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The Constants to Measure the Differences Between Isosceles and $\alpha-\beta$ Orthogonalities

Qichuan Ni, Qi Liu*, Yin Zhou, Qin Qian

School of Mathematics and Physics, Anqing Normal University, Anqing 246133, P. R. China
niqichuan111@163.com, liuq67@aqnu.edu.cn, zhouyin0330@163.com, 15212956918@163.com
Correspondence: liuq67@aqnu.edu.cn

Abstract. In this paper, by combining the isosceles orthogonality and $\alpha-\beta$ orthogonality of Banach spaces, we first introduce a new geometric constant. We demonstrate some basic properties about it, such as calculating its value in the common norm spaces. Moreover, the necessary and sufficient conditions for the new constant to characterize Hilbert spaces are given. Finally, only consider the points on the unit sphere, we introduce another new geometric constant and some basic properties are also obtained.

1. Introduction

Traditional orthogonality plays a key role in the geometry of Banach spaces and is a geometric feature of Hilbert spaces. We repeat the definitions of the next three orthogonality types. In 1945, James [8] introduced isosceles orthogonality as follows:

$$x \perp_I y \text{ if and only if } \|x+y\| = \|x-y\|.$$

Balestro [4] introduced the orthogonality of Pythagoras as follows:

$$x \perp_P y \text{ if and only if } \|x-y\|^2 = \|x\|^2 + \|y\|^2.$$

Birkhoff defined the following Birkhoff orthogonality [5] in linear metric spaces:

$$x \perp_B y \text{ if and only if } \|x+\alpha y\| \geq \|x\| \text{ for all } \alpha \in \mathbb{R}.$$

These three orthogonalities have been investigated in several papers (see [2], [9] and so on). Over the years, many scholars have introduced the concept of extended orthogonality. The family of orthogonalities introduced by Carlson [6] in 1961, covering isosceles and Pythagorean orthogonalities, is known as Carlson orthogonality:

$$x \perp_C y \text{ if and only if } \sum_{i=1}^{n} a_i \|b_i x + c_i y\|^2 = 0.$$

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A further special case of Carlson orthogonality was introduced by Dimitini et al. [7] in 1983 as follows:

$$x \perp_{\alpha} y \text{ if and only if } (1+\alpha^2)\|x-y\|^2 = \|x-\alpha y\|^2 + \|y-\alpha x\|^2,$$

where fixed $\alpha \neq 1$. Two years later, this orthogonality was generalized by the same author [3] as:

$$x \perp_{\alpha\beta} y \text{ if and only if } \|x-y\|^2 + \|\alpha x - \beta y\|^2 = \|x-\beta y\|^2 + \|y-\alpha x\|^2.$$

It is well known that in the general Banach spaces, these orthogonalities are not the same. For example, isosceles orthogonality has symmetry, but Birkhoff orthogonality does not have symmetry, which means that $x \perp_B y$ cannot show that $y \perp_B x$. Because of this difference, measuring the difference between the two types of orthogonality is of great significance. Many scholars have defined and studied many novel orthogonality geometric constants and given a large number of results, including well-known constants (see [11], [12]):

$$BI(X) = \sup\left\{\frac{\|x+y\| - \|x-y\|}{\|x\|} : x, y \in S_X, x, y \neq 0, x \perp_B y\right\}$$

and

$$BR(X) = \sup_{\alpha>0}\left\{\frac{\|x+\alpha y\| - \|x-\alpha y\|}{\alpha} : x, y \in S_X, x \perp_B y\right\}.$$

For more information, refer to references [1], [13–15]. Throughout the article, we use X to represent a real Banach space with norm $\|\cdot\|$, the unit ball is denoted as