Eur. J. Math. Anal. 4 (2024) 8
doi: 10.28924/ada/ma.4.8

Hybrid Inertial Iterative Method for Fixed point, Variational Inequality and Generalized Mixed Equilibrium Problems in Banach Space

Lawal Umar[1,*], Yusuf Ibrahim[2], M.S. Lawan[3]

[1] *Department of Mathematics, Federal College of Education, Zaria Kaduna, Nigeria*
lawalu4@gmail.com

[2] *Department of Mathematics, Saadatu Rimi University of Education, Kumbotso Kano, Nigeria*
danustazz@gmail.com

[3] *Department of Mathematics and Statistics, Kaduna Polytechnic Kaduna, Nigeria*
mslawankh@yahoo.com

[*] *Correspondence: lawalu4@gmail.com*

Abstract. In this paper, we introduced a hybrid inertial iterative method which converges strongly to a common element of solution of generalized mixed equilibrium, variational inequality and fixed point problems in a two uniformly smooth and uniformly convex Banach space. Our hybrid inertial iterative method, techniques of proof and corollaries improves, extends and generalizes many results in the literature.

1. INTRODUCTION

Let B denotes a real Banach space with B^* as the dual space of B. We consider $\langle \tau_1, j\rangle$ as the value of the functional $j \in B^*$ at $\tau_1 \in B$ and $\| \, . \, \|$ as the norm of B or B^*. Let $c \neq \emptyset$ be subset of B. A mapping $J : B \longrightarrow 2^{B^*}$ is called normalized duality provided that

$$J\tau_1 = \{\tau_2 \in B^* : \langle \tau_2, \tau_1\rangle = \|\tau_1\|^2 = \|\tau_2\|^2\}, \forall \tau_1 \in B.$$

We denotes the short form $GMEP$ as generalized mixed equilibrium problem: Find $v_1 \in C$ such that

$$D(v_1, v_2) + \langle Gv_1, v_2 - v_1\rangle + \vartheta(v_1, v_2) - \vartheta(v_1, v_1) \geq 0, \ \forall v_2 \in C, \tag{1.1}$$

where D, $\vartheta : C \times C \longrightarrow \mathbb{R}$ and $G : C \longrightarrow B^*$ denotes the bifunctions and a nonlinear mapping respectively, also $\mathbb{R}$ is consider as the set of all real numbers. Then, $Sol(GMEP(1.1))$ is consider as the solution set of $GMEP.(1.1)$.

Received: 16 Jan 2024.
Key words and phrases. Hybrid Inertial Iterative Method; Fixed point problem; Variational Inequality problem; Generalized Mixed Equilibrium Problem.

If $G \equiv 0$, $GMEP$(1.1) reduces to generalized equilibrium problem (with GEP as the short form): Find $v_1 \in C$ such that

$$D(v_1, v_2) + \vartheta(v_1, v_2) - \vartheta(v_1, v_1) \geq 0, \forall v_2 \in C. \tag{1.2}$$

Then, $Sol(GEP(1.2))$ is represent the solution set of $GEP(1.2)$.
If $G \equiv 0$ and $\vartheta \equiv 0$, $GMEP$(1.1) becomes equilibrium problem (with EP as the short form) [3]: Find $v_1 \in C$ such that

$$D(v_1, v_2) \geq 0, \forall v_2 \in C. \tag{1.3}$$

Then, $Sol(EP(1.3))$ is consider as the solution set of $EP.(1.3)$.
If $D \equiv 0$ and $\vartheta \equiv 0$, $GMEP$(1.1) reduces to variational inequality problem (with VIP as the short form): Find $v_1 \in C$ such that

$$\langle Gv_1, v_2 - v_1 \rangle \geq 0, \forall v_2 \in C. \tag{1.4}$$

Then, $Sol(VIP(1.4))$ is consider as the solution set of $VIP(1.4)$.

Definition 1.1. Let $T : C \longrightarrow C$ be a mapping [6], then
(i) a point $v_1 \in C$ is called fixed point of T provided that $F(T) = \{v_1 \in C : Tv_1 = v_1\} \neq \emptyset$;
(ii) a point $v_0 \in C$ is called an asymptotic fixed point of T provided that $\{v_n\} \subset C$, $v_n \rightharpoonup v_0$ such that

$$\lim_{n\to\infty} \| v_n - Tv_n \| = 0.$$

The set of asymptotic fixed point of T is denoted by $\hat{F}(T)$;
(iii) T is called quasi$-\phi-$nonexpansive provided that $\phi(v_0, Tv) \leq \phi(v_0, v)$ and $F(T) \neq \emptyset$, $\forall v \in C$, $v_0 \in F(T)$;
(iv) T is called quasi$-\phi-$asymptotically nonexpansive provided that $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \longrightarrow 1$ as $n \to \infty$ such that

$$\phi(v_0, T^n v) \leq k_n \phi(v_0, v), \ \forall v \in C, \ v_0 \in F(T), \ n \geq 1.$$

Definition 1.2. A function $T : C \longrightarrow B^*$ is said to be [6] :
(i) Monotone if $\langle \tau_1 - \tau_2, T\tau_1 - T\tau_2 \rangle \geq 0$, $\forall \tau_1, \tau_2 \in B$;
(ii) $\gamma-$inverse strongly monotone (with ism as short form) if $\exists \gamma > 0$ such that

$$\langle \tau_1 - \tau_2, T\tau_1 - T\tau_2 \rangle \geq \gamma \| T\tau_1 - T\tau_2 \|^2, \ \forall \tau_1, \tau_2 \in B;$$

(iii) Lipschitz continuous if $\exists L > 0$ such that $\| T\tau_1 - T\tau_2 \| \leq L \| \tau_1 - \tau_2 \|$, $\forall \tau_1, \tau_2 \in B$. If T is $\gamma - ism$, then it is Lipschitz continuous with $\frac{1}{\gamma}$ as a constant.

Definition 1.3. A mapping $\Pi_C : B \longrightarrow C$ is called generalized projection [6], provided that $\Pi_C \tau_1 = v_0$, for any $\tau_1 \in B$ and v_0 be the solution of $\phi(v_0, \tau_1) = \inf_{v \in C} \phi(v, \tau_1)$.

An inertial-type algorithm is a method for speeding the convergence of the sequence of an algorithm introduced by Polyak [16]. Numerous problems have been approximated by using inertial algorithms (for more details see, [4,5,12] and the references therein). Mainge [13] proposed and studied the development of an inertial- type algorithm method as follows:

$$\begin{cases} u_n = \omega_n + \theta_n(\omega_n - \omega_{n-1}), \\ \omega_{n+1} = (1-\delta_n)u_n + \delta_n T u_n. \end{cases}$$

Takahashi and Zembayashi [17] Proposed an iterative process which converges strongly to a common element of solution of equilibrium problem and fixed point problem of relatively nonexpansive mapping. Furthermore, the generalization of the proposed iterative process [17] have been carried out by many researchers (for more details see, [7,8,11,18,20] and the references therein). Kazmi and Ali [10] introduced an iterative algorithm for solving a common solution of EP.(1.3). and fixed point problemof quasi$-\phi-$ asymptotically nonexpansive mapping.

Alansari et al. [1] studied an inertial iterative method for finding a common solution of generalized equilibrium, variational inequality and fixed point problems using the sequences $\{x_n\}$ and $\{z_n\}$ generated by the iterative algorithm:

$$\begin{cases} x_0 = x_1, \; z_0 \in C, \; C_0 := C; \\ \mu_n = x_n + \alpha_n(x_n - x_{n-1}); \\ y_n = \Pi_C J^{-1}(J\mu_n - w_n G\mu_n); \\ u_n = J^{-1}(\delta_n J z_n + (1-\delta_n) J T y_n); \\ z_{n+1} = T_{r_n} u_n; \\ C_n = \{u \in C : \phi(u, z_{n+1}) \leq \delta_n \phi(u, z_n) + (1-\delta_n)\phi(u, \mu_n); \\ Q_n = \langle u \in C : x_n - u, Jx_n - Jx_0 \rangle \leq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{w_n\} \subset (0,\infty)$, $\{\delta_n\} \subset [0,1]$ and $\{r_n\} \subset [a,\infty)$, for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\varpi = \Pi_\Gamma x_0$.

