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Best Proximity Point of Generalized $\theta - \phi-$Proximal Non-self Contractions

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Abstract. In this manuscript, motivated and inspired by results of Best proximity point of generalized F-proximal non-self contractions, we introduce the concept of generalized $\theta-\phi-$proximal contraction and prove new best proximity results for these contractions in the setting of a metric space. Our results generalize and extend many recent results appearing in the literature. An example is being given to demonstrate the usefulness of our results.

1. Introduction

It is well known that the Banach contraction theorem is the first outstanding result in the field of the fixed point theory that ensure the existence of unique fixed point in complete metric spaces. Due to its importance, various mathematics steadied many interesting extensions and generalizations [7,8,12,14]. One of the famous generalizations of the Banach contraction principle [2] for existence of fixed point for self-mapping on metric space is the theorem by Zheng et al. [14] and the contraction introduced by Jleli and Samet in [6].

Best proximity point theorem analyses the condition under which the optimisation problem, namely $\inf_{x\in A} d(x, Tx)$, has a solution. The point x is called the best proximity of $T : A \to B$, if $d(x, Tx) = d(A, B)$, where $\{d(A, B) = \inf d(x, y) : x \in A, y \in B\}$. Note that the best proximity point reduces to a fixed point if T is a self-mapping. Various best proximity point results were established on such spaces [1,9,12].

Sankar Raj [10] and Zhang et al. [13] defined the notion of $P-$property and weak $P-$property respectively. Beg et al. [4] defined the concept of generalized F-proximal non-self contractions and obtained some best proximity point theorems for self-mappings.

In this paper, inspired by the idea of generalized F-proximal non-self contractions, introduced by Beg et al. [4] in metric spaces, we prove a new existence of best proximity point for generalized

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$\theta - \phi-$proximal contraction defined on a closed subset of a complete metric space. Our theorems extend, generalize and improve many existing results.

2. Preliminaries

Let (A, B) be a pair of non empty subsets of a metric space (X, d). We adopt the following notations:

$d(A, B) = \{\inf d(a, b) : a \in A, b \in B\}$;

$A_0 = \{\ a \in A$ there exists $b \in A$ such that $d(a, b) = d(A, B)\}$;

$B_0 = \{\ b \in B$ there exists $a \in A$ such that $d(a, b) = d(A, B)\}$.

Definition 2.1. [5] Let $T : A \to B$ be a mapping. An element x^* is said to be a best proximity point of T if

$$d(x^*, Tx^*) = d(A, B).$$

Definition 2.2. [10] Let (A, B) be a pair of non empty subsets of a metric space (X, d) such that A_0 is non empty. Then the pair (A, B) is to have P-property if and only if

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Definition 2.3. [3] A set B is called approximately compact with respect to A if every sequence $\{x_n\}$ of B with $d(y, x_n) \to d(y, B)$ for some $y \in A$ has a convergent subsequence.

Definition 2.4. [6] Let Θ be the family of all functions $\theta :]0, +\infty[\to]1, +\infty[$ such that

(θ_1) θ is strictly increasing;

(θ_2) For each sequence $x_n \in]0, +\infty[$;

$$\lim_{n\to 0} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

(θ_3) θ is continuous.

Definition 2.5. [14] Let Φ be the family of all functions ϕ: $[1, +\infty[\to [1, +\infty[$, such that

(ϕ_1) ϕ is increasing;

(ϕ_2) For each $t \in]1, +\infty[$, $lim_{n\to\infty}\phi^n(t) = 1$;

(ϕ_3) ϕ is continuous.

Lemma 2.6. *[14] If $\phi \in \Phi$ Then $\phi(1)$=1, and $\phi(t) < t$.*

Definition 2.7. [14]. Let (X, d) be a metric space and $T : X \to X$ be a mapping.

T is said to be a $\theta - \phi-$contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \theta[d(Tx, Ty)] \leq \phi[\theta(d(x, y))],$$

3. Main result

In this section, inspired by the notion of F-proximal contraction of the first kind and second kind, we introduce new generalized $\theta-\phi$-proximal first kind and second kind on complete metric space.

Definition 3.1. The mapping $T: A \to B$ is said to be a generalized $\theta-\phi$-proximal contraction of first kind if there exist $\theta \in \Theta$, $\phi \in \Phi$ and $a, b, c, h \geq 0$ with $a+b++2ch$, $c \neq 1$ such that

$$\begin{cases} d\left(u_1, Tv_1\right) = d\left(A, B\right) \\ d\left(u_2, Tv_2\right) = d\left(A, B\right) \end{cases} \Rightarrow \theta(d(u_1, u_2))$$

$$\leq \phi\left[\theta\left[ad\left(v_1, v_2\right) + bd\left(u_1, v_1\right) + cd\left(u_2, v_2\right) + h\left(d\left(v_1, u_2\right) + d\left(v_2, u_1\right)\right)\right]\right]$$

for all $u_1, u_2, v_1, v_2 \in A$ and $u_1 \neq v_1$.

