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Duals of Continuous Frames in Hilbert C^*-Modules

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Abstract. The concept of frame is an exciting, dynamic, and fast-paced subject with applications in numerous fields of mathematics and engineering. The purpose of this paper is to introduce equivalent $*$-continuous frames and to present ordinary duals of constructed $*$-continuous frames by an adjointable and invertible operator. Also, we establish some properties.

1. Introduction

Frames in Hilbert spaces have been introduced by Duffin and Schaeffer [3] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper, by Daubechies, Grossman and Meyer [2], frame theory began to be widely used, particularly in the more specialized context of wavelet frame and Gabor frame [4]. Frames have been used in signal processing, image processing, data compression and sampling theory. For more about frames, see [5,7–11].

In this paper, we introduce the notions of continuous frame on a Hilbert C^*-module over a unital C^*-algebra which is a generalization of discrete frame, the $*$-continuous frame, which are a generalization of $*$-frame in Hilbert C^*-modules and we establish some new results.

The paper is organized as follows. We continue this introductory section and briefly recall the definitions and basic properties of Hilbert C^*-modules. In Section 2, the generalized duals for a given $*$-continuous frame will be considered. Also, we study their properties and characterize all operator dual $*$-continuous frames associated with the given $*$-continuous frame in Hilbert C^*-modules. In Section 3, we extend this notion for sequences (continuous frames) in Hilbert C^*-modules. Also, some properties of them will be studied. In Section 4, a $*$-continuous frame is constructed by an orthogonal projection.

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Definition 1.1. [6] Suppose that $\mathcal{A}$ is a C^*-algebra. A linear space $\mathcal{H}$ which is also an algebraic left $\mathcal{A}$-module together with an $\mathcal{A}$-inner product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \longrightarrow \mathcal{A}$ and possesses the following properties is called a pre-Hilbert C^*-module:

(1) $\langle f, f\rangle \geq 0$, for any $f \in \mathcal{H}$;
(2) $\langle f, f\rangle = 0$ if and only if $f = 0$;
(3) $\langle f, g\rangle = \langle g, f\rangle^*$, for any $f, g \in \mathcal{H}$;
(4) $\langle \lambda f, h\rangle = \lambda\langle f, h\rangle$, for any $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}$;
(5) $\langle af + bg, h\rangle = a\langle f, h\rangle + b\langle g, h\rangle$, for any $a, b \in \mathcal{A}$ and $f, g, h \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = ||\langle x, x\rangle||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}$ for $x \in \mathcal{H}$. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

Lemma 1.2. *[12] Let (Ω, μ) be a measure space, X and Y be two Banach spaces, $\lambda : X \to Y$ be a bounded linear operator and $f : \Omega \to Y$ be a measurable function. Then*

$$\lambda\left(\int_\Omega f d\mu\right) = \int_\Omega (\lambda f) d\mu.$$

Lemma 1.3. *[1] Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*(\mathcal{H}, \mathcal{K})$.*

(i) *If T is injective and T has a closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

The following definition was introduced in [11].

Definition 1.4. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. A map $F : \Omega \to \mathcal{H}$ is called a $*$-continuous frame with respect to (Ω, μ) if

1. for all $f \in \mathcal{H}$, $w \to \langle f, F_w\rangle$ is a measurable function on Ω,
2. there exist two strictly nonzero elements $A, B > 0$ in $\mathcal{A}$ such that

$$A\langle f, f\rangle A^* \leq \int_\Omega \langle f, F_w\rangle\langle F_w, f\rangle d\mu(w) \leq B\langle f, f\rangle B^*, \forall f \in \mathcal{H}. \tag{1.1}$$

The elements A and B are called $*$-continuous frame bounds. If $A = B$, we call this $*$-continuous frame a tight $*$-continuous frame, and if $A = B = 1$, it is called a Parseval $*$-continuous frame. If only the right-hand inequality of (1.1) is satisfied, we call $F : \Omega \to \mathcal{H}$ a $*$-continuous Bessel map with Bessel bound B.

Let X be a Banach space, (Ω, μ) be a measure space and $f : \Omega \to X$ be a measurable function. Integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Since every C^*-algebra and Hilbert C^*-module is a Banach space, we can use this integral and its properties.

Let (Ω, μ) be a measure space. We define

$$L^2(\Omega, \mathcal{A}) = \left\{\varphi : \Omega \to \mathcal{A} : \left\| \int_\Omega \varphi(\omega)\varphi(\omega)^* d\mu(\omega) \right\| < \infty \right\}.$$

For any $\varphi, \psi \in L^2(\Omega, \mathcal{A})$, if the $\mathcal{A}$-valued inner product is defined by

$$\langle \varphi, \psi \rangle = \int_\Omega \varphi(\omega)\psi(\omega)^* d\mu(w),$$

the norm is defined by $\|\varphi\| = \|\langle \varphi, \varphi \rangle\|^{\frac{1}{2}}$, then $L^2(\Omega, \mathcal{A})$ is a Hilbert C^*-module.

The frame transform or pre-frame operator $T : \mathcal{H} \longrightarrow L^2(\Omega, \mathcal{A})$ is defined by $T(f) = \{\langle f, F_w \rangle\}_{w\in\Omega}$ and it is an injective and closed range adjointable $\mathcal{A}$-module map and $\|T\| \leq \|B\|$. The adjoint operator T^* is surjective and it is given by $T^*(e_w) = F_w$ for $w \in \Omega$, where $\{e_w\}_{w\in\Omega}$ is the standard basis for $L^2(\Omega, \mathcal{A})$.

Definition 1.5. Let F be a continuous frame for $\mathcal{H}$ with respect to (Ω, μ). We define the frame operator $S : \mathcal{H} \to \mathcal{H}$ by $Sx = T_F^* T_F x = \int_\Omega \langle x, F_w \rangle F_w d\mu(w), \forall x \in \mathcal{H}$, that is positive, invertible and adjointable and the inequality

$$\left\|A^{-1}\right\|^{-2} \leq \|S\| \leq \|B\|^2$$

holds, and the reconstruction formula $f = \int_\Omega \left\langle f, S^{-1}F_w \right\rangle F_w d\mu(w)$ holds for all $f \in \mathcal{H}$.