Farid et al. [6] proposed the following inertial algorithm for approximating a common solution of generalized mixed equilibrium problem, variational inequality problem and fixed point problem for family of quasi$-\phi-$nonexpansive mappings:

$$\begin{cases} x_0, x_1 \in q, \ \ q_1 := q; \\ \omega_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = \Pi_q J^{-1}(J\omega_n - w_n Q\omega_n); \\ v_n = J^{-1}(\delta_{n,0} J\omega_n + \sum_{i=1}^{N} \delta_{n,i} JT_i\omega_n); \\ z_n = J^{-1}(\alpha_n Jy_n + (1-\alpha_n)Jv_n); \\ u_n = T_{r_n} z_n; \\ q_n = \{u \in q : \phi(u, u_n) \leq \phi(u, \omega_n); \\ Q_n = \langle u \in q : x_n - u, Jx_n - Jx_0\rangle \leq 0\}; \\ x_{n+1} = \Pi_{q_n \cap Q_n} x_0, \forall n \geq 1. \end{cases}$$

Consider $\{\delta_{n,i}\}$ and $\{\alpha_n\} \subset [0,1]$, $\{w_n\} \subset (0,\infty)$, $\{\theta_n\} \subset (0,1)$ and $\{r_n\} \subset [a,\infty)$, for some $a > 0$. It has been proved that $\{x_n\}$ is a strong convergent to $\hat{x} = \Pi_\Omega x_0$.

Motivated and inspired by the work of Kazmi and Ali [10], Alansari et al. [1] and Farid et al. [6]. We proposed a hybrid inertial iterative algorithm for approximating a common solution of $GMEP$.(1.1), VIP(1.4) and fixed point problem for a family of two quasi$-\phi-$asymptotically nonexpansive map-pings in two- uniformly convex and uniformly smooth Banach spaces. Our result extends and improves the results of Kazmi and Ali [10], Alansari et al. [1] and Farid et al. [6], many results in the literature.

2. Preliminaries

Let $W = \{\tau_1 \in B :\| \tau_1 \|= 1\}$ be the unit sphere of B. If for any $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that $\| \tau_1 - \tau_2 \| \geq \varepsilon \Longrightarrow \frac{\| \tau_1 + \tau_2 \|}{2} \leq 1 - \delta$, $\forall \tau_1, \tau_2 \in W$, then B is called uniformly convex. B is called strictly convex if $\frac{\| \tau_1 + \tau_2 \|}{2} < 1$, $\forall \tau_1, \tau_2 \in W$ and $\tau_1 \neq \tau_2$. The space B is called smooth if $\lim_{t \to 0} \frac{\| \tau_1 + t\tau_2 \| - \| \tau_1 \|}{t}$ exists, $\forall \tau_1, \tau_2 \in W$ and also is said to be uniformly smooth if the limit is attained uniformly, $\forall \tau_1, \tau_2 \in W$.
A function $\phi : B \times B \longrightarrow \mathbb{R}$ defined by

$$\phi(\tau_1, \tau_2) =\| \tau_1 \|^2 - 2\langle \tau_1, J\tau_2\rangle + \|\tau_2\|^2, \ \forall \tau_1, \tau_2 \in B.$$

is consider as Lyapunov functional. From the definition of ϕ, the following properties can be verified [6]:
(L_1) $(\| \tau_1 \| - \| \tau_2 \|)^2 \leq \phi(\tau_1, \tau_2) \leq (\| \tau_1 \| + \| \tau_2 \|)^2$, $\forall \tau_1, \tau_2 \in B$;
(L_2) $\phi(\tau_1, J^{-1}(\lambda J\tau_2 + (1-\lambda)J\tau_3)) \leq \lambda\phi(\tau_1, \tau_2) + (1-\lambda)\phi(\tau_1, \tau_3)$, $\forall \tau_1, \tau_2, \tau_3 \in B$,
(L_3) $\phi(\tau_1, \tau_2) =\| \tau_1 \| \| J\tau_1 - J\tau_2 \| + \| \tau_2 \| \| \tau_1 - \tau_2 \|$, $\forall \tau_1, \tau_2 \in B$.

Remark 2.1. Consider B as smooth, strictly convex and reflexive Banach space, then

$$\phi(\tau_1, \tau_2) = 0 \iff \tau_1 = \tau_2, \ \forall \tau_1, \tau_2 \in B.$$

Lemma 2.2. *[9] Let $C \neq \emptyset$ be closed convex subset of a stricly convex, reflexive and smooth Banach space B. Then, $\exists$ a unique element $\tau_0 \in C$ such that $\phi(\tau_0, \tau_1) = \inf_{v \in C} \phi(v, \tau_1)$, for $\tau_1 \in B$.*

Lemma 2.3. *[15] Let B be a uniformly convex and smooth Banach space, $C \subset B$ be closed convex and $T : C \longrightarrow C$ be closed and quasi$-\phi-$asymptotically nonexpansive mapping. Then, $F(T)$ is closed and convex.*

Lemma 2.4. *[14] Let $C \neq \emptyset$ be closed convex subset of B and $Q : C \longrightarrow B^*$ be monotone and hemicontinuous function. Then VIP(1.4). is closed and convex*

Lemma 2.5. *[19] Let B be a $2-$uniformly convex and smooth Banach space. Then, $\tau_1, \tau_2 \in B$, $\phi(\tau_1, \tau_2) \geq \delta \parallel \tau_1 - \tau_2 \parallel^2$, where $0 < \delta \leq 1$ and called two-uniformly convex constant.*

Lemma 2.6. *[19] Let B be a two-uniformly convex Banach space, then*

$$\parallel \tau_1 - \tau_2 \parallel \leq \frac{2}{\delta} \parallel J\tau_1 - J\tau_2 \parallel, \ \forall \tau_1, \tau_2 \in B,$$

where $0 < \delta \leq 1$.

Lemma 2.7. *[9] Let E be a smooth and uniformly convex Banach space and let $\{u_n\}$ and $\{v_n\}$ be sequences in E such that either $\{u_n\}$ or $\{v_n\}$ is bounded. If $\lim_{n\to\infty} \phi(u_n, v_n) = 0$, then $\lim_{n\to\infty} \parallel u_n - v_n \parallel = 0$.*

Remark 2.8. By considering (L_3), it is observe that the converse of Lemma 2.7 is true, provided that $\{u_n\}$ and $\{v_n\}$ are bounded

Lemma 2.9. *[2] Let $C \neq \emptyset$ be closed convex subset of a stricly convex, reflexive and smooth Banach space B. Then,*

$$\phi(v, \Pi_C \tau_1) + \phi(\Pi_C \tau_1, \tau_1) \leq (v, \tau_1), \ \forall v \in C, \ \tau_1 \in B.$$

And, so for any $\tau_1 \in B$ and $v \in C$,

$$u = \Pi_C \tau_1 \iff \langle v - u, J\tau_1 - Jv \rangle, \ \forall u \in C.$$

Assumption 1: Consider $D : C \times C \longrightarrow \mathbb{R}$ as a bifunction satisfies the following assumptions [3]:
(D_1) $D(v, v) = 0, \forall v \in C$;
(D_2) D is monotone, 1.e, $D(v, u) + D(u, v) \leq 0, \ \forall v, u \in C$;
(D_3) the mapping $v \mapsto D(v, u)$ is upper hemicontinuity, $\forall \ u \in C$.
(D_4) the mapping $u \mapsto D(v, u)$, $u \in C$ is convex and lower semicontinuous.
Assumption 2: Also consider $\vartheta : C \times C \longrightarrow \mathbb{R}$ as a bifunction satisfying the following assumptions:

(ϑ_1) ϑ is skew-symmetric, i.e., $\vartheta(v,v)-\vartheta(v,u)-\vartheta(u,v)+\vartheta(u,u)\geq 0, \forall v,u\in C$;

(ϑ_2) ϑ is convex in the second argument;

(ϑ_3) ϑ is continuous.

