R] \subseteq P$, then $B(g(v_1 + w_1) - \beta(v_1 + w_1)) \subseteq P$.

It gives $B(g(w_1) - \beta(w_1)) \subseteq P$.

We obtain $B(g(w_1)-\beta(w_1)) \subseteq P$ for every $w_1 \in B$ and hence $[B, B](g(w_1)-\beta(w_1)) \subseteq P \, \forall \, w_1 \in B$. As $P$ is arbitrary and $\bigcap P_\theta = 0$, this implies $[B, B](g(w_1) - \beta(w_1)) = 0$ for all $w_1 \in B$. Similarly, we can show that $(g(w_1) - \beta(w_1))[B, B] = 0$ for all $w_1 \in B$. This implies that $(g(w_1) - \beta(w_1)) \in C_R[B, B]$, for all $w_1 \in B$. By Lemma 2.10 and [6], we have $(g(u_1), \beta(u_1)) \in C_R(B)$, $\forall \, w_1 \in B$. Thus we have $[g(u_1) - u_1, \beta(u_1)] = 0 \; \forall \; u_1 \in B$. This implies that $[g(u_1), \beta(u_1)] = 0 \; \forall \; u_1 \in B$. This shows that $g$ is commuting on $B$.

**Theorem 2.13.** Suppose a derivation, $d : R \to R$ where $0 \neq d$, in $R$ and a generalized $(\alpha, \beta)$-reverse derivation $g$ on left ideal $B \neq 0$. If $g$ is a homomorphism on $B$, then commutativity exists in $R$.

**Proof.** By our hypothesis

$$[g(u_1), u_1]_{\alpha,\beta} = 0, \; for\, all\, u_1 \in B. \tag{8}$$

We replace $u_1$ by $u_1 + v_1$, in equation (2.1),we get

$[g(u_1 + v_1), u_1 + v_1]_{\alpha,\beta} = 0$,

we have

$[g(u_1) + g(v_1), u_1 + v_1]_{\alpha,\beta} = 0$,

we arrives to

$[g(u_1) + g(v_1), u_1]_{\alpha,\beta} + [g(u_1) + g(v_1), v_1]_{\alpha,\beta} = 0$,

this gives

$[g(u_1), u_1]_{\alpha,\beta} + [g(v_1), u_1]_{\alpha,\beta} + [g(u_1), v_1]_{\alpha,\beta} + [g(v_1), v_1]_{\alpha,\beta} = 0$.

By equation (), we obtain

$$[g(u_1), v_1]_{\alpha,\beta} + [g(v_1), u_1]_{\alpha,\beta} = 0, \; for\, all\, u_1 \in B. \tag{9}$$

By substituting $v_1 = u_1 v_1$ in equation (9), we have

$[g(u_1), u_1 v_1]_{\alpha,\beta} + [g(u_1 v_1), u_1]_{\alpha,\beta} = 0$,

we have

$\beta(u_1)[g(u_1), v_1]_{\alpha,\beta} + [g(u_1), u_1]_{\alpha,\beta}\,\alpha(v_1) + [g(v_1)\alpha(u_1) + \beta(v_1)d(u_1), u_1]_{\alpha,\beta} = 0$.

this implies us by the equation (2.1),

$\beta(u_1)[g(u_1), v_1]_{\alpha,\beta} + [g(v_1)\alpha(u_1), u_1]_{\alpha,\beta} + [\beta(v_1)d(u_1), u_1]_{\alpha,\beta} = 0$.

This gives us by $[\alpha(u_1), \alpha(u_1)] = 0$,

$\beta(u_1)[g(u_1), v_1]_{\alpha,\beta} + [g(v_1), u_1]_{\alpha,\beta}\,\alpha(u_1) + [\beta(v_1)d(u_1), u_1]_{\alpha,\beta} = 0$,

Since $g$ is commuting on $B$, we have

$$[\beta(v_1)d(u_1), u_1]_{\alpha,\beta} = 0, \; for\, all\, u_1 \in B. \tag{10}$$

We replace $v_1$ by $r_1 v_1$ in equation (10), we have

$[\beta(r_1 v_1)d(u_1), u_1]_{\alpha,\beta} = 0$, we get

$\beta(r_1)[\beta(v_1)d(u_1), u_1]_{\alpha,\beta} + [\beta(r_1), \beta(u_1)]\beta(v_1)d(u_1) = 0.$

By equation (10), we have

$[\beta(r_1), \beta(u_1)]\beta(v_1)d(u_1) = 0,$

this gives

$[\beta(r_1), \beta(u_1)]Bd(u_1) = 0$, for all $u_1 \in B$ and $r_1 \in R$,

By the semi-primeness of $R$, $\exists$ a set $\omega = \{P_\alpha/\alpha \in \bigwedge\}$ of prime ideals and $\bigcap P_\alpha = (0)$.