2. $*$-Continuous operator duals

Definition 2.1. Let $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ be two $*$-continuous frames for $\mathcal{H}$. If there exists an invertible adjointable $\mathcal{A}$-module map on $\mathcal{H}$ such that

$$x = \int_\Omega \langle \Gamma x, g_w \rangle F_w d\mu(\omega), \quad \forall x \in \mathcal{H}, \tag{2.1}$$

then $\{g_w\}_{w\in\Omega}$ is called a $*$-continuous operator dual of $\{F_w\}_{w\in\Omega}$.

Remark 2.2. Every $*$-continuous frame $\{F_w\}_{w\in\Omega}$ with continuous frame operator S is a $*$-continuous operator dual for itself. To see this, set $\Gamma := S^{-1}$ and the reconstruction formula concludes it.

Remark 2.3. Every dual $*$-continuous frame $\{g_w\}_{w\in\Omega}$ of $*$-continuous frame $\{F_w\}_{w\in\Omega}$ is a $*$-continuous operator dual when $\Gamma = I$, I is the identity operator on $\mathcal{H}$.

Remark 2.4. Let $\mathcal{G} = \{g_w\}_{w\in\Omega}$ be an operator dual of a $*$-continuous frame $\mathcal{F} = \{F_w\}_{w\in\Omega}$ in $\mathcal{H}$. Then for some invertible adjointable map $\Gamma \in B_*(\mathcal{H})$

$$x = \int_\Omega \langle \Gamma x, g_w \rangle F_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

The equality shows that $I = \left(T_{\mathcal{F}}^{*}T_{\mathcal{G}}\right)\Gamma$, where I is the identity map on $\mathcal{H}$, and $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$ are pre-frame operators of $\mathcal{F}$ and $\mathcal{G}$, respectively. Therefore, the operator Γ is unique and $\Gamma^{-1} = T_{\mathcal{F}}^{*}T_{\mathcal{G}}$.

By Remark 2.4, we say that $\{g_w\}_{w\in\Omega}$ is an operator dual for $\{F_w\}_{w\in\Omega}$ with the corresponding operator Γ. Moreover, we mention that the operator duality relation of x-frames is symmetric. It is considered in the next remark.

Remark 2.5. If $\mathcal{G} = \{g_w\}_{w\in\Omega}$ is an operator dual of a given $*$-continuous frame $\mathcal{F} = \{F_w\}_{w\in\Omega}$ with the corresponding operator Γ, then $\{F_w\}_{w\in\Omega}$ is an operator dual for $\{g_w\}_{w\in\Omega}$ with the corresponding operator Γ^{*}. In order to see this, assume that $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$ are pre-frame operators of $\mathcal{F}$ and $\mathcal{G}$, respectively. By the definition of operator duals, we have

$$I = \int_{\Omega} \langle \Gamma x, g_w\rangle F_w d\mu(\omega) = (T_{\mathcal{F}}^{*}T_{\mathcal{G}})\Gamma.$$

Since Γ is invertible, $\Gamma^{-1} = T_{\mathcal{F}}^{*}T_{\mathcal{G}}$ and

$$I = \Gamma\,(T_{\mathcal{F}}^{*}T_{\mathcal{G}}) = \left(T_{\mathcal{G}}^{*}T_{\mathcal{F}}\right)\Gamma^{*} = \int_{\Omega} \langle \Gamma^{*} f, F_w\rangle g_w d\mu(\omega).$$

The following lemma is obtained by using the last remark and some properties of pre-frame operators.

Lemma 2.6. *Let $\mathcal{F} = \{F_w\}_{w\in\Omega}$ and $\mathcal{G} = \{g_w\}_{w\in\Omega}$ be $*$-Bessel sequences for $\mathcal{H}$ with the pre-frame operators $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$, respectively. Assume that Γ is an invertible and adjointable $\mathcal{A}$-module map on $\mathcal{H}$. Then for $x \in \mathcal{H}$, the following statements are equivalent:*

(i) $x = \int_{\Omega} \langle \Gamma x, g_w\rangle F_w d\mu(\omega)$.

(ii) $x = \int_{\Omega} \langle \Gamma^{*} x, F_w\rangle g_w d\mu(\omega)$.

In case that one of the above equalities is satisfied, $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ are operator dual $*$-frames. Moreover, if B is an upper bound for $\{F_w\}_{w\in\Omega}$ and S is frame operator of $\{F_w\}_{w\in\Omega}$, then $B\left\|S^{-1}\right\|^{-\frac{1}{2}}\|T_{\mathcal{F}}\|^{-1}\|\Gamma\|^{-1}$ is a lower bound for $\{g_w\}_{w\in\Omega}$.

Proof. The equivalency of the two conditions is given from Remark 2.5.

Now, let B be a $*$-Bessel bound for $\{F_w\}_{w\in\Omega}$ and (i) holds. By the definition of $*$-Bessel sequence $\{Fw\}_{w\in\Omega}$ and $T_{\mathcal{F}}^{*}T_{\mathcal{G}}\Gamma = id_{\mathcal{H}}$, we can write, for $x \in \mathcal{H}$,

$$\begin{aligned} \langle T_{\mathcal{F}}x, T_{\mathcal{F}}x\rangle &\leq B\langle x, x\rangle B^{*} \\ &= B\left\langle T_{\mathcal{F}}^{*}T_{\mathcal{G}}\Gamma x, T_{\mathcal{F}}^{*}T_{\mathcal{G}}\Gamma x\right\rangle B^{*} \leq B\,\|T_{\mathcal{F}}\|^{2}\left\langle T_{\mathcal{G}}\Gamma x, T_{\mathcal{G}}\Gamma x\right\rangle B^{*}. \end{aligned} \tag{2.2}$$

Using Lemma 1.3, we have

$$\left\|(T_{\mathcal{F}}^{*}T_{\mathcal{F}})^{-1}\right\|^{-1}\langle x, x\rangle \leq \langle T_{\mathcal{F}}x, T_{\mathcal{F}}x\rangle, \quad \forall f \in \mathcal{H}. \tag{2.3}$$