Lemma 2.10. *[1, 6, 21] Let B a uniformly smooth, strictly convex and reflexive Banach space and $C\subset B$ be closed. Let $G: C\longrightarrow B^*$ be a continuous and monotone mapping, $D: C\times C\longrightarrow \mathbb{R}$ be a bifunction satisfying Assumptions 1 and $\vartheta: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying Assumptions 2. For any given number $r>0$ and $\tau_1\in B$, define a mapping $T_r: B\longrightarrow C$ by*

$$T_r(\tau_1)=\{u\in C: D(u,v)+\langle v-u, Gu\rangle+\frac{1}{r}\langle v-u, Ju-J\tau_1\rangle+\psi(u,v)-\psi(u,u)\geq 0, \forall y\in C\},$$

$\forall v\in B$.

The mapping T_r has the following properties:

(p_1) *T_r is single-valued;*

(p_2) *T_r is a firmly nonexpansive - type mapping, for all $\tau_1,\tau_2\in B$,*

$$\langle T_r\tau_1-T_r\tau_2, JT_r\tau_1-JT_r\tau_2\rangle\leq\langle T_r\tau_1-T_r\tau_2, J\tau_1-J\tau_2\rangle,$$

(p_3) *$F(T_r)=Sol(GMEP(1.1))$ is closed convex set of C;*

(p_4) *T_r is quasi$-\phi-$ nonexpansive;*

(p_5) *$\phi(v_0, T_r\tau_1)+\phi(T_r\tau_1,\tau_1)\leq\phi(v_0,\tau_1),\ \forall v_0\in F(T_r),\ \tau_1\in B$.*

Furthermore, consider the map $\Phi: B\times B^*\longrightarrow\mathbb{R}$, defined by

$$\Phi(\tau_1,\tau_1^*)=\parallel\tau_1\parallel^2-\langle\tau_1,\tau_1^*\rangle+\parallel\tau_1^*\parallel^2$$

Observe that $\Phi(\tau_1,\tau_1^*)=\Phi(\tau_1, J^{-1}\tau_1^*)$

Lemma 2.11. *[2] Let B be a strictly convex, smooth and reflexive Banach space. Then*

$$\Phi(\tau_1,\tau_1^*)+2\langle J^{-1}\tau_1^*-\tau_1,\tau_2^*\rangle\leq\Phi(\tau_1,\tau_1^*+\tau_2^*),\ \forall\tau_1\in B,\ \tau_1^*,\tau_2^*\in B^*.$$

3. Main Results

Theorem 3.1. *Let C be a nonempty closed and convex subset of a $2-$uniformly smooth and uniformly convex Banach space B with B^* as the dual space of B. Let $Q:\longrightarrow B^*$ be a $\gamma-$ism mapping with $\gamma\in(0,1)$ as a constant. Let $D: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying Assumption 1, $\vartheta: C\times C\longrightarrow\mathbb{R}$ be a bifunction satisfying Assumption 2 and $G: C\longrightarrow B^*$ be a monotone and continuous mapping. Let $T_i: C\longrightarrow C$ and $S_i: C\longrightarrow C$, for each $i=1,2,...,N$ be two finite family of closed l_i-Lipschitz continuous and uniformly quasi$-\phi-$asymptotically nonexpansive mappings such that $\Omega:=\big(\cap_{i=1}^N F(T_i)\big)\cap\big(\cap_{i=1}^N F(S_i)\big)\cap Sol\big(VIP(1.4)\big)\cap Sol\big(GMEP(1.1)\big)\neq\emptyset$. Let $\{x_n\}$*

generated by algorithm :

$$\begin{cases} x_0, x_1 \in C, \ C_1 := C, \\ \omega_n = x_n + \alpha_n(x_n - x_{n-1}), \\ v_n = \Pi_C J^{-1}(J\omega_n - \beta_n Q\omega_n), \\ y_n = J^{-1}(\mu_{n,0}J\omega_n + \sum_{i=1}^{N} \mu_{n,i}JT_i^n\omega_n); \\ z_n = J^{-1}(\eta_{n,0}Jv_n + \sum_{i=1}^{N} \eta_{n,i}JS_i^n y_n), \\ u_n = T_{r_n}z_n, \\ C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq k_n^2\phi(u, \omega_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \ \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\mu_{n,i}\} \subset [0,1]$ *and* $\{\eta_{n,i}\} \subset (0,1]$ *satisfying the following conditions:*
(S_1) $\sum_{i=0}^{N} \mu_{n,i} = 1$;
(S_2) $\sum_{i=0}^{N} \eta_{n,i} = 1$;
(S_3) $\limsup_{n\to\infty} \eta_{n,0} < 1$;
(S_4) *for same* $a > 0, \ r_n \in [a, \infty)$;
(S_5) $\{\beta_n\} \subset (0,\infty)$ *satisfying the condition* $0 < \liminf_{n\to\infty} \beta_n < \frac{\delta^2\gamma}{2}$, *where* $0 < \delta \leq 1$.
Then, $\{x_n\}$ *converges strongly to* ϖ, *where* $\varpi = \Pi_\Omega x_0$ *is consider as the generalized projection of* ϖ *onto* Ω.

Proof. We consider the proof in the following steps:

Step 1 : We show that C_{n+1} is closed and convex for each $n \geq 1$ and $\{x_n\}$ is well defined. Observe clearly that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Now, we know from 3.1 that for any $u \in C_n$,

$$\begin{aligned} \phi(u, u_n) \leq k_n^2\phi(u, \omega_n) \iff & (1 - k_n^2)\big[\parallel u \parallel^2 - 2(1 - k_n^2)\langle u, Ju_n\rangle + 2k_n^2\langle u, J\omega_n - Ju_n\rangle\big] \\ \leq \quad & k_n^2 \parallel \omega_n \parallel^2 - \parallel u_n \parallel^2 . \end{aligned}$$

Then, C_{n+1} is closed and convex. Implies that $\Pi_{C_{n+1}}x_0$ is well defined $\forall n \geq 1$, also $\{x_n\}$ is well defined. Furthermore since $\Omega \neq \emptyset$, by considering Lemma 2.3, 2.4 and 2.10 we conclude that Ω is closed and convex, and so $\Pi_\Omega x_0$ is well defined.
Step 2 : we show that $\Omega \subset C_n, \ \forall n \geq 1$. It is Obvious that $\Omega \subset C_1 = C$. Suppose that $\Omega \subset C_n$ for some $n \geq 1$. Let $\hat{x} \in \Omega$, from the definition of ϕ, quasi$-\phi-$asymptotically nonexpansive mapping

of S_i and convexity of $\| \,.\, \|^2$ we have the following estimate:

$$\begin{aligned}
\phi(\hat{x}, u_n) &= \phi(\hat{x}, T_{r_n} z_n) \\
&\leq \phi(\hat{x}, z_n) \qquad (3.2) \\
&= \phi\big(\hat{x}, J^{-1}(\eta_{n,0} J v_n + \sum_{i=1}^{N} \eta_{n,i} J S_i^n y_n)\big) \\
&= \| \hat{x} \|^2 - 2(\langle \hat{x}, \eta_{n,0} J v_n + \sum_{i=1}^{N} \eta_{n,i} J S_i^n y_n \rangle) + \|\eta_{n,0} J v_n + \sum_{i=1}^{N} \eta_{n,i} J S_i^n y_n\|^2 \\
&\leq \| \hat{x} \|^2 - 2\eta_{n,0} \langle \hat{x}, J v_n \rangle - 2 \sum_{i=1}^{N} \eta_{n,i} \langle \hat{x}, J S_i^n y_n \rangle + \eta_{n,0} \|J v_n\|^2 \\
&+ \sum_{i=1}^{N} \eta_{n,i} \|J S_i^n y_n\|^2 \\
&= \eta_{n,0}\big(\|\hat{x}\|^2 - 2\langle \hat{x}, J v_n \rangle + \|v_n\|^2\big) + \sum_{i=1}^{N} \eta_{n,i}\big(\|\hat{x}\|^2 - 2\langle \hat{x}, J S_i^n y_n \rangle + \|S_i^n y_n\|^2\big) \\
&= \eta_{n,0} \phi(\hat{x}, v_n) + \sum_{i=1}^{N} \eta_{n,i} \phi(\hat{x}, S_i^n y_n) \\
&\leq \eta_{n,0} \phi(\hat{x}, v_n) + k_n \sum_{i=1}^{N} \eta_{n,i} \phi(\hat{x}, y_n) \qquad (3.3)
\end{aligned}$$