Definition 3.2. The mapping $T: A \to B$ is said to be a generalized $\theta-\phi$-proximal contraction of second kind if there exist $\theta \in \Theta$, $\phi \in \Phi$ and $a, b, c, h \geq 0$ with $a+b++2ch$, $c \neq 1$ such that

$$\begin{cases} d\left(u_1, Tv_1\right) = d\left(A, B\right) \\ d\left(u_2, Tv_2\right) = d\left(A, B\right) \end{cases} \Rightarrow \theta(d(Tu_1, Tu_2))$$

$$\leq \phi\left[\theta\left[ad\left(Tv_1, Tv_2\right) + bd\left(Tu_1, Tv_1\right) + cd\left(Tu_2, Tv_2\right) + h\left(d\left(Tv_1, Tu_2\right) + d\left(Tv_2, Tu_1\right)\right)\right]\right]$$

for all $u_1, u_2, v_1, v_2 \in A$ and $Tu_1 \neq Tv_1$.

Theorem 3.3. *Let (X, d) be a complete metric space and (A, B) be a pair of non-void closed subsets of (X, d). If B is approximately compact with respect to A and $T: A \to B$ satisfy the following conditions :*

(i) *$T\left(A_0\right) \in B_0$ and the pair (A, B) satisfies the weak P-property;*
(ii) *T is a generalized $\theta-\phi$-proximal contraction of first kind.*

Then there exists a unique $u \in A$ such that $d(u, Tu) = d(A, B)$. In addition, for any fixed element $u_0 \in A_0$, sequence $\{u_n\}$ defined by

$$d(u_{n+1}, Tu_n) = d(A, B),$$

converges to the proximity point.

Proof. Choose an element $u_0 \in A_0$. As, $T\left(A_0\right) \in B_0$, therefore there is an element $u_1 \in A_0$ satisfying

$$d(u_1, Tu_0) = d(A, B).$$

Since $T(A_0) \in B_0$, there exists $u_2 \in A_0$ such that

$$d(u_2, Tu_1) = d(A, B).$$

Again, since $T(A_0) \in B_0$, there exists $u_3 \in A_0$ such that

$$d(u_3, Tu_2) = d(A, B).$$

Continuing this process, by induction, we construct a sequence $x_n \in A_0$ such that

$$d\left(u_{n+1}, Tu_n\right) = d(A, B), \forall n \in \mathbb{N}.$$

Since (A, B) satisfies the P property, we conclude that

$$d(u_n, u_{n+1}) = d(Tu_n, Tu_{n+1}), \forall n \in \mathbb{N}. \tag{3.1}$$

If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, from (3) one obtains

$$d\left(u_{n_0}, Tu_{n_0}\right) = d\left(u_{n_0+1}, Tu_{n_0}\right) = d(A, B) \tag{3.2}$$

that is, $u_{n_0} \in BPP$. Thus, we suppose that $d(u_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.
We shall prove that the sequence u_n is a Cauchy sequence. Let us first prove that

$$\lim_{n \to \infty} d\left(u_n, u_{n+1}\right) = 0.$$

As T is generalized (θ, ϕ)-proximal contraction of the first kind, we have that

$$\begin{aligned}
&\theta\left(d\left(u_n, u_{n+1}\right)\right) \\
&\leq \phi\left[\theta\left[ad\left(u_{n-1}, u_n\right) + bd\left(u_{n-1}, u_n\right) + cd\left(u_n, u_{n+1}\right) + h\left(d\left(u_{n-1}, u_{n+1}\right) + d\left(u_n, u_n\right)\right)\right]\right] \\
&= \phi\left[\theta\left[ad\left(u_{n-1}, u_n\right) + bd\left(u_{n-1}, u_n\right) + cd\left(u_n, u_{n+1}\right) + h\left(d\left(u_{n-1}, u_{n+1}\right)\right)\right]\right] \\
&\leq \phi\left[\theta\left[ad\left(u_{n-1}, u_n\right) + bd\left(x_{n-1}, x_n\right) + cd\left(u_n, u_{n+1}\right) + h\left(d\left(u_{n-1}, u_n\right) + d\left(u_n, u_{n+1}\right)\right)\right]\right] \\
&= \phi\left[\theta\left[(a+b+h)d\left(u_{n-1}, u_n\right) + (c+h)d\left(u_n, u_{n+1}\right)\right]\right]
\end{aligned}$$