It follows from Lemma 1.3, (2.2), and (2.3) that for $x \in \mathcal{H}$,

$$\left\|S^{-1}\right\|^{-1} \|\Gamma\Gamma^*\|^{-1} \langle f, f\rangle \leq \left\|S^{-1}\right\|^{-1} \left\langle \Gamma^{-1} f, \Gamma^{-1} f\right\rangle \leq B \|T_{\mathcal{F}}\|^2 \langle T_{\mathcal{G}} f T_{\mathcal{G}} f\rangle B^*,$$

$$\left(B^{-1} \left\|S^{-1}\right\|^{-\frac{1}{2}} \|T_{\mathcal{F}}\|^{-1} \|\Gamma\|^{-1}\right) \langle f, f\rangle \left(B^{-1} \left\|S^{-1}\right\|^{-\frac{1}{2}} \|T_{\mathcal{F}}\|^{-1} \|\Gamma\|^{-1}\right)^*$$
$$\leq \langle T_{\mathcal{G}} f, T_{\mathcal{G}} f\rangle .$$

Therefore, $B \left\|S^{-1}\right\|^{-\frac{1}{2}} \|\Gamma\|^{-1} \|T_{\mathcal{F}}\|^{-1}$ is a lower $*$-frame bound for $\left\{g_j\right\}_{j\in J}$ and $\left\{g_j\right\}_{j\in J}$ is a $*$-frame.

Similarly, $\left\{f_j\right\}_{j\in J}$ is also a $*$-frame. □

Proposition 2.7. *Let $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ be an operator dual of a $*$-continuous frame $\{F_w\}_{w\in\Omega}$.*

1. *For a strictly nonzero element α in the center of $\mathcal{A}$, the pair $\left(\{\alpha g_w\}_{w\in\Omega}, \alpha^{-1}\Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$.*
2. *If Υ is an invertible and adjointable operator on $\mathcal{H}$, then $\left(\{\Upsilon g_w\}_{w\in\Omega}, (\Upsilon)^{-1}\Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$.*
3. *The sequence $\{g_w\}_{w\in\Omega}$ is a dual of $\{\Gamma^* F_w\}_{w\in\Omega}$.*
4. *Assume that $\left(\{h_w\}_{w\in\Omega}, \Lambda\right)$ is another operator dual of $\{Fw\}_{w\in\Omega}$. Then $\left(\{g_w + h_w\}_{w\in J}, \left(\Gamma^{-1} + \Lambda^{-1}\right)^{-1}\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$.*

Proposition 2.8. *Let $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ be an operator dual of $\{F_w\}_{w\in\Omega}$ for $\mathcal{H}$. If f is an element of $\mathcal{H}$ such that $\langle x, x\rangle$ is a strictly nonzero element in the center of $\mathcal{A}$, then $\left\{\left\langle g_w, (\langle x, x\rangle)^{-1}\Gamma x\right\rangle\right\}_{w\in\Omega}$ is a dual of $\{\langle F_w, x\rangle\}_{w\in\Omega}$.*

Proof. Suppose that $a \in \mathcal{A}$. Then

$$\begin{aligned}\int_\Omega \left\langle a, \left\langle g_w, \langle x, x\rangle^{-1}\Gamma x\right\rangle\right\rangle \langle F_w, x\rangle \, d\mu(\omega) &= \int_\Omega a \left\langle \langle x, x\rangle^{-1}\Gamma x, g_w\right\rangle \langle F_w, x\rangle \, d\mu(\omega) \\ &= a\langle x, x\rangle^{-1} \left\langle \int_\Omega \langle \Gamma x, g_w\rangle F_w d\mu(\omega), x\right\rangle \\ &= a\langle x, x\rangle^{-1}\langle x, x\rangle = a.\end{aligned}$$

This completes the proof. □

Proposition 2.9. *Let $\{F_w\}_{w\in\Omega}$ be a $*$-continuous frame for $\mathcal{H}$ with frame operator S. If θ is an adjointable and invertible operator on $\mathcal{H}$, then $\left(\{\theta F_w\}_{w\in\Omega}, \left(\theta^{-1}\right)^* S^{-1}\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$.*

Proof. Let $x \in \mathcal{H}$. Then

$$\begin{aligned}\int_\Omega \left\langle \left(S^{-1}\theta^{-1}\right) x, F_w\right\rangle \theta F_w d\mu(\omega) &= \theta \left(\int_\Omega \left\langle \left(S^{-1}\theta^{-1}\right) x d\mu(\omega), F_w\right\rangle F_w\right) \\ &= \theta\left(\theta^{-1} x\right) = x.\end{aligned}$$

This completes the proof. □

Proposition 2.10. *Let $\{F_w\}_{w\in\Omega}$ be a $*$-continuous frame and θ be an adjointable and invertible operator on $\mathcal{H}$. Then the sets of operator duals of $\{F_w\}_{w\in\Omega}$ and $\{\theta F_w\}_{w\in\Omega}$ are in one to one correspondence.*

Proof. First, suppose that $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$. For $x \in \mathcal{H}$, we obtain

$$\begin{aligned} x &= \int_\Omega \langle \Gamma^* x, F_w\rangle g_w d\mu(\omega) = \int_\Omega \left\langle \theta^* \left(\theta^{-1}\right)^* \Gamma^* x, F_w \right\rangle g_w d\mu(\omega) \\ &= \int_\Omega \left\langle \left(\theta^{-1}\right)^* \Gamma^* x, \theta F_w \right\rangle g_w d\mu(\omega). \end{aligned}$$

So $\left(\{g_w\}_{w\in\Omega}, \Gamma\theta^{-1}\right)$ is an operator dual for $\{\theta F_w\}_{w\in\Omega}$.