Similarly, by quasi$-\phi-$asymptotically nonexpansive of T_i, definition of ϕ and convexity of $\| \,.\, \|^2$, we estimate as follows:

$$\begin{aligned}
\phi(\hat{x}, y_n) &= \phi\big(\hat{x}, J^{-1}(\mu_{n,0} J \omega_n + \sum_{i=1}^{N} \mu_{n,i} J T_i^n \omega_n)\big) \\
&= \| \hat{x} \|^2 - 2(\langle \hat{x}, \mu_{n,0} J \omega_n + \sum_{i=1}^{N} \mu_{n,i} J T_i^n \omega_n \rangle) + \| \mu_{n,0} J \omega_n + \sum_{i=1}^{N} \mu_{n,i} J T_i^n \omega_n \|^2 \\
&\leq \| \hat{x} \|^2 - 2\mu_{n,0} \langle \hat{x}, J \omega_n \rangle - 2 \sum_{i=1}^{N} \mu_{n,i} \langle \hat{x}, J T_i^n \omega_n \rangle + \mu_{n,0} \|J \omega_n\|^2 \\
&+ \sum_{i=1}^{N} \mu_{n,i} \|J T_i^n \omega_n\|^2 \\
&= \mu_{n,0}\big(\|\hat{x}\|^2 - 2\langle \hat{x}, J \omega_n \rangle + \|\omega_n\|^2\big) + \sum_{i=1}^{N} \mu_{n,i}\big(\| \hat{x} \|^2 - 2\langle \hat{x}, J T_i^n \omega_n \rangle + \|T_i^n \omega_n\|^2\big) \\
&= \mu_{n,0} \phi(\hat{x}, \omega_n) + \sum_{i=1}^{N} \mu_{n,i} \phi(\hat{x}, T_i^n \omega_n) \\
&\leq \mu_{n,0} \phi(\hat{x}, \omega_n) + k_n \sum_{i=1}^{N} \mu_{n,i} \phi(\hat{x}, \omega_n) \\
&\leq k_n \mu_{n,0} \phi(\hat{x}, \omega_n) + k_n \sum_{i=1}^{N} \mu_{n,i} \phi(\hat{x}, \omega_n) \\
&= k_n \phi(\hat{x}, \omega_n) \qquad (3.4)
\end{aligned}$$

It has been observe from (3.3) and (3.4) that

$$\begin{aligned}\phi(\hat{x}, u_n) &\leq \eta_{n,0}\phi(\hat{x}, v_n) + k_n \sum_{i=1}^{N} \eta_{n,i}\big[k_n\phi(\hat{x}, \omega_n)\big] \\ &= \eta_{n,0}\phi(\hat{x}, v_n) + k_n^2 \sum_{i=1}^{N} \eta_{n,i}\phi(\hat{x}, \omega_n) \\ &\leq k_n^2\eta_{n,0}\phi(\hat{x}, v_n) + k_n^2 \sum_{i=1}^{N} \eta_{n,i}\phi(\hat{x}, \omega_n) \qquad (3.5)\end{aligned}$$

Also, by Lemma 2.6 and 2.11, we estimate as:

$$\begin{aligned}\phi(\hat{x}, v_n) &= \phi\big(\hat{x}, \Pi_C J^{-1}(J\omega_n - \beta_n Q\omega_n)\big) \\ &\leq \phi\big(\hat{x}, J^{-1}(J\omega_n - \beta_n Q\omega_n)\big) \\ &= \Phi\big(\hat{x}, J\omega_n - \beta_n Q\omega_n\big) \\ &\leq \Phi\big(\hat{x}, (J\omega_n - \beta_n Q\omega_n) + \beta_n Q\omega_n\big) - 2\langle J^{-1}(J\omega_n - \beta_n Q\omega_n) - \hat{x}, \beta_n Q\omega_n\rangle \\ &= \Phi(\hat{x}, J\omega_n) - 2\beta_n\langle J^{-1}(J\omega_n - \beta_n Q\omega_n) - \hat{x}, Q\omega_n\rangle \\ &= \phi(\hat{x}, \omega_n) - 2\langle \omega_n - \hat{x}, Q\omega_n\rangle - 2\beta_n\langle J^{-1}(J\omega_n - \beta_n Q\omega_n) - \omega_n, Q\omega_n\rangle \\ &= \phi(\hat{x}, \omega_n) - 2\langle \omega_n - \hat{x}, Q\omega_n - Q\hat{x}\rangle - 2\beta_n\langle J^{-1}(J\omega_n - \beta_n Q\omega_n) - \omega_n, Q\omega_n\rangle \\ &\leq \phi(\hat{x}, \omega_n) - 2\beta_n\gamma \parallel Q\omega_n\|^2 + 2\beta_n \parallel J^{-1}(J\omega_n - Q\omega_n) - J^{-1}J\omega_n\| \|Q\omega_n\|^2 \\ &\leq \phi(\hat{x}, \omega_n) - 2\beta_n\gamma \parallel Q\omega_n \parallel^2 + \frac{4\beta_n^2}{\delta^2} \parallel Q\omega_n \parallel^2 \\ &= \phi(\hat{x}, \omega_n) - 2\beta_n\big(\gamma - \frac{2\beta_n}{\delta^2}\big) \parallel Q\omega_n \parallel^2, \qquad (3.6)\end{aligned}$$

if follows by combined with $\beta_n < \dfrac{\delta^2}{2}$ that

$$\phi(\hat{x}, v_n) \leq \phi(\hat{x}, \omega_n) \qquad (3.7)$$

Now, putting (3.7) in (3.5) leads to

$$\begin{aligned}\phi(\hat{x}, u_n) &\leq k_n^2\eta_{n,0}\phi(\hat{x}, \omega_n) + k_n^2 \sum_{i=1}^{N} \eta_{n,i}\phi(\hat{x}, \omega_n) \\ &= (\eta_{n,0} + \sum_{i=1}^{N} \eta_{n,i})k_n^2\phi(\hat{x}, \omega_n) \\ &= k_n^2\phi(\hat{x}, \omega_n),\end{aligned}$$

which gives

$$\phi(\hat{x}, u_n) \leq k_n^2\phi(\hat{x}, \omega_n), \qquad (3.8)$$

Therefore $\hat{x} \in C_{n+1}$, implies that $\Omega \subset C_{n+1}$. Hence $\Omega \subset C_n$, $\forall n \geq 1$.

Step 3 : we show that $\{x_n\}$, $\{\omega_n\}$, $\{v_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded and $\{x_n\}$ is Cauchy. We consider $x_n = \Pi_{C_n} x_0$ and $C_{n+1} \subset C_n$, $\forall n \geq 1$. Then from Lemma 2.9, we observe that

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$$

Hence $\{\phi(x_n, x_0)\}$ is non decreasing. Also it has been observe that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(\hat{x}, x_0) - \phi(\hat{x}, x_n) \leq \phi(\hat{x}, x_0),$$

which gives that $\{\phi(x_n, x_0)\}$ is bounded and $\{x_n\}$ is also bounded. Therefore, since $\{\phi(x_n, x_0)\}$ non decreasing. $\{\phi(x_n, x_0)\}$ convergent. Taking the advantage of $\{x_n\}$ as a bounded sequence implies that $\{\omega_n\}$, $\{v_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are all bounded. Also by Lemma 2.9, we have

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(x_n, x_0) \longrightarrow 0 \; as \; n, m \to \infty. \end{aligned} \tag{3.9}$$

By Lemma 2.7, we have $\lim_{n\to\infty} \| x_m - x_n \| = 0$. Hence $\{x_n\}$ is a Cauchy sequence.