Since θ is strictly increasing and by Lemma 2.6, we deduce

$$d\left(x_n, x_{n+1}\right) < (a+b+h)d\left(x_{n-1}, x_n\right) + (c+h)d\left(x_n, x_{n+1}\right).$$

Thus

$$d\left(u_n, u_{n+1}\right) < \frac{a+b+h}{1-c-h}\left(d\left(u_{n-1}, x_n\right)\right).$$

If $b+b+c+2h = 1$, we have $0 < 1-c-h$ and so

$$d\left(u_n, u_{n+1}\right) \leq \frac{a+b+h}{1-c-h}\left(d\left(u_{n-1}, u_n\right)\right) = d\left(u_{n-1}, u_n\right), \forall n \in \mathbb{N};$$

Consequently,

$$\theta\left(d\left(u_n, u_{n+1}\right)\right) \leq \phi\left[\theta\left(d\left(u_{n-1}, u_n\right)\right)\right]$$

If $b+b+c+2h < 1$, we have $0 < 1-c-h$ and so

$$d\left(u_n, u_{n+1}\right) < d\left(u_{n-1}, u_n\right), \forall n \in \mathbb{N};$$

Consequently,

$$\theta\left(d\left(u_n, u_{n+1}\right)\right) \leq \phi\left[\theta\left(d\left(u_{n-1}, u_n\right)\right)\right]$$

It implies

$$\begin{aligned}\theta\left(d\left(u_n, u_{n+1}\right)\right) &\leq \phi\left[\theta\left(d(x_{n-1}, u_n\right)\right] \\ &\leq \phi^2\left[\theta\left(d(u_{n-2}, u_{n-1}\right)\right] \\ &\leq \ldots \leq \phi^n\left[\theta\left(d(u_0, u_1\right)\right].\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$1 \leq \theta(d\left(u_n, u_{n+1}\right)) \leq \lim_{n\to\infty} \phi^n\left[\theta(d\left(u_0, u_1\right))\right] = 1.$$

Since $\theta \in \Theta$, we obtain

$$\lim_{n\to\infty} d\left(u_n, u_{n+1}\right) = 0. \tag{3.3}$$

Next, we shall prove that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n\to\infty} d\left(u_{n,} u_m\right) = 0$, for all $n \in \mathbb{N}$. Suppose to the contrary that exists $\varepsilon > 0$ and sequences $n_{(k)}$ and $m_{(k)}$ of natural numbers such that

$$m_{(k)} > n_{(k)} > k, \quad d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon, \quad D\left(u_{m_{(k)-1}}, u_{n_{(k)}}\right) < \varepsilon. \tag{3.4}$$

Using the triangular inequality, we find that,

$$\varepsilon \leq d\left(u_{m_{(k)}}, u_{n_{(k)}}\right) \leq d\left(u_{m_{(k)}}, u_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n_{(k)}}\right) \tag{3.5}$$

$$< \varepsilon + d\left(u_{n(k)-1}, u_{n_{(k)}}\right). \tag{3.6}$$

Then, by 3.4 and 3.22, it follows that

$$\lim_{k\to\infty} d\left(u_{m_{(k)}}, u_{n_{(k)}}\right) = \varepsilon. \tag{3.7}$$

Using the triangular inequality, we find that,

$$\varepsilon \leq d\left(u_{m_{(k)}}, u_{n_{(k)}}\right) \leq d\left(u_{m_{(k)}}, u_{n(k)+1}\right) + d\left(x_{n(k)+1}, u_{n_{(k)}}\right) \tag{3.8}$$

and

$$\varepsilon \leq d\left(u_{m_{(k)}}, u_{n_{(k)+1}}\right) \leq d\left(u_{m_{(k)}}, u_{n(k)}\right) + d\left(u_{n(k)}, u_{n_{(k)+1}}\right) \tag{3.9}$$

Then, by (3.25) and (3.9), it follows that

$$\lim_{k\to\infty} d\left(u_{m_{(k)}}, u_{n_{(k)+1}}\right) = \varepsilon. \tag{3.10}$$

Similarly method, we conclude that

$$\lim_{k\to\infty} d\left(u_{m_{(k)+1}}, u_{n_{(k)}}\right) = \varepsilon. \tag{3.11}$$

Using again the triangular inequality,

$$d\left(u_{m_{(k)+1}}, u_{n_{(k)+1}}\right) \leq d\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + d\left(u_{m(k)}, u_{n_{(k)}}\right) + d\left(u_{n_{(k)}}, u_{n_{(k)+1}}\right). \tag{3.12}$$