Now, if $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual of $\{\theta F_w\}_{w\in\Omega}$, then $\left(\{g_w\}_{w\in\Omega}, \Gamma^*\theta\right)$ is an operator dual of $\{F_w\}_{w\in\Omega}$, since

$$x = \int_\Omega \langle \Gamma x, \theta F_w\rangle g_w d\mu(\omega) = \int_\Omega \langle \theta^* \Gamma x, F_w\rangle g_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

This completes the proof. □

Proposition 2.11. *Let $\mathcal{F} = \{F_w\}_{w\in\Omega}$ be a $*$-continuous frame for $\mathcal{H}$ with pre-frame operator $T_\mathcal{F}$ and frame operator S. Then the set of all the operator duals of $\{F_w\}_{w\in\Omega}$ is precisely the following*

$$\{g_w\}_{w\in\Omega} = \left\{\Gamma F_w + \varphi e_w - \int_\Omega \left\langle S^{-1} F_w, F_i\right\rangle \varphi e_w d\mu(\omega)\right\}_{w\in\Omega},$$

where $\{e_w\}_{w\in\Omega}$ is the standard orthonormal basis for $L^2(\Omega, \mathcal{A})$, $\varphi \in B_*\left(\mathcal{H}, L^2(\Omega, \mathcal{A})\right)$, and Γ is an invertible adjointable operator on $\mathcal{H}$.

Proof. Assume that $\{g_w\}_{w\in\Omega}$ is a sequence as above. Then its pre-frame operator is $T_\mathcal{G} = T_\mathcal{F}\Gamma + \varphi - T_\mathcal{F} S^{-1} T_\mathcal{F}^* \varphi$ and so

$$\begin{aligned} (S\Gamma)^{-1}\left(T_\mathcal{F}^* T_\mathcal{G}\right) &= (S\Gamma)^{-1}\left(T_\mathcal{F}^* T_\mathcal{F}\Gamma + T_\mathcal{F}^*\varphi - T_\mathcal{F}^* T_\mathcal{F} S^{-1} T_\mathcal{F}^*\varphi\right) \\ &= (S\Gamma)^{-1}\left(T_\mathcal{F}^* T_\mathcal{F} S^{-1} S\Gamma + T_\mathcal{F}^*\varphi - T_\mathcal{F}^* T_\mathcal{F} S^{-1} T_\mathcal{F}^*\varphi\right) \\ &= (S\Gamma)^{-1}(S\Gamma) = I. \end{aligned}$$

By a similar relation with the given equality in Remark 2.5, we can conclude that $\{g_w\}_{w\in\Omega}$ is an operator dual for $\{F_w\}_{w\in\Omega}$ with the corresponding operator $(S\Gamma)^{-1}$. □

Theorem 2.12. *Let $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ be an operator dual of $*$-continuous frame $\{F_w\}_{w\in\Omega}$ for $\mathcal{H}$. Then there exist a Hilbert $\mathcal{A}$-module $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{u_w\}_{w\in\Omega}$ of $\mathcal{K}$ which has a unique dual $\{v_w\}_{w\in\Omega}$ and satisfies $(Pu)u_w = F_w$ and $(Pv_0)v_w = g_w$ for all $w \in \Omega$, where P is the projection from $\mathcal{K}$ onto $\mathcal{H}$.*

Proof. Assume that $T_{\mathcal{F}}, T_{\mathcal{G}}$ and $S_{\mathcal{F}}, S_{\mathcal{G}}$ are pre-frame operators and frame operators of $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$, respectively. Also, the orthogonal projections onto the range of $T_{\mathcal{F}}$, $R(T_{\mathcal{F}})$, and the range of $T_{\mathcal{G}}$, $R(T_{\mathcal{G}})$, are $P_{\mathcal{F}}$ and P_G, respectively. Now, for $x \in \mathcal{H}$,

$$\begin{aligned}\left\langle T_{\mathcal{G}}x, T_{\mathcal{G}}S_{\mathcal{G}}^{-1}g_w\right\rangle &= \left\langle T_{\mathcal{G}}^*T_{\mathcal{G}}x, S_{\mathcal{G}}^{-1}g_w\right\rangle = \left\langle S_{\mathcal{G}}^{-1}S_{\mathcal{G}}x, g_w\right\rangle \\ &= \langle x, g_w\rangle = \langle T_{\mathcal{G}}x, e_w\rangle = \langle T_{\mathcal{G}}x, P_{\mathcal{G}}e_w\rangle'\end{aligned}$$

and so

$$P_G e_w = T_G S_{\mathcal{G}}^{-1} g_w, \quad \forall w \in \Omega, \tag{2.4}$$

where $\{e_w\}_{w\in\Omega}$ is the standard orthonormal basis of $L^2(\Omega, \mathcal{A})$. By (2.4), for $x \in \mathcal{H}$, we give

$$\begin{aligned}P_{\mathcal{G}}T_{\mathcal{F}}(\Gamma^*x) &= P_{\mathcal{G}}\left(\int_\Omega \langle \Gamma^*x, F_w\rangle e_w\right) \\ &= \int_\Omega \langle \Gamma^*x, F_w\rangle P_{\mathcal{G}}e_w \\ &= \int_\Omega \langle \Gamma^*F, F_w\rangle T_{\mathcal{G}}S_{\mathcal{G}}^{-1}g_w \\ &= T_{\mathcal{G}}S_{\mathcal{G}}^{-1}\left(\int_\Omega \langle \Gamma^*x, F_w\rangle g_w\right) = T_{\mathcal{G}}S_{\mathcal{G}}^{-1}x.\end{aligned}$$

Set

$$\mathcal{K} = \mathcal{H} \oplus P_{\mathcal{G}}^{\perp} l_2(\mathcal{A}), \quad u_w = F_w \oplus P_{\mathcal{G}}^{\perp} e_w, \quad \forall w \in \Omega.$$