If $P \in \omega$ and $u_1 \in B$, then by equation (10), $[R, \beta(u_1)] \subseteq P$ or $P \supseteq d(u_1)$. Since $0 \neq d$ on $R$, so by [7], $0 \neq d$ on $B$. Consider $d(u_1)P$, where $u_1 \in B$, then $P \supseteq [R, \beta(u_1)]$. Suppose $w_1 \in B$, we see that $w_1$ not in $Z$, then $d(w_1) \subseteq P$ and $u_1 + w_1$ not in $Z$. This gives that $d(u_1 + w_1) \subseteq P$ and then $d(u_1) \subseteq P$, which contradicts to our consideration that $d(u_1)P$. So, this gives us $w_1 \in Z$, $\forall$ $w_1 \in B$.

This implies that $B$ is commutative also that by the [7], then commutativity holds in $R$.

**Theorem 2.14.** Suppose a semi-prime ring $R$ and a left ideal $B$ of $R$, s.t. $B \bigcap Z \neq 0$ for center $Z$ of $R$. Let a generalized $(\alpha, \beta)$-reverse derivation $g$ on $R$ and $d \neq 0$ a derivation and $g$ is centralizing on $B$. Then commutativity holds in $R$.

**Proof.** If $Z \neq 0$ and $g$ is commutation on $B$, so our proof is complete.

As $g$ is centralizing $B$ and by Theorem 2.12, we get

$$[g(u_1), u_1]_{\alpha,\beta} \in Z, \ \forall\, u_1 \in B. \tag{11}$$

Put $u_1 = (u_1 + v_1)$ in equation (11), then

$[g(u_1 + v_1), u_1 + v_1]_{\alpha,\beta} \in Z$, for all $u_1 \in B$,

this relates to

$[g(u_1), u_1 + v_1]_{\alpha,\beta} + [g(v_1), u_1 + v_1]_{\alpha,\beta} \in Z, \ \forall\ u_1 \in B.$

It implies

$\beta(u_1)[g(u_1), u_1]_{\alpha,\beta} + \beta(u_1)[g(u_1), v_1]_{\alpha,\beta} + [g(v_1), u_1]_{\alpha,\beta}\,\alpha(u_1) + [g(v_1), v_1]_{\alpha,\beta}\,\alpha(u_1) \in Z, \ \forall\ u_1 \in B.$

By the equation (11), we get

$$\beta(u_1)[g(u_1), v_1]_{\alpha,\beta} + [g(v_1), u_1]_{\alpha,\beta}\,\alpha(u_1) \in Z,\ for\, all\, u_1, v_1 \in B. \tag{12}$$

Replace $u_1$ by $v_1 w_1$ in equation (12), we obtain

$\beta(u_1)[g(v_1 w_1), v_1]_{\alpha,\beta} + [g(v_1), v_1 w_1]_{\alpha,\beta}\,\alpha(u_1) \in Z,$

we get

$\beta(u_1)[g(w_1)\alpha(v_1) + \beta(w_1)d(v_1), v_1]_{\alpha,\beta} + \beta(v_1)[G(v_1), w_1]_{\alpha,\beta}\,\alpha(u_1)$

$+ [g(v_1), v_1]\alpha(w_1)\alpha(u_1) \in Z$, this implies

$\beta(u_1)[g(w_1)\alpha(v_1), v_1]_{\alpha,\beta} + \beta(u_1)[\beta(w_1)d(v_1), v_1]_{\alpha,\beta} + \beta(v_1)[g(v_1), w_1]_{\alpha,\beta} \in Z.$

This equalized to

$\beta(u_1)[g(w_1), v_1]\alpha(v_1) + \beta(u_1)g(w_1)[\alpha(v_1), \alpha(v_1)]_{\alpha,\beta} + \beta(u_1)[\beta(w_1), \beta(v_1)]d(v_1)$
$+ \beta(u_1)\beta(w_1)[d(v_1), v_1]_{\alpha,\beta} + \beta(v_1)[g(v_1), w_1]_{\alpha,\beta}\alpha(u_1) \in Z.$

As we know for any $u_1, v_1, w_1 \in Z$ can commute with each one of $R$, by equation (12) and $[\alpha(w_1), \alpha(v_1)] = 0$, we get

$\beta(u_1)\beta(w_1)[d(v_1), v_1]_{\alpha,\beta} \in Z.$

Since $\beta(w_1) \neq 0$, this implies by Lemma 2.8, we get

$[d(v_1), v_1]_{\alpha,\beta} \in Z$, for every $v_1 \in B$.

Then we have $d$ is centralizing on $B$, hence by the reference [3], $R$ is commutative. This completes our proof.

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