On the other hand, using triangular inequality, we have

$$d\left(u_{m_{(k)}}, u_{n_{(k)}}\right) \leq d\left(u_{m_{(k)}}, u_{m_{(k)+1}}\right) + d\left(u_{m_{(k)+1}}, u_{n_{(k)+1}}\right) + d\left(u_{n_{(k)+1}}, u_{n_{(k)}}\right). \tag{3.13}$$

Letting $k \to \infty$ in inequality (3.12) and (3.13), we obtain

$$\lim_{k\to\infty} d\left(u_{m_{(k)+1}}, u_{n_{(k)+1}}\right) = \varepsilon. \tag{3.14}$$

Substituting $u_1 = x_{m_{(k)+1}}, u_2 = x_{n_{(k)+1}}, v_1 = u_{m_{(k)}}$ and $v_1 = u_{n_{(k)}}$ in assumption of the theorem, we get

$$\theta\left(d\left(u_{m_{(k)+1}}, u_{n_{(k)+1}}\right)\right) \leq \phi\left\{\theta\left\{\begin{array}{l} ad\left(u_{m_{(k)}}, u_{n_{(k)}}\right) \\ + bd\left(u_{m_{(k)}1}, u_{n_{(k)}}\right) \\ + cd\left(u_{n_{(k)}+1}, u_{n_{(k)}}\right) \\ + h(d\left(u_{m_{(k)}}, u_{n_{(k)}+1}\right) + d\left(u_{n_{(k)}}, u_{m_{(k)}+1}\right)) \end{array}\right\}\right\} \tag{3.15}$$

Letting Letting $k \to \infty$ in (3.15), and using (θ_1), (θ_3) , (ϕ_3) and Lemma (2.6) we obtain

$$\theta(\varepsilon) \leq \phi\left[\theta\left(a\varepsilon + b\varepsilon + c\varepsilon + 2h\varepsilon\right)\right].$$

We derive

$$\varepsilon < \varepsilon.$$

Which is a contradiction. Thus $\lim_{n,m\to\infty} d(u_n, u_m) = 0$, which shows that $\{x_n\}$ is a Cauchy sequence. Then there exists $z \in A$ such that

$$\lim_{n\to\infty} d(u_n, u) = 0.$$

Also,

$$\begin{aligned} d(u, B) &\leq d(u, Tu_n) \\ &\leq d(u, x_{n+1}) + d(u_{n+1}, Tu_n) \\ &= d(u, u_{n+1}) + d(A, B) \\ &\leq d(u, u_{n+1}) + d(u, B). \end{aligned}$$

Therefore, $d(u, Tu_n) \to d(u, B)$. In spite of the fact that B is approximately compact with respect to A , the sequence $\{Tu_n\}$ has a subsequence $\{Tu_{n_k}\}$ converging to some element $v \in B$. So it turns out that

$$d(u, v) = \lim_{n\to\infty} d\left(u_{n_k+1}, Tu_{n_k}\right) = d(A, B). \tag{3.16}$$

Thus u must be an element of A_0. Again, since $T(A_0) \in B_0$, there exists $t \in A_0$ such that

$$d(t, Tu) = d(A, B) \tag{3.17}$$

for some element t in A. Using the weak p-property and (3.33) we have

$$d(u_{n_k+1}, t) = d(Pu_{n_k}, Pu), \forall n_k \in \mathbb{N}.$$

If for some n_0, $d(t, u_{n_0+1}) = 0$, consequently $d(Pu_{n_0}, Tu) = 0$. So $Pu_{n_0} = Tu$, hence $d(A, B) = d(u, Tu)$. Thus the conclusion is immediate. So let for any $n \geq 0$, $d(t, u_{n+1}) > 0$. Since T is a generalized (θ, ϕ)-proximal contraction of the first kind, it follows from this that

$$\theta(d(t, u_{n+1})) \leq \phi\left[\theta\left[ad\left(u, u_n\right) + bd\left(t, u\right) + cd\left(u_n, u_{n+1}\right) + h\left(d\left(u, u_{n+1}\right) + d\left(u_n, t\right)\right)\right]\right] \quad (3.18)$$

Since θ and ϕ are two continuous functions, by letting $n \to \infty$ in inequality (3.18), we obtain

$$\begin{aligned}\theta(d(t, u)) &\leq \phi\left[\theta\left[(b+h)\left(d\left(u, t\right)\right)\right]\right] \\ &\leq \phi\left[\theta\left[\left(d\left(u, t\right)\right)\right]\right] \\ &< \theta(d(t, u)).\end{aligned}$$

It is a contradiction. Therefore, $u = t$, that

$$d(u, Tu) = d(t, Tu) = d(A, B).$$

Uniqueness: Suppose that there is another best proximity point z of the mapping T such that

$$d(z, Tz) = d(A, B).$$

Since T is a generalized (θ, ϕ)-proximal contraction of the first kind, it follows from this that

$$\begin{aligned}\theta(d(z, u)) &\leq \phi\left[\theta\left[ad\left(z, u\right) + bd\left(z, z\right) + cd\left(u, u\right) + h\left(d\left(z, u\right) + d\left(z, u\right)\right)\right]\right] \\ &= \phi\left[\theta\left[(a+2h)d\left(z, u\right)\right]\right],\end{aligned}$$

which is a contradiction. Thus, z and u must be identical. Hence, T has a unique best proximity point. □

Next, we state and prove the best proximity point theorem for non-self generalized (θ, ϕ)-proximal contraction of the second kind.