If T_U is a pre-frame operator of the sequence $\{u_w\}_{w\in\Omega}$, then $T_U(x \oplus v) = T_{\mathcal{F}}x + v$ and

$$\|T_U(x \oplus v)\| = \|T_{\mathcal{F}}f + w\| \leq B(\|x\| + \|v\|) = B\|x \oplus v\|, \quad \forall x \oplus v \in \mathcal{K}$$

for some $B > 0$ and so $\{u_w\}_{w\in\Omega}$ is a Bessel sequence. We show that T_U has a closed range. Suppose $\{\eta_n\}_{n\in\mathbb{N}} \subseteq R(T_U)$ such that $\eta_n \overset{n\to\infty}{\longrightarrow} \eta$. Since Γ is invertible and adjointable, there exists $\Gamma^*F_n \oplus v_n \in \mathcal{H} \oplus P_{\mathcal{G}}^{\perp}(L^2(\Omega, \mathcal{A}); T_U(\Gamma^*F_n \oplus v_n) = \eta_n$. On the other hand,

$$T_U(\Gamma^*F_n \oplus v_n) = T_{\mathcal{F}}(\Gamma^*F_n) + v_n = \eta_n \overset{n\to\infty}{\longrightarrow} \eta$$

and Remark 2.4 gives

$$F_n = T_{\mathcal{G}}^*T_{\mathcal{F}}\Gamma^*F_n = T_{\mathcal{G}}^*(T_{\mathcal{F}}\Gamma^*f_n + v_n) \overset{n\to\infty}{\longrightarrow} T_{\mathcal{G}}^*\eta.$$

$R(T_U)$ is closed, since $R(T_{\mathcal{F}})$ is closed. Also, T_U^* has a closed range. This step will obtain the injectivity of T_U^*. If $T_U^*\left(\int_\Omega a_w e_w\right) = 0$, then

$$0 = \int_\Omega a_w\left(F_w \oplus P_G^{\perp}e_w\right) = \int_\Omega a_w F_w \oplus P_G^{\perp}\left(\int_\Omega a_w e_w\right).$$

It concludes that (i) $\int_\Omega a_w F_w = 0$ and (ii) $P_G^{\perp}\left(\int_\Omega a_w e_w\right) = 0$. From (ii), we have

$$\int_\Omega a_w e_w \in R(T_{\mathcal{G}}) \Longrightarrow \exists h \in \mathcal{H}; T_{\mathcal{G}}h = \int_\Omega a_w e_w$$

and on the other hand,

$$T_{\mathcal{G}}h=\int_{\Omega}\langle h,g_w\rangle e_w \Longrightarrow a_w=\langle h,g_w\rangle, \quad \forall w\in\Omega.$$

From (i), we have

$$0=\int_{\Omega}a_wF_w=\int_{\Omega}\langle h,g_w\rangle F_w=\int_{\Omega}\left\langle \Gamma\Gamma^{-1}h,g_w\right\rangle F_w=\Gamma^{-1}h.$$

Since Γ is injective, $h=0$ and $a_w=0$ for $w\in\Omega$. So $\int_{\Omega}a_we_w=0$, and T_U^* is injective. The operator $T_U^*T_U$ is an invertible selfadjoint operator such that it has an upper bound and a lower bound by Lemma 1.3, and also $\{u_w\}_{w\in\Omega}$ is a frame for $\mathcal{K}$ with frame operator $S_U=T_U^*T_U$. □

3. Equivalent $*$-continuous frames

Definition 3.1. Two sequences $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ in $\mathcal{H}$ are said to be equivalent sequences if there exists an adjointable and invertible operator Λ on $\mathcal{H}$ such that $\Lambda F_w=g_w$, for $w\in\Omega$.

Theorem 3.2. *Let $\{F_w\}_{w\in\Omega}$ be a $*$-continuous frame for $\mathcal{H}$ and ξ be an adjointable and invertible operator on $\mathcal{H}$. Then every dual of the $*$-continuous frame $\{\xi F_w\}_{w\in\Omega}$ is equivalent to a dual of $\{F_w\}_{w\in\Omega}$, and the converse of the relation is valid.*

Proof. First, suppose that $\{g_w\}_{w\in\Omega}$ is a dual of $\{F_w\}_{w\in\Omega}$. Then for $x\in\mathcal{H}$, we obtain

$$\begin{aligned} x&=\xi\left(\xi^{-1}\right)x\\ &=\xi\left(\int_{\Omega}\left\langle\xi^{-1}x,g_w\right\rangle F_w d\mu(\omega)\right)\\ &=\int_{\Omega}\left\langle x,\left(\xi^{-1}\right)^*g_w\right\rangle \xi F_w d\mu(\omega). \end{aligned}$$

So $\left\{\left(\xi^{-1}\right)^*g_w\right\}$ is a dual for $\{\xi F_w\}_{w\in\Omega}$, and it is also equivalent to $\{g_w\}_{w\in\Omega}$.

Now, suppose that $\{h_w\}_{w\in\Omega}$ is a dual frame for $\{\xi F_w\}_{w\in\Omega}$. Set $g_w=\xi^*h_w$, for $w\in\Omega$. Then for $x\in\mathcal{H}$,

$$\begin{aligned} \int_{\Omega}\langle x,g_w\rangle F_w d\mu(\omega)&=\int_{\Omega}\langle x,\xi^*h_w\rangle \xi^{-1}\xi F_w d\mu(\omega)\\ &=\xi^{-1}\left(\int_{\Omega}\langle \xi f,h_w\rangle \xi F_w d\mu(\omega)\right)\\ &=\xi^{-1}\xi x\\ &=x. \end{aligned}$$

Thus $\{g_w\}_{w\in\Omega}$ is a dual for $\{F_w\}_{w\in\Omega}$ and $h_w=\left(\xi^{-1}\right)^*g_w$. □

Theorem 3.3. *If $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ are $*$-continuous frames with the continuous frame operators $S_{\mathcal{F}}$ and $S_{\mathcal{G}}$, respectively, then there exists a $*$-continuous frame that is equivalent to $\{g_w\}_{w\in\Omega}$ and its frame operator is $S_{\mathcal{F}}$.*

Proof. For the adjointable and invertible operator

$$\xi = S_{\mathcal{F}}^{\frac{1}{2}} S_{\mathcal{G}}^{-\frac{1}{2}},$$

the sequence $\{\xi g_w\}_{w\in\Omega}$ is a $*$-continuous frame with the frame operator $S_\xi = \xi S_{\mathcal{G}} \xi^*$. So

$$\begin{aligned} S_\xi &= \xi S_{\mathcal{G}} \xi^* \\ &= \left(S_{\mathcal{F}}^{\frac{1}{2}} S_{\mathcal{G}}^{-\frac{1}{2}} \right) S_{\mathcal{G}} \left(S_{\mathcal{F}}^{\frac{1}{2}} S_{\mathcal{G}}^{-\frac{1}{2}} \right)^* \\ &= S_{\mathcal{F}}. \end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ be $*$-continuous frames for $\mathcal{H}$. Then the following statements are valid.*