Step 4 : we show that $x_n \longrightarrow \varpi$, $\omega_n \longrightarrow \varpi$, $u_n \longrightarrow \varpi$, $z_n \longrightarrow \varpi$, $y_n \longrightarrow \varpi$ and $v_n \longrightarrow \varpi$ $(as\ n \to \infty)$. Since $\{x_n\}$ is a Cauchy sequence, then by the closedness of C and the completeness of B, we can assume that there exists $\varpi \in C$ such that

$$\lim_{n\to\infty} x_n = \varpi. \tag{3.10}$$

Now, setting $m = n + 1$ in (3.9), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.11}$$

Using Lemma 2.7, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

We observe from (3.1) that

$$\| \omega_n - x_n \| = \| \alpha_n (x_n - x_{n-1}) \| \leq \| x_n - x_{n-1} \|$$

Using (3.12), we arrive at

$$\lim_{n\to\infty} \|\omega_n - x_n\| = 0. \tag{3.13}$$

By (3.10) and (3.13), we conclude that

$$\lim_{n\to\infty} \omega_n = \varpi. \tag{3.14}$$

Taking the advantage of Remark 2.8, (3.13) and boundedness of $\{\omega_n\}$, we get

$$\lim_{n\to\infty} \phi(\omega_n, x_n) = 0. \tag{3.15}$$

Also, by (3.12) and (3.13), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - \omega_n \|= 0. \tag{3.16}$$

Using Remark 2.8, we present (3.16) as

$$\lim_{n\to\infty} \phi(x_{n+1}, \omega_n) = 0. \tag{3.17}$$

We observe from $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and definition of C_n that

$$\phi(x_{n+1}, u_n) \leq k_n^2 \phi(x_{n+1}, \omega_n)$$

Using (3.17,) we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Applying Lemma 2.7, we get

$$\lim_{n\to\infty} \| x_{n+1} - u_n \|= 0. \tag{3.18}$$

Taking the advantage of triangular inequality, we present

$$\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|$$

By (3.12) and (3.18), we obtain

$$\lim_{n\to\infty} \| x_n - u_n \|= 0. \tag{3.19}$$

It follows from (3.10) and (3.19) that

$$\lim_{n\to\infty} u_n = \varpi. \tag{3.20}$$

Similarly, by definition of C_n and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we also present that

$$\phi(x_{n+1}, z_n) \leq k_n^2 \phi(x_{n+1}, \omega_n)$$

By applying (3.17,) we arrive at

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0.$$

Using Lemma 2.7, we have

$$\lim_{n\to\infty} \| x_{n+1} - z_n \|= 0. \tag{3.21}$$

Taking into account that

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|$$

Using (3.12) and (3.21), we get

$$\lim_{n\to\infty} \| x_n - z_n \|= 0. \tag{3.22}$$

By considering(3.10) and (3.22), we obtain

$$\lim_{n\to\infty} z_n = \varpi. \tag{3.23}$$

Also from the definition of C_n and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we estimate as

$$\phi(x_{n+1}, y_n) \leq k_n^2 \phi(x_{n+1}, \omega_n)$$

By (3.17,) we get

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

It follows from Lemma 2.7 that

$$\lim_{n\to\infty} \| x_{n+1} - y_n \| = 0. \tag{3.24}$$

By triangular inequality, we obtain

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$$

Also by (3.12) and (3.24), we get

$$\lim_{n\to\infty} \| x_n - y_n \| = 0. \tag{3.25}$$

Using (3.10) and (3.25), we obtain

$$\lim_{n\to\infty} y_n = \varpi. \tag{3.26}$$

Finally, by considering $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and definition of C_n, we present that

$$\phi(x_{n+1}, v_n) \leq k_n^2 \phi(x_{n+1}, \omega_n)$$

Applying (3.17,) we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, v_n) = 0.$$

By Lemma 2.7, we get

$$\lim_{n\to\infty} \| x_{n+1} - v_n \| = 0. \tag{3.27}$$

We consider the following estimate using triangular inequality

$$\|x_n - v_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - v_n\|$$

Using (3.12) and (3.27), we obtain

$$\lim_{n\to\infty} \| x_n - v_n \| = 0. \tag{3.28}$$

Using (3.10) and (3.28), we obtain

$$\lim_{n\to\infty} v_n = \varpi. \tag{3.29}$$

Step 4 : we show that $\| \omega_n - T_i^n\omega_n \| = \| y_n - S_i^n y_n \| = 0$. Now, taking the advantage of J as uniformly continuity on bounded sets, then it follows from (3.16) and (3.24) that

$$\| J\omega_n - Jx_{n+1} \| = \| Jx_{n+1} - Jy_n \| = 0. \tag{3.30}$$

From (3.1), we observe that

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \| Jx_{n+1} - \big(\mu_{n,0}J\omega_n + \sum_{i=1}^{N}\mu_{n,i}JT_i^n\omega_n\big) \| \\ &= \| \sum_{i=1}^{N}\mu_{n,i}Jx_{n+1} - \sum_{i=1}^{N}\mu_{n,i}JT_i^n\omega_n + \mu_{n,0}Jx_{n+1} - \mu_{n,0}J\omega_n \| \\ &= \| \sum_{i=1}^{N}\mu_{n,i}\big(Jx_{n+1} - JT_i^n\omega_n\big) + \mu_{n,0}\big(Jx_{n+1} - J\omega_n\big) \| \\ &\geq \sum_{i=1}^{N}\mu_{n,i} \| Jx_{n+1} - JT_i^n\omega_n \| - \mu_{n,0} \| J\omega_n - Jx_{n+1} \|, \end{aligned}$$

this gives

$$\| Jx_{n+1} - JT_i^n\omega_n \| \leq \frac{1}{\sum_{i=1}^{N}\mu_{n,i}}\Big[\| Jx_{n+1} - Jy_n \| + \mu_{n,0} \| J\omega_n - Jx_{n+1} \| \Big].$$

By (3.30), we arrive at

$$\lim_{n\to\infty} \| Jx_{n+1} - JT_i^n\omega_n \| = 0.$$

As J^{-1} is uniform norm-to-norm continuous on bounded sets, we present that

$$\lim_{n\to\infty} \| x_{n+1} - T_i^n\omega_n \| = 0. \tag{3.31}$$

Taking into account that

$$\| \omega_n - T_i^n\omega_n \| \leq \| \omega_n - x_{n+1} \| + \| x_{n+1} - T_i^n\omega_n \|$$

By (3.16) and (3.31), we obtain

$$\lim_{n\to\infty} \| \omega_n - T_i^n\omega_n \| = 0. \tag{3.32}$$

Similarly, we observe from (3.21), (3.27) and by continuity of J that

$$\| Jx_{n+1} - Jz_n \| = \| Jx_{n+1} - Jv_n \| = 0. \tag{3.33}$$

Also by (3.1), we observe that

$$\begin{aligned}\| Jx_{n+1} - Jz_n \| &= \| Jx_{n+1} - \big(\eta_{n,0}Jv_n + \sum_{i=1}^{N} \eta_{n,i}JS_i^n y_n\big) \| \\ &= \| \sum_{i=1}^{N} \eta_{n,i}Jx_{n+1} - \sum_{i=1}^{N} \eta_{n,i}JS_i^n y_n + \eta_{n,0}Jx_{n+1} - \eta_{n,0}Jv_n \| \\ &= \| \sum_{i=1}^{N} \eta_{n,i}\big(Jx_{n+1} - JS_i^n y_n\big) + \eta_{n,0}\big(Jx_{n+1} - Jv_n\big) \| \\ &\geq \sum_{i=1}^{N} \eta_{n,i} \| Jx_{n+1} - JS_i^n y_n \| - \eta_{n,0} \| Jv_n - Jx_{n+1} \|,\end{aligned}$$

this implies

$$\|Jx_{n+1} - JS_i^n y_n\| \leq \frac{1}{\sum_{i=1}^{N} \eta_{n,i}} \big[\|Jx_{n+1} - Jz_n\| + \eta_{n,0}\|Jv_n - Jx_{n+1} \| \big].$$