Theorem 3.4. *Let (X, d) be a complete metric space and (A, B) be a pair of non-void closed subsets of (X, d). If A is approximately compact with respect to B and $T : A \to B$ satisfy the following conditions :*

(i) *$T(A_0) \in B_0$ and the pair (A, B) satisfies the weak P-property;*
(ii) *T is continuous generalized (θ, ϕ)-proximal contraction of second kind.*

Then there exists a unique $u \in A$ such that $d(u, Tu) = d(A, B)$ and $u_n \to u$, where u_0 is any fixed point in A_0 and $d(u_{n+1}, Tu_n) = d(A, B)$ for $n \geq 0$. Further, if z is another best proximity point of T, then $Tu = Tz$.

Proof. Similar to Theorem 3.3, we can find a sequence $\{u_n\}$ in A_0 such that

$$d(u_{n+1}, Tu_n) = d(A, B). \tag{3.19}$$

for all non-negative integral values of n. From the p-property and (3.19) we get

$$d(u_n, u_{n+1}) = d(Tu_{n-1}, Tu_n), \forall n \in \mathbb{N}.$$

If for some n_0, $d(u_{n_{0+1}}, u_{n_{0+2}}) = 0$, consequently $d(Tu_{n_0}, Tu_{n_0+1}) = 0$. So $Tu_{n_0} = Tu_{n_0+1}$, hence $d(A, B) = d(Tu_{n_0}, T_{n_0+1})$. Thus the conclusion is immediate. So let for any $n \geq 0$, $d(Tu_n, Tu_{n+1}) > 0$. We shall prove that the sequence u_n is a Cauchy sequence. Let us first prove that

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0.$$

As T is generalized (θ, ϕ)-proximal contraction of the second kind, we have that

$$\begin{aligned}
&\theta(d(Tu_n, Tu_{n+1}))\\
&\leq \phi[\theta[ad(Tu_{n-1}, Tu_n) + bd(Tu_{n-1}, Tu_n) + cd(Tu_n, Tu_{n+1}) + h(d(Tu_{n-1}, Tu_{n+1}) + d(Tu_n, Tu_n))]]\\
&= \phi[\theta[ad(Tu_{n-1}, Tu_n) + bd(Tu_{n-1}, Tu_n) + cd(Tu_n, Tu_{n+1}) + h(d(Tu_{n-1}, Tu_{n+1}))]]\\
&\leq \phi[\theta[ad(Tu_{n-1}, Tu_n) + bd(Tu_{n-1}, Tu_n) + cd(Tu_n, Tu_{n+1}) + h(d(Tu_{n-1}, Tu_n) + d(Tu_n, Tu_{n+1}))]]\\
&= \phi[\theta[(a+b+h)d(Tu_{n-1}, Tu_n) + (c+h)d(Tu_n, Tu_{n+1})]]
\end{aligned}$$

Since θ is strictly increasing and by Lemma 2.6, we deduce

$$d(Tu_n, Tu_{n+1}) < (a+b+h)d(Tu_{n-1}, Tu_n) + (c+h)d(Tu_n, Tu_{n+1}).$$

Thus

$$d(Tu_n, Tu_{n+1}) < \frac{a+b+h}{1-c-h}(d(Tu_{n-1}, Tu_n)).$$

If $b + b + c + 2h = 1$, we have $0 < 1 - c - h$ and so

$$d(Tu_n, Tu_{n+1}) \leq \frac{a+b+h}{1-c-h}(d(Tu_{n-1}, Tu_n)) = d(Tu_{n-1}, Tu_n), \forall n \in \mathbb{N};$$