(1) *$\{g_w\}_{w\in\Omega}$ is equivalent to a dual frame of $\{F_w\}_{w\in\Omega}$ if and only if there exists an adjointable and invertible operator ξ on $\mathcal{H}$ such that*

$$\xi f x = \int_\Omega \langle f, F_w \rangle g_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

(2) *$\{g_w\}_{w\in\Omega}$ is equivalent to a dual frame of $\{F_w\}_{w\in\Omega}$ if and only if there exists an adjointable and invertible operator Γ such that $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$.*

Proof. First, assume that $\{g_w\}_{w\in\Omega}$ is equivalent to a dual frame of $\{F_w\}_{w\in\Omega}$. Then there exists an adjointable and invertible operator Γ on $\mathcal{H}$ such that $\{\Gamma g_w\}_{w\in\Omega}$ is a dual for $\{F_w\}_{w\in\Omega}$. Now, for $x \in \mathcal{H}$,

$$= \int_\Omega \langle x, F_w \rangle \Gamma g_w d\mu(\omega).$$

Set $\xi = \Gamma^{-1}$. Then it concludes

$$\begin{aligned} \xi x &= \Gamma^{-1} x \\ &= \Gamma^{-1} \left(\int_\Omega \langle x, F_w \rangle \Gamma g_w d\mu(\omega) \right) \\ &= \int_\Omega \langle x, F_w \rangle g_w d\mu(\omega). \end{aligned}$$

In the second step, the adjointable and invertible operator ξ on $\mathcal{H}$ satisfies the following property

$$\xi x = \int_\Omega \langle x, F_w \rangle g_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

Since ξ is invertible,

$$x = \int_\Omega \langle x, F_w \rangle \xi^{-1} g_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

It shows that $\left\{\xi^{-1} g_w\right\}_{w\in\Omega}$ is a dual for $\{F_w\}_{w\in\Omega}$, and is equivalent to $\{g_w\}_{w\in\Omega}$.

For the proof of "if" part, assume that there exists a dual frame $\{h_w\}_{w\in\Omega}$ for $\{F_w\}_{w\in\Omega}$ such that $\{h_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ are equivalent. Then there is an adjointable and invertible operator

$\Lambda : \mathcal{H} \to \mathcal{H}$ such that $\Lambda g_w = h_w$ for all $w \in \Omega$. By Theorem 3.2, the sequence $\{g_w\}_{w\in\Omega}$ is a $*$-continuous frame.

On the other hand, for $x \in \mathcal{H}$,

$$\begin{aligned} x &= \int_\Omega \langle x, h_w\rangle F_w d\mu(\omega) \\ &= \int_\Omega \langle x, \Lambda g_w\rangle F_w d\mu(\omega) \\ &= \int_\Omega \langle \Lambda^* x, g_w\rangle F_w d\mu(\omega) \end{aligned}$$

and so

$$x = \int_\Omega \langle \Lambda^* x, g_w\rangle F_w d\mu(\omega).$$

This shows that $\{(g_w, \Lambda^*)\}$ is an operator dual for $\{F_w\}_{w\in\Omega}$. The converse part is clear by the last equalities. □

Theorem 3.5. *Let $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ be $*$-frames for $\mathcal{H}$. Then $\{g_w\}_{w\in\Omega}$ is equivalent to an operator dual frame of $\{F_w\}_{w\in\Omega}$ if and only if there exists an adjointable and invertible operator ξ on $\mathcal{H}$ such that*

$$\xi x = \int_\Omega \langle x, g_w\rangle F_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

Proof. Suppose that $\{g_w\}_{w\in\Omega}$ is equivalent to $\{h_w\}_{w\in\Omega}$, where $\left(\{h_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$. Then there exists an adjointable and invertible operator θ on $\mathcal{H}$ such that $\theta g_w = h_w$ for all $w \in \Omega$ and for $x \in \mathcal{H}$,

$$\begin{aligned} x &= \int_\Omega \langle \Gamma x, h_w\rangle F_w \\ &= \int_\Omega \langle \Gamma x, \theta g_w\rangle F_w d\mu(\omega) \\ &= \int_\Omega \langle \theta^* \Gamma x, g_w\rangle F_w d\mu(\omega). \end{aligned}$$

Set $\xi = (\theta^*\Gamma)^{-1}$. Then the result is obtained.

For the converse, let ξ be an adjointable and invertible operator on $\mathcal{H}$ that

$$\xi x = \int_\Omega \langle x, g_w\rangle F_w d\mu(\omega).$$

Then

$$x = \int_\Omega \left\langle \xi^{-1} x, g_w\right\rangle F_w d\mu(\omega), \quad \forall x \in \mathcal{H}$$

and $\left(\{g_w\}_{w\in\Omega}, \xi^{-1}\right)$ is an operator dual of $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ is equivalent to itself. □

Moreover, some equivalence frames have the same Grammian matrices. These frames are introduced in the following proposition.