Also by (3.33), we get

$$\lim_{n\to\infty} \| Jx_{n+1} - JS_i^n y_n \| = 0.$$

Applying J^{-1} as uniform norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \| x_{n+1} - S_i^n y_n \| = 0. \tag{3.34}$$

By triangular inequality, we obtain

$$\| y_n - S_i^n y_n \| \leq \| y_n - x_{n+1} \| + \| x_{n+1} - S_i^n y_n \|$$

By (3.24) and (3.34), we get

$$\lim_{n\to\infty} \| y_n - S_i^n y_n \| = 0. \tag{3.35}$$

Therefore by (3.32) and (3.35), we conclude that

$$\lim_{n\to\infty} \| \omega_n - T_i^n \omega_n \| = \lim_{n\to\infty} \| y_n - S_i^n y_n \| = 0.$$

$Step$ 5 : we show that $\varpi \in \Omega$. To show this we claim as follows:

We claim that $\varpi \in \big(\cap_{i=1}^N F(T_i)\big) \cap \big(\cap_{i=1}^N F(S_i)\big)$. By triangular inequality for $i \geq 1$, we have

$$\| T_i^n \omega_n - \varpi \| \leq \| T_i^n \omega_n - \omega_n \| + \| \omega_n - \varpi \| .$$

Using (3.14) and (3.32), we arrive at

$$\lim_{n\to\infty} \| T_i^n \omega_n - \varpi \| = 0. \tag{3.36}$$

By the assumption that for each T_i is uniformly L_i-Lipschitz continuous, we obtain

$$\begin{aligned}\|T_i^{n+1}\omega_n - T_i^n\omega_n\| &\leq \|T_i^{n+1}\omega_n - T_i^{n+1}\omega_{n+1}\| + \|T_i^{n+1}\omega_{n+1} - \omega_{n+1}\| \\ &+ \|\omega_{n+1} - \omega_n\| + \|\omega_n - T_i^n\omega_n\| \\ &\leq (L_i+1)\|\omega_{n+1} - \omega_n\| + \|T_i^{n+1}\omega_{n+1} - \omega_{n+1}\| + \|\omega_n - T_i^n\omega_n\|.\end{aligned}$$

By (3.12) and (3.32,) we get

$$\lim_{n\to\infty} \|T_i^{n+1}\omega_n - T_i^n\omega_n\| = 0.$$

Which yields from (3.36) that

$$\lim_{n\to\infty} \|T_i^{n+1}\omega_n - \varpi\| = 0, \ \forall i \geq 1.$$

Consequently, we get $T_i(T_i^n)\omega_n \longrightarrow \varpi$ ($as\ n \to \infty$). In view of the closedness of T_i, we arrive at $T_i\varpi = \varpi, \ \forall i \geq 1$. Thus $\varpi \in \cap_{i=1}^N F(T_i)$. Furthermore, following similar argument as above, one can also claim that $\varpi \in \cap_{i=1}^N F(S_i)$. Hence

$$\varpi \in \big(\cap_{i=1}^n F(T_i)\big) \cap \big(\cap_{i=1}^n F(S_i)\big).$$

Next, we claim that $\varpi \in Sol(VIP(1.4))$. Consider the triangular inequality

$$\| \omega_n - z_n \| \leq \| \omega_n - x_n \| + \| x_n - z_n \| .$$

Using (3.13) and (3.22,) leads to

$$\lim_{n\to\infty} \| \omega_n - z_n \| = 0. \tag{3.37}$$

From the uniform continuity of J on bounded set, we get

$$\lim_{n\to\infty} \| J\omega_n - Jz_n \| = 0. \tag{3.38}$$

Since $\hat{x} \in \Omega$, then it follows from (3.2), (3.3), (3.4) and (3.6) that

$$\begin{aligned}\phi(\hat{x}, z_n) &\leq \eta_{n,0}\big[\phi(\hat{x}, \omega_n) - 2\beta_n\big(\gamma - \frac{2\beta_n}{\delta^2}\big)\|Q\omega_n\|^2\big] + k_n\sum_{i=1}^N \eta_{n,i}\big[k_n\phi(\hat{x}, \omega_n)\big] \\ &\leq k_n^2\eta_{n,0}\phi(\hat{x}, \omega_n) + k_n^2\sum_{i=1}^N \eta_{n,i}\phi(\hat{x}, \omega_n) - 2\beta_n\eta_{n,0}\big(\gamma - \frac{2\beta_n}{\delta^2}\big) \| Q\omega_n \|^2 \\ &= k_n^2\phi(\hat{x}, \omega_n) - 2\beta_n\eta_{n,0}\big(\gamma - \frac{2\beta_n}{\delta^2}\big) \| Q\omega_n \|^2,\end{aligned}$$

implies that

$$2\beta_n\eta_{n,0}\big(\gamma - \frac{2\beta_n}{\delta^2}\big) \| Q\omega_n \|^2 \leq k_n^2\phi(\hat{x}, \omega_n) - \phi(\hat{x}, z_n) \tag{3.39}$$

But

$$\begin{aligned} k_n^2\phi(\hat{x},\omega_n)-\phi(\hat{x},z_n) &= k_n^2[\parallel \hat{x}\parallel^2 -2\langle \hat{x}, J\omega_n\rangle+\parallel \omega_n\parallel^2]-[\parallel \hat{x}\parallel^2 -2\langle \hat{x}, Jz_n\rangle+\parallel z_n\parallel^2] \\ &= (k_n^2-1)\|\hat{x}\|^2-2(k_n^2-1)\langle \hat{x}, Jz_n\rangle-2k_n^2\langle \hat{x}, J\omega_n-J\omega_n\rangle \\ &+ k_n^2\|\omega_n\parallel^2-\parallel z_n\parallel^2 \\ &= (k_n^2-1)\parallel \hat{x}\parallel^2 -2(k_n^2-1)\langle \hat{x}, Jz_n\rangle-2k_n^2\langle \hat{x}, J\omega_n-Jz_n\rangle \\ &+ (k_n^2-1)\|\omega_n\parallel^2+\parallel \omega_n\parallel^2-\parallel z_n\parallel^2 \\ &\leq |(k_n^2-1)\parallel \hat{x}\parallel^2|+|2(k_n^2-1)\langle \hat{x}, Jz_n\rangle|+|2k_n^2\langle \hat{x}, J\omega_n-Jz_n\rangle| \\ &+ |(k_n^2-1)\|\omega_n\parallel^2|+|\|\omega_n\parallel^2+\parallel z_n\parallel^2| \\ &\leq (k_n^2-1)\parallel \hat{x}\parallel^2 +2(k_n^2-1)\parallel \hat{x}\parallel\parallel Jz_n\parallel +2k_n^2\parallel \hat{x}\parallel\parallel J\omega_n-Jz_n\parallel \\ &+ (\parallel \omega_n-z_n\parallel)(\parallel \omega_n\parallel+\parallel z_n\parallel). \end{aligned}$$

Since $k_n \longrightarrow 1$ *as* $n \longrightarrow \infty$, then by (3.37) and (3.38,) we obtain

$$\lim_{n\to\infty}\big(k_n^2\phi(\hat{x},\omega_n)-\phi(\hat{x},z_n)\big)=0. \tag{3.40}$$

Also since $\beta_n\eta_{n,0}\big(\gamma-\frac{2\beta_n}{\delta^2}\big)>0$, by (3.39) and (3.40), we have

$$\lim_{n\to\infty}\parallel Q\omega_n\parallel=0. \tag{3.41}$$

Taking the advantage of Q as $\gamma-ism$ and so $\frac{1}{\gamma}-$Lipschitz continuous. Therefore, it follows from (3.38) and (3.40) that $\varpi\in Q^{-1}(0)$. Hence, $\varpi\in Sol(VIP(1.4))$.