Consequently,

$$\theta(d(Tu_n, Tu_{n+1})) \leq \phi[\theta(d(Tu_{n-1}, Tu_n))]$$

If $b + b + c + 2h < 1$, we have $0 < 1 - c - h$ and so

$$d(Tu_n, Tu_{n+1}) < d(Tu_{n-1}, Tu_n), \forall n \in \mathbb{N};$$

Consequently,

$$\theta(d(Tu_n, Tu_{n+1})) \leq \phi[\theta(d(Tu_{n-1}, Tu_n))]$$

It implies

$$\begin{aligned}\theta\left(d\left(Tu_n, Tu_{n+1}\right)\right) &\leq \phi\left[\theta\left(d(Tu_{n-1}, Tu_n)\right]\right. \\ &\leq \phi^2\left[\theta\left(d(Tu_{n-2}, Tu_{n-1})\right]\right. \\ &\leq \ldots \leq \phi^n\left[\theta\left(d(Tu_0, Tu_1)\right]\right. .\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$1 \leq \theta(d\left(Tu_n, Tu_{n+1}\right)) \leq \lim_{n\to\infty} \phi^n\left[\theta(d\left(Tu_0, Tu_1\right))\right] = 1.$$

Since $\theta \in \Theta$, we obtain

$$\lim_{n\to\infty} d\left(Tu_n, Tu_{n+1}\right) = 0. \tag{3.20}$$

Next, we shall prove that $\{Tu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n\to\infty} d\left(Tu_{n,}Tu_m\right) = 0$, for all $n \in \mathbb{N}$. Suppose to the contrary that exists $\varepsilon > 0$ and sequences $Tn_{(k)}$ and $Tm_{(k)}$ of natural numbers such that

$$Tm_{(k)} > Tn_{(k)} > k, \quad d\left(Tu_{m_{(k)}}, Tu_{n_{(k)}}\right) \geq \varepsilon, \quad d\left(Tu_{m_{(k)-1}}, Tu_{n_{(k)}}\right) < \varepsilon. \tag{3.21}$$

Using the triangular inequality, we find that,

$$\varepsilon \leq d\left(Tu_{m_{(k)}}, Tu_{n_{(k)}}\right) \leq d\left(Tu_{m_{(k)}}, Tx_{n(k)-1}\right) + d\left(Tu_{n(k)-1}, Tu_{n_{(k)}}\right) \tag{3.22}$$

$$< \varepsilon + d\left(Tu_{n(k)-1}, Tu_{n_{(k)}}\right). \tag{3.23}$$

Then, by 3.4 and 3.22, it follows that

$$\lim_{k\to\infty} d\left(Tu_{m_{(k)}}, Tu_{n_{(k)}}\right) = \varepsilon. \tag{3.24}$$

Using the triangular inequality, we find that,

$$\varepsilon \leq d\left(Tu_{m_{(k)}}, Tu_{n_{(k)}}\right) \leq d\left(Tu_{m_{(k)}}, Tu_{n(k)+1}\right) + d\left(Tu_{n(k)+1}, Tu_{n_{(k)}}\right) \tag{3.25}$$

and

$$\varepsilon \leq d\left(Tu_{m_{(k)}}, Tu_{n_{(k)+1}}\right) \leq d\left(Tu_{m_{(k)}}, Tu_{n(k)}\right) + d\left(Tu_{n(k)}, Tu_{n_{(k)+1}}\right) \tag{3.26}$$

Then, by (3.25) and (3.9), it follows that

$$\lim_{k\to\infty} d\left(Tu_{m_{(k)}}, Tu_{n_{(k)+1}}\right) = \varepsilon. \tag{3.27}$$

Similarly method, we conclude that

$$\lim_{k\to\infty} d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)}}\right) = \varepsilon. \tag{3.28}$$

Using again the triangular inequality,

$$d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)+1}}\right) \leq d\left(u_{m_{(k)+1}}, Tu_{m_{(k)}}\right) + d\left(Tu_{m(k)}, Tu_{n_{(k)}}\right) + d\left(Tu_{n_{(k)}}, Tu_{n_{(k)+1}}\right). \tag{3.29}$$

On the other hand, using triangular inequality, we have

$$d\left(Tu_{m_{(k)}}, Tu_{n_{(k)}}\right) \leq d\left(Tu_{m_{(k)}}, Tu_{m_{(k)+1}}\right) + d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)+1}}\right) + d\left(Tu_{n_{(k)+1}}, Tu_{n_{(k)}}\right). \quad (3.30)$$

Letting $k \to \infty$ in inequality (3.29) and (3.30), we obtain

$$\lim_{k\to\infty} d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)+1}}\right) = \varepsilon. \quad (3.31)$$