Proposition 3.6. *Let $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ be equivalent Parseval frames for $\mathcal{H}$ and let $\mathcal{G}_\mathcal{F}$ and $\mathcal{G}_\mathcal{G}$ be Grammian matrices of $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$, respectively. Then $\mathcal{G}_\mathcal{F} = \mathcal{G}_\mathcal{G}$.*

Proof. Since two frames $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ are equivalent, there exists an adjointable and invertible operator $\xi : \mathcal{H} \longrightarrow \mathcal{H}$ by $\xi F_w = g_w$ for $w \in \Omega$. Since their frame operators are the identity operator on $\mathcal{H}$, by Theorem 3.2,

$$\xi\xi^* = \xi id\xi^* = id.$$

So ξ is a unitary operator and then for $i, w \in \Omega$,

$$\langle g_i, g_w\rangle = \langle \xi F_i, \xi F_w\rangle = \langle F_i, F_w\rangle ,$$

which shows that $\mathcal{G}_{\mathcal{F}} = [\langle F_i, F_w\rangle]_{w\in\Omega} = [\langle g_i, g_w\rangle]_{w\in\Omega} = \mathcal{G}_{\mathcal{G}}$. □

4. Constructed $*$-continuous frames and some properties

Theorem 4.1. *Let $\{F_w\}_{w\in\Omega}$ be a $*$-continuous frame for $\mathcal{H}$ and ξ be an adjointable and invertible operator on $\mathcal{H}$. Then the set $\left(\{g_w\}_{w\in\Omega}, \Gamma\xi^{-1}\right)$ is all of operator duals of $\{\xi F_w\}_{w\in\Omega}$, where $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$.*

Proof. Let $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ be an operator dual of $\{F_w\}_{w\in\Omega}$. Then for $x \in \mathcal{H}$,

$$\begin{aligned}\int_\Omega \left\langle \Gamma\xi^{-1}x, g_w\right\rangle \xi x_w d\mu(\omega) &= \xi\left(\int_\Omega \left\langle \Gamma\xi^{-1}f, g_w\right\rangle F_w d\mu(\omega)\right)\\ &= \xi\left(\xi^{-1}\right)x\\ &= x.\end{aligned}$$

This shows that $\left(\{g_j\}_{j\in J}, \Gamma\xi^{-1}\right)$ is an operator dual of $\{\xi f_j\}_{j\in J}$.

Now, if $\left(\{g_j\}_{j\in J}, \Gamma\right)$ is an operator dual for $\{\xi f_j\}_{j\in J}$, then it is enough to set $\Gamma := \Gamma\xi$ in the last equalities which follows that $\left(\{g_j\}_{j\in J}, \Gamma\right)$ is an operator dual for $\{f_j\}_{j\in J}$. □

An orthogonal projection will obtain a $*$-frame, and relation will also be given for this projection. To see this, we must show that the inverse of the frame operator is unique in the reconstruction formula. So, firstly this fact will be considered.

Theorem 4.2. *If $\{F_w\}_{w\in\Omega}$ is a $*$-continuous frame for $\mathcal{H}$, then there exists a unique adjointable operator Λ on $\mathcal{H}$ such that*

$$x = \int_\Omega \langle x, \Lambda F_w\rangle F_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

Proof. By the reconstruction formula, there exists $\Lambda = S^{-1}$. For the uniqueness of S^{-1} with this property, we know that $\left\{S^{-\frac{1}{2}}F_w\right\}_{w\in\Omega}$ is a continuous Parseval frame for $\mathcal{H}$. Set $g_w = S^{-\frac{1}{2}}F_w$. Then $F_w = S^{\frac{1}{2}}g_w$. Now, suppose that Λ is an adjointable operator such that

$$x = \int_\Omega \langle x, \Lambda F_w\rangle F_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

Then we have

$$\begin{aligned} x &= \int_\Omega \langle x, \Lambda F_w \rangle F_w d\mu(\omega) \\ &= \int_\Omega \left\langle x, \Lambda S^{\frac{1}{2}} g_w \right\rangle S^{\frac{1}{2}} g_w d\mu(\omega) \\ &= S^{\frac{1}{2}} \left(\int_\Omega \left\langle x, \Lambda S^{\frac{1}{2}} g_w \right\rangle g_w d\mu(\omega) \right) \\ &= S^{\frac{1}{2}} \left(\int_\Omega \left\langle S^{\frac{1}{2}} \Lambda^* x, g_w \right\rangle g_w \right) \\ &= S^{\frac{1}{2}} \left(S^{\frac{1}{2}} \Lambda^* x \right) \\ &= S\Lambda^* x, \quad \forall x \in \mathcal{H}. \end{aligned}$$

This concludes that $S\Lambda^* = id$ and then $\Lambda^* = S^{-1}$. More precisely, Λ is self-adjoint, positive and invertible. □

Now, a $*$-continuous frame is constructed by an orthogonal projection.

Proposition 4.3. *Let $\{F_w\}_{w\in\Omega}$ be a $*$-continuous frame for $\mathcal{H}$ with the frame operator S and $*$-continuous frame bounds A and B. Also, suppose that P is an orthogonal projection on $\mathcal{H}$. Then $\{PF_w\}_{w\in\Omega}$ is a $*$-continuous frame for R_P with $*$-continuous frame bounds A and B. Moreover, if $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual of $\{F_w\}_{w\in\Omega}$, then $\{P\Gamma^* g_w\}_{w\in\Omega}$ is a dual $*$-continuous frame for $\{Pf_w\}_{w\in\Omega}$.*

Proof. For $x \in R_P$,

$$\begin{aligned} \int_\Omega \langle x, PF_w \rangle \langle PF_w, x \rangle d\mu(\omega) &= \int_\Omega \langle Px, F_w \rangle \langle F_w, Px \rangle d\mu(\omega) \\ &= \int_\Omega \langle x, F_w \rangle \langle F_w, x \rangle d\mu(\omega) \end{aligned}$$

and by the definition of $*$-continuous frame $\{F_w\}_{w\in\Omega}$, we have

$$A\langle x, x\rangle A^* \le \int_\Omega \langle x, PF_w \rangle \langle PF_w, x \rangle d\mu(\omega) \le B\langle x, x\rangle B^*.$$

Now, if $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual of $\{F_w\}_{w\in\Omega}$, then for $x \in R_P$,

$$\begin{aligned} x &= Px \\ &= P\left(\int_\Omega \langle \Gamma Px, g_w \rangle F_w d\mu(\omega) \right) \\ &= \int_\Omega \langle x, P\Gamma^* g_w \rangle PF_w d\mu(\omega). \end{aligned}$$

If $\{g_w\}_{w\in\Omega}$ is also a dual of $\{F_w\}_{w\in\Omega}$, then $\{Pg_w\}_{w\in\Omega}$ is a dual of $\{PF_w\}_{w\in\Omega}$. So the result is clear by $\Gamma = id_{\mathcal{H}}$. □

By the last theorem, a necessary and sufficient condition is found for commutating a projection with the inverse of the frame operator of a given $*$-frame.