We also claim that $\varpi\in Sol(GMEP(1.1))$. Consider the triangular inequality

$$\parallel u_n-z_n\parallel\leq\parallel u_n-x_n\parallel+\parallel x_n-z_n\parallel .$$

By (3.19) and (3.22), we get

$$\lim_{n\to\infty}\parallel u_n-z_n\parallel=0.$$

From uniform continuity of J on bounded sets, we obtain

$$\lim_{n\to\infty}\parallel Ju_n-Jz_n\parallel=0. \tag{3.42}$$

Since $r_n\geq a$ and by (3.42), we have

$$\lim_{n\to\infty}\frac{\parallel Ju_n-Jz_n\parallel}{r_n}=0. \tag{3.43}$$

Equation $u_n=T_{r_n}z_n$ implies that

$$H(u_n,v)+\frac{1}{r_n}\langle v-u_n, Ju_n-Jz_n\rangle+\vartheta(v,u_n)-\vartheta(u_n,u_n)\geq 0,\ \forall v\in C.$$

where

$$H(u_n, v) = D(u_n, v) + \langle Gu_n, v - u_n\rangle.$$

By applying Assumption (D_2), we obtain

$$\begin{aligned} \frac{1}{r_n}\langle v - u_n, Ju_n - Jz_n\rangle &\geq -H(u_n, v) - \vartheta(v, u_n) + \vartheta(u_n, u_n) \\ &\geq H(v, u_n) - \vartheta(v, u_n) + \vartheta(u_n, u_n). \end{aligned}$$

Letting $n \longrightarrow \infty$, by Assumption (D_4) and (3.43), we get

$$H(v, \varpi) - \vartheta(v, \varpi) + \vartheta(\varpi, \varpi) \leq 0, \ \forall v \in C.$$

For all $s \in (0, 1]$ and $v \in C$, setting $v_s := sv + (1 - s)\varpi$. Therefore $v_s \in C$ and then,

$$H(v_s, \varpi) - \vartheta(v_s, \varpi) + \vartheta(\varpi, \varpi) \leq 0.$$

By Assumption $(D_1) - (D_4)$, we estimate as

$$\begin{aligned} 0 &= H(v_s, v_s) \\ &\leq sH(v_s, v) + (1 - s)H(v_s, \varpi) \\ &\leq sH(v_s, v) + (1 - s)\big[\vartheta(v_s, \varpi) - \vartheta(\varpi, \varpi)\big] \\ &\leq sH(v_s, v) + (1 - s)\big[\vartheta(v, \varpi) - \vartheta(\varpi, \varpi)\big] \end{aligned}$$

As $s > 0$, from Assumption (D_3), we conclude that

$$H(\varpi, v) + \vartheta(v, \varpi) - \vartheta(\varpi, \varpi) \geq 0, \ \forall v \in C.$$

Hence, $\varpi \in Sol(GMEP(1.1))$.

$Step$ 6 : Finally we show that $\varpi = \Pi_\Omega x_0$ and so $x_n \longrightarrow \Pi_\Omega x_0$ as $n \longrightarrow \infty$. Putting $x^* = \Pi_\Omega x_0$, since $x^* \in \Omega \subset C_n$ and $x_n = \Pi_\Omega x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x^*, x_0), \ \forall n \geq 0.$$

Then

$$\phi(\varpi, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(x^*, x_0),$$

implies that $\varpi = x^*$ and since $x^* = \Pi_\Omega x_0$, then we conclude that $x_n \longrightarrow \varpi = \Pi_\Omega x_0$, *as* $n \to \infty$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed and convex subset of a 2−uniformly smooth and uniformly convex Banach space B with B^* as the dual space of B. Let $D : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1, $\vartheta : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 2 and $G : C \longrightarrow B^*$ be a monotone and continuous mapping. Let $T_i : C \longrightarrow C$ and $S_i : C \longrightarrow C$, for each $i = 1, 2, ..., N$ be two finite family of closed l_i−Lipschitz continuous and uniformly* *quasi−ϕ−asymptotically nonexpansive mappings such that* $\Omega := \big(\cap_{i=1}^{N} F(T_i)\big) \cap \big(\cap_{i=1}^{N} F(S_i)\big) \cap \cap Sol\big(GMEP(1.1)\big) \neq \emptyset$. *Let* $\{x_n\}$ *generated by algorithm :*

$$\begin{cases} x_0, x_1 \in C, \ C_1 := C, \\ \omega_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J^{-1}(\mu_{n,0} J\omega_n + \sum_{i=1}^{N} \mu_{n,i} JT_i^n \omega_n); \\ z_n = J^{-1}(\eta_{n,0} J\omega_n + \sum_{i=1}^{N} \eta_{n,i} JS_i^n y_n), \\ u_n = T_{r_n} z_n, \\ C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq k_n^2 \phi(u, \omega_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\mu_{n,i}\} \subset [0,1]$ *and* $\{\eta_{n,i}\} \subset (0,1]$ *satisfying the following conditions:*

(S_1) $\sum_{i=0}^{N} \mu_{n,i} = 1$;

(S_2) $\sum_{i=0}^{N} \eta_{n,i} = 1$;

(S_3) $\limsup_{n\to\infty} \eta_{n,0} < 1$;

(S_4) *for same* $a > 0$, $r_n \in [a, \infty)$.

Then, $\{x_n\}$ *converges strongly to* ϖ, *where* $\varpi = \Pi_\Omega x_0$ *is consider as the generalized projection of* ϖ *onto* Ω.

Corollary 3.3. *Let* C *be a nonempty closed and convex subset of a* $2-$*uniformly smooth and uniformly convex Banach space* B *with* B^* *as the dual space of* B. *Let* $D : C \times C \longrightarrow \mathbb{R}$ *be a bifunction satisfying Assumption 1 and* $G : C \longrightarrow B^*$ *be a monotone and continuous mapping. Let* $T_i : C \longrightarrow C$ *and* $S_i : C \longrightarrow C$, *for each* $i = 1, 2, ..., N$ *be two finite family of closed* l_i-*Lipschitz continuous and uniformly quasi−ϕ−asymptotically nonexpansive mappings such that* $\Omega := \big(\cap_{i=1}^{N} F(T_i)\big) \cap \big(\cap_{i=1}^{N} F(S_i)\big) \cap \cap Sol\big(GEP(1.2)\big) \neq \emptyset$. *Let* $\{x_n\}$ *generated by algorithm :*

$$\begin{cases} x_0, x_1 \in C, \ C_1 := C, \\ \omega_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J^{-1}(\mu_{n,0} J\omega_n + \sum_{i=1}^{N} \mu_{n,i} JT_i^n \omega_n); \\ z_n = J^{-1}(\eta_{n,0} J\omega_n + \sum_{i=1}^{N} \eta_{n,i} JS_i^n y_n), \\ u_n = T_{r_n} z_n, \\ C_{n+1} = \{u \in C_n : \phi(u, u_n) \leq k_n^2 \phi(u, \omega_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\mu_{n,i}\} \subset [0,1]$ *and* $\{\eta_{n,i}\} \subset (0,1]$ *satisfying the following conditions:*

(S_1) $\sum_{i=0}^{N} \mu_{n,i} = 1;$

(S_2) $\sum_{i=0}^{N} \eta_{n,i} = 1;$

(S_3) $\limsup_{n\to\infty} \eta_{n,0} < 1;$

(S_4) *for same* $a > 0$, $r_n \in [a, \infty)$.

Then, $\{x_n\}$ *converges strongly to* ϖ, *where* $\varpi = \Pi_\Omega x_0$ *is consider as the generalized projection of* ϖ *onto* Ω.

4. Numerical Example

Let $B = \mathbb{R}$ and $C = [0,1]$. Let $Q : C \to C$ be defined by $Qu = 2u \forall u \in C$. Define $\vartheta : C \times C \to \mathbb{R}$, $D : C \times C \to \mathbb{R}$, $G : C \to \mathbb{R}$, $Q : C \to \mathbb{R}$, $T_i : C \to C$ and $S_i : C \to C$ by $\vartheta(u,v) = 0$, $D(u,v) = (u+v)(v-u)$, $G(u) = u$, $Q(u) = 2u$ and $T_i(u) = S_i(u) = \frac{1}{i+1}u$, respectively.
Setting $\{\beta_n\} = \{\frac{0.9}{2n}\}$, $r_n = \frac{1}{2}$, $\{\alpha_n\} = 0.9$, $\mu_{0,n} = \frac{1}{2}$, $\sum_{i=1}^{N} \mu_{n,i} = \frac{1}{2}$ such that $\sum_{i=0}^{N} \mu_{i,n} = 1$ and $\eta_{0,n} = \frac{1}{3}$, $\sum_{i=1}^{N} \eta_{n,i} = \frac{2}{3}$ so that $\sum_{i=0}^{N} \eta_{i,n} = 1$.
Let $\{x_n\}$ be generated by the hybrid inertial iterative algorithm (3.1) converges to $x^* = \{0\} \in \Omega$.