Substituting $u_1 = Tu_{m_{(k)+1}}, u_2 = Tu_{n_{(k)+1}}, v_1 = Tu_{m_{(k)}}$ and $v_1 = Tu_{n_{(k)}}$ in assumption of the theorem, we get

$$\theta\left(d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)+1}}\right)\right) \leq \phi\left\{\theta\left[\begin{cases} ad\left(Tu_{m_{(k)}}, Tu_{n_{(k)}}\right) \\ + bd\left(Tu_{m_{(k)}1}, Tu_{n_{(k)}}\right) \\ + cd\left(Tu_{n_{(k)}+1}, Tu_{n_{(k)}}\right) \\ + h(d\left(Tu_{m_{(k)}}, Tu_{n_{(k)}+1}\right) + d\left(Tu_{n_{(k)}}, Tu_{m_{(k)}+1}\right)) \end{cases}\right]\right\} \quad (3.32)$$

Letting Letting $k \to \infty$ in (3.32), and using (θ_1), (θ_3) , (ϕ_3) and Lemma (2.6) we obtain

$$\theta(\varepsilon) \leq \phi\left[\theta\left(a\varepsilon + b\varepsilon + c\varepsilon + 2h\varepsilon\right)\right].$$

We derive

$$\varepsilon < \varepsilon.$$

Which is a contradiction. Thus $\lim_{n,m\to\infty} d\left(Tu_n, Tu_m\right) = 0$, which shows that $\{Tu_n\}$ is a Cauchy sequence. Then there exists $v \in B$ such that

$$\lim_{n\to\infty} d\left(Tu_n, v\right) = 0.$$

Also,

$$\begin{aligned} d(v, A) &\leq d(v, Tu_n) \\ &\leq d(v, u_{n+1}) + d(u_{n+1}, Tu_n) \\ &= d(v, u_{n+1}) + d(A, B) \\ &\leq d(v, u_{n+1}) + d(v, A). \end{aligned}$$

Therefore, $d(v, Tu_n) \to d(v, A)$. Since A is approximately compact with respect to B , the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ converging to some element $u \in A$. So it turns out that

$$d(u, v) = \lim_{n\to\infty} d\left(u_{n_k+1}, Tu_{n_k}\right) = d(A, B). \quad (3.33)$$

Because T is a continuous mapping,

$$d(u, Tu) = \lim_{n\to\infty} d(u_{n+1}, Tu_n) = d(A, B).$$

Uniqueness: Suppose that there is another best proximity point z of the mapping T such that

$$d(z, Tz) = d(A, B).$$

Since T is a generalized (θ, ϕ)-proximal contraction of the first second, it follows from this that

$$\begin{aligned}\theta(d(Tz, Tu)) &\leq \phi\left[\theta\left[ad\left(Tz, Tu\right) + bd\left(Tz, Tz\right) + cd\left(Tu, Tu\right) + h\left(d\left(Tz, Tu\right) + d\left(Tz, Tu\right)\right)\right]\right] \\ &= \phi\left[\theta\left[(a + 2h)d\left(Tz, Tu\right)\right]\right],\end{aligned}$$

which is a contradiction. Thus, z and u must be identical. Hence, T has a unique best proximity point. □

Theorem 3.5. *Let (X, d) be a complete metric space and (A, B) be a pair of non-void closed subsets of (X, d). Let $T : A \to B$ satisfy the following conditions :*

(i) *$T(A_0) \in B_0$ and the pair (A, B) satisfies the weak P-property;*

(ii) *T is a generalized (θ, ϕ)-proximal contraction of the first kind as well as a generalized (θ, ϕ)-proximal contraction of the second kind.*

Then there exists a unique $u \in A$ such that $d(u, Tu) = d(A, B)$ and $u_n \to u$, where u_0 is any fixed point in A_0 and $d(u_{n+1}, Tu_n) = d(A, B)$ for $n \geq 0$.

Proof. Similar to Theorem 3.3, we find a sequence $\{u_n\}$ in A_0 such that

$$d(u_{n+1}, Tu_n) = d(A, B)$$

for all non-negative integral values of n. Similar to Theorem 3.3, we can show that sequence $\{u_n\}$ is a Cauchy sequence. Thus converges to some element u in A. As in Theorem 3.4, it can be shown that the sequence $\{Tu_n\}$ is a Cauchy sequence and converges to some element v in B. Therefore,

$$d(u, v) = \lim_{n\to\infty} d(u_{n+1}, Tu_n) = d(A, B). \tag{3.34}$$

Eventually, u becomes an element of A_0. In light of the fact that $T(A_0) \in B_0$,

$$d(t, Tu) = d(A, B)$$

for some element t in A. From the p-property framework and (3.34,) we get

$$d(u_{n+1}, t) = d(Tu_n, Tu), \forall n \in \mathbb{N}.$$

If for some n_0, $d(t, u_{n_0+1}) = 0$, consequently $d(Tu_{n_0}, Tu) = 0$. So $Tu_{n_0} = Tu$, hence $d(A, B) = d(u, Tu)$. Thus the conclusion is immediate. So let for any $n \geq 0$, $d(t, u_{n+1}) > 0$. Since T is a generalized (θ, ϕ)-proximal contraction of the first kind, it can be seen that

$$\theta(d(t, u_{n+1})) \leq \phi\left[\theta\left(ad(u, u_n) + bd(t, u) + cd(u_n, u_{n+1}) + h[d(u, u_{n+1}) + d(u_n, t)\right)\right]. \tag{3.35}$$