Theorem 4.4. *Let $\{F_w\}_{w\in\Omega}$ be a $*$-continuous frame for $\mathcal{H}$ with the frame operator S. Suppose that P is an orthogonal projection on $\mathcal{H}$. Then $PS^{-1}F_w = S_P^{-1}PF_w$, for all $w \in \Omega$ if and only if $PS^{-1} = S^{-1}P$, where S_P is the continuous frame operator of the $*$-continuous frame $\{PF_w\}_{w\in\Omega}$.*

Proof. First, assume that $PS^{-1}F_w = S_P^{-1}PF_w$, for all $w \in \Omega$. Now, let $x \in \mathcal{H}$. Then we have

$$\begin{aligned} S_P^{-1}Px &= S_P^{-1}P\left(\int_\Omega \left\langle x, S^{-1}F_w\right\rangle F_w d\mu(\omega)\right) \\ &= \int_\Omega \left\langle x, S^{-1}F_w\right\rangle S_P^{-1}PF_w d\mu(\omega) \\ &= \int_\Omega \left\langle x, S^{-1}F_w\right\rangle PS^{-1}F_w d\mu(\omega) \\ &= PS^{-1}\left(\int_\Omega \left\langle x, S^{-1}F_w\right\rangle F_w d\mu(\omega)\right) \\ &= PS^{-1}x. \end{aligned}$$

Therefore, $PS^{-1}x = S_P^{-1}Px$, for all $x \in \mathcal{H}$, and so

$$S_P^{-1}P = PS^{-1}P \Rightarrow PS^{-1}P = PS^{-1}.$$

Thus

$$PS^{-1}P = \left(PS^{-1}P\right)^* = \left(PS^{-1}\right)^* = S^{-1}P.$$

For the proof of converse, suppose that $PS^{-1} = S^{-1}P$. Let $x \in R_P$. Then we have

$$\begin{aligned} x &= Px \\ &= P\left(\int_\Omega \left\langle x, S^{-1}F_w\right\rangle F_w d\mu(\omega)\right) \\ &= \int_\Omega \left\langle x, S^{-1}F_w\right\rangle PF_w d\mu(\omega) \\ &= \int_\Omega \left\langle Px, S^{-1}F_w\right\rangle PF_w d\mu(\omega) \\ &= \int_\Omega \left\langle x, PS^{-1}F_w\right\rangle PF_w d\mu(\omega) \\ &= \int_\Omega \left\langle x, S^{-1}PF_w\right\rangle PF_w d\mu(\omega). \end{aligned}$$

By Theorem 4.2 and the assumption, for $w \in \Omega$,

$$S^{-1}PF_w = S_P^{-1}F_w = S_P^{-1}PF_w = PS^{-1}F_w$$

and the proof is complete. □

Theorem 4.5. *Let $\{F_w\}_{w\in\Omega}$ be a continuous Parseval frame of $\mathcal{H}$ with pre-frame operator $\theta_{\mathcal{F}}$ and let $\{g_w\}_{w\in\Omega}$ be a $*$-continuous frame with the pre-frame operator $\theta_{\mathcal{G}}$. Then $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$ if and only if $P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma = \theta_{\mathcal{F}}$, where $P_{\theta_{\mathcal{F}}}$ is the orthogonal projection on the range of $\theta_{\mathcal{F}}$.*

Proof. Suppose that $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$. Since $\{F_w\}_{w\in\Omega}$ is a continuous Parseval frame, the pre-frame operator $\theta_{\mathcal{F}}$ is an isometry

$$\langle \theta_{\mathcal{F}} x, \theta_{\mathcal{F}} x\rangle = \int_\Omega \langle x, F_w\rangle \langle F_w, x\rangle \, d\mu(\omega) = \langle x, x\rangle, \quad \forall x \in \mathcal{H}.$$

Then for $x \in \mathcal{H}$,

$$\begin{aligned}\left\langle P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma x, \theta_{\mathcal{F}} x\right\rangle &= \left\langle \theta_{\mathcal{G}}\Gamma x, P_{\theta_{\mathcal{F}}}\theta_{\mathcal{F}} x\right\rangle \\ &= \langle \theta_{\mathcal{G}}\Gamma f, \theta_{\mathcal{F}} x\rangle \\ &= \left\langle \int_\Omega \langle \Gamma x, g_w\rangle F_w d\mu(\omega), x\right\rangle \\ &= \langle x, x\rangle \\ &= \langle \theta_{\mathcal{F}} x, \theta_{\mathcal{F}} x\rangle .\end{aligned}$$

Thus $P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma = \theta_{\mathcal{F}}$.

Conversely, since $\theta_{\mathcal{F}}$ is an isometry, by a similar method, we have

$$\begin{aligned}\langle x, g\rangle &= \langle \theta_{\mathcal{F}} x, \theta_{\mathcal{F}} g\rangle \\ &= \left\langle P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma x, \theta_{\mathcal{F}} g\right\rangle \\ &= \left\langle \int_\Omega \langle \Gamma x, g_w\rangle F_w d\mu(\omega), g\right\rangle, \quad \forall x \in \mathcal{H},\end{aligned}$$

and so

$$x = \int_\Omega \langle \Gamma x, g_w\rangle F_w d\mu(\omega), \quad \forall x \in \mathcal{H}.$$

Thus $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is a dual for $\{F_w\}_{w\in\Omega}$. □

Corollary 4.6. *Let $\{F_w\}_{w\in\Omega}$ and $\{g_w\}_{w\in\Omega}$ be two $*$-continuous frames for $\mathcal{H}$ with pre-frame operators $\theta_{\mathcal{F}}$ and $\theta_{\mathcal{G}}$, respectively. Then $\left(\{g_w\}_{w\in\Omega}, \Gamma\right)$ is an operator dual for $\{F_w\}_{w\in\Omega}$ if and only if $\theta_{\mathcal{F}}^* P_{\theta_{\mathcal{F}}}\theta_{\mathcal{G}}\Gamma = id$.*

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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