Proof. Clearly ϑ and D satisfy assumptions 1 and 2, respectively, and G is continuous and monotone so that $Sol(GMEP(eq1.1)) = \{0\} \neq \emptyset$, $Sol(VIP(eq1.4)) = \{0\} \neq \emptyset$. Obviously Q is $\frac{1}{2} - ism$, and T_i and S_i are two finite families of closed 1-Lipschitz continuous and uniformly quisi-ϕ-asymptotically nonexpansive mappings with $Fix(T_i) = Fix(S_i) = \{0\}$. Thus $\Omega = Sol(GMEP(eq1.1)) \cap Sol(VIP(eq1.4)) \cap Fix(T_i) \cap Fix(S_i) = \{0\} \neq \emptyset$. Hence, the iterative scheme (3.1) becomes the following scheme (4.1) after simplification:

$$\begin{cases} x_0, x_1 \in C, \ C_1 := C, \\ \omega_n = x_n + 0.9(x_n - x_{n-1}), \\ y_n = \frac{1}{2}\omega_n + \frac{1}{2(n+1)}\omega_n, \\ z_n = \frac{1}{3}y_n + \frac{2}{3(n+1)}v_n, \\ u_n = \frac{2z_n}{7}, \\ C_{n+1} = \left[0, \frac{u_n+\omega_n}{2}\right], \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 1, \\ where, \ for \ \Pi_C \ a \ metric \ projection \ onto \ C, \\ v_n = \Pi_C(\omega_n - \beta_n Q\omega_n) = \begin{cases} 0, \ \omega_n - \frac{0.9}{2n}\omega_n < 0 \\ 1, \ \omega_n - \frac{0.9}{2n}\omega_n > 1 \\ \omega_n - \frac{0.9}{2n}\omega_n, \ otherwise. \end{cases} \end{cases} \tag{4.1}$$

Finally, using the software Matlab 7.8.0, we have the following figure which shows that $\{x_n\}$ converges to $\{0\}$ as $n \to \infty$.

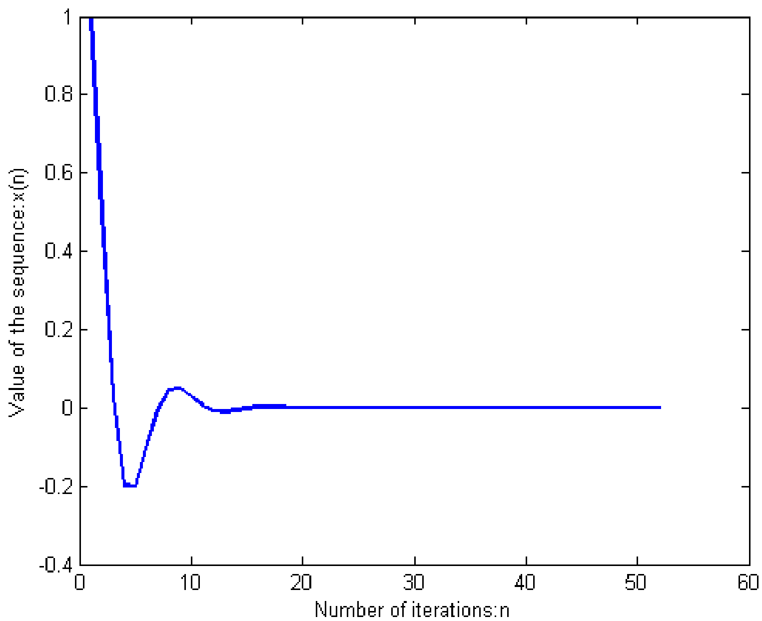

Figure 1. Convergence of $\{x_n\}$ when $x_0 = 1.0$ and $x_1 = 0.5$

References

[1] M. Alansari, R. Ali and M. Farid, Strong convergence of an inertial iterative algorithm for variational inequality problem, generalized equilibrium problem and fixed point problem in a Banach space, J. Ineq. Appl. 2020 (2020) 42.

[2] Y.I. Alber, Metric and generalized projection operators in Banach spaces, In: Properties and Applications, Lect. Note, Pure. Appl. Math. 8(1996), 15–50.

[3] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123–145.

[4] R.I. Bot, E.R. Csetnek and C. Hendrich, Inertial Douglas-Racheord splitting for monotone inclusion problems, Appt. Math. Comp. 256 (2015) 472–487.

[5] R.I. Bot and E.R. Csetnek, An inertial forward- backward - forward primal-dual splitting algorithm for solving monotone inclusions problems, Numer. Algor. 71 (2016) 519–540.

[6] M. Farid, R. Ali and K.R. Kazmi, Inertial iterative method for a generalized mixed equilibrium, variational inequality and a fixed point problems for a family of quasi$-\phi-$nonexpansive mappings, Filomat. 37 (2023), 6133-6150.

[7] S. Gupta, S. Husain and V.N. Mishra, Variationa inclusion governed by $\alpha\beta - H((.,.),(.,.)-$ mixed accretive mapping, Filomat, 31 (2017), 6529–6542.

[8] J. Iqbal, V.N. Mashra W.A. Mir, A.H. Dar, M. Ishiyak and L. Rathour, Generalized resolvent operator involving $G(.,.)-$ Co-monotone mapping for solving generalized variational inclusion problem, Georgian Math. J. 3 (2022), 533–542.

[9] S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002) 938–945.

[10] K.R. Kazmi and R. Ali, Common solution to an problem and fixed point problem for an asymptotically quasi$-\phi-$nonexpansive mapping in intermediate sense, RACSAM, 111 (2017) 877–889.

[11] L. Umar, Y. Ibrahim and T.M. Kabir, Hybrid algorithm for solving fixed point problem and generalized mixed equilibrium problem in Banach spaces, Uzbek Math. J. 66 (2022) 101–118.

[12] L. Umar, T.M. Kabir and I.U. Haruna, An inertial algorithm of generalized f- projection for maximal monotone operators and generalized mixed equilibrium problem in Banach spaces, Afr. Sci. Rep. 1 (2022) 32–47.

[13] P.E. Mainge, Convergence theorem for inertial KM-type algorithms J. Comp. Appl. Math. 219 (2008) 223–236.

[14] K. Nakajo, Strong convergence for gradient projection method and relatively nonexpansive mappings in Banach spaces, Appl. Math. Comp. 271 (2017) 251–258.

[15] X.L. Qin, Y.J. Cho and S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in a Banach spaces, J. Comp. Appl. Math. 225 (2009) 20–30.

[16] B.T. Polyak, Some methods of speeding up the convergence of iteration methods. USSR Comp. Math. Phys. 4 (1964) 1–7.

[17] W. Takahashi and K. Zembayashi, Strong and weak convergence theorem for equilibrium problem and relatively nonexpansive mappings in Banach space, Nonlinear Anal. TMA. 70 (2009) 45–57.

[18] Vandana, R. Dubey, Deepmala, L.N. Mishra and V.N. Mishra, Duality relations for a class of a multiobjective faction programming problem involving support functions, Amer. J. Oper. Res. 8 (2018) 293–311.

[19] H.K. Xu, Inequality in Banach space with application, Nonlinear. Anal. TMA. 16 (1991) 1127- 1138.

[20] C. Zalinesco, On uniformly convex function, J. Math. Anal. Appl. 95 (1983) 344–374.

[21] H. Zegeye, A hybrid iterative scheme for equilibrium problems, variational inequality problems and common fixed point problems in Banach spaces, Nonlinear Anal. TMA. 72 (2010) 2136–2146.