Since θ and ϕ are two continuous functions, by letting $n \to \infty$ in inequality (3.35), we obtain, $d(u, Tu) = d(t, Tu) = d(A, B)$. Also, as in the theorem 3.3, the uniqueness of the best proximity point of mapping T follows. □

Example 3.6. Let $X = \{\lambda_n : n \in \mathbb{N}\}$ with the metric $d(x, y) = |x - y|$ for all $x, y \in X$, where the sequence G_n, defined by

$$\lambda_1 = 1$$

$$\lambda_2 = 1 + 2$$

$$\lambda_3 = 1 + 2 + 3$$

$$\ldots$$

$$\lambda_n = 1 + 2 + 3 + \ldots + n.$$

We know, (X, d) is a complete metric space. Let $A = G_{3n} : n \in \mathbb{N}$ and $B = G_{3n-1} : n \in \mathbb{N}$. It is easy to see that $d(A, B) = 3$, $A_0 = A$ and $B_0 = B$. Define a mappings $T : A \to B$, by $T(\lambda_{3n}) = \lambda_{3n-1}$ for all $n \geq 1$. It is clear that A is approximately compact with respect to B, (A, B) satisfies the p-property, T is continuous and $T(A_0) \subseteq B_0$. We will show that T is an (θ, ϕ)-proximal contraction with $\theta \in \Theta$ and $\phi \in \Phi$ that is $\theta(t) = e^t$ and $\phi(t) = t^{\frac{1}{2}}$. Observe that, With out of generality, we may assume that $n < m$, and since

$$\lambda_{3n-1} = 1 + 2 + 3 + \ldots + 3n - 1,$$

$$\lambda_{3m-1} = 1 + 2 + 3 + \ldots + 3m - 1,$$

$$\lambda_{3n} = 1 + 2 + 3 + \ldots + 3n - 1 + 3n,$$

$$\lambda_{3m} = 1 + 2 + 3 + \ldots + 3m - 1 + 3m.$$

It follow that,

$$\begin{aligned} d(T(\lambda_{3n}), T(\lambda_{3m})) &= |\lambda_{3n-1} - \lambda_{3m-1}| \\ &= 3n + (3n + 1) + \ldots + (3m - 1), \end{aligned}$$

$$\begin{aligned} d(\lambda_{3n}, \lambda_{2m}) &= |\lambda_{2n} - \lambda_{2m}| \\ &= 3n + (3n + 1) + \ldots + (3m), \end{aligned}$$

and

$$\begin{aligned} d(T(\lambda_{2n}), T(\lambda_{3m})) - d(\lambda_{3n}, \lambda_{3m}) &= |\lambda_{3n-1} - \lambda_{3m-1}| - |\lambda_{3n} - \lambda_{3m}| \\ &= 3n - 3m. \end{aligned}$$

So that,

$$
\begin{aligned}
e^{d(T(\lambda_{3n}),T(\lambda_{3m}))-d(\lambda_{3n},\lambda_{3m})}) &= \frac{e^{d(T(\lambda_{3n}),T(\lambda_{3m}))}}{e^{d(\lambda_{3n},\lambda_{3m})}} \\
&= e^{3n-3m}) \\
&= e^{-3(m-n)}) \\
&\leq e^{-3} = \frac{1}{e^3}.
\end{aligned}
$$

So that,

$$
\begin{aligned}
e^{d(T(\lambda_{3n}),T(\lambda_{3m}))}+1 = \theta(d(T(\lambda_{3n}),T(\lambda_{3m}))) &\leq e^{d(\lambda_{3n},\lambda_{3m})}\frac{1}{e^3}+1 \\
&\leq \frac{e^{d(\lambda_{3n},\lambda_{3m})}+2}{2} \\
&= \phi\left[\theta(d(\lambda_{3n},\lambda_{3m}))\right].
\end{aligned}
$$

Consequently, T is an generalized (θ,ϕ)-proximal contraction of the second kind with $a=1$, $b=c=h=0$. Thus, all the conditions of Theorem 3.4 are satisfied. Hence, T has a unique best proximity point and there exist $\lambda_3 \in A$ such that

$$
d(\lambda_3, T\lambda_3) = d(\lambda_3,\lambda_2) = 3 = d(A,B)
$$

Conflict of interest

The authors declare that they have no competing interests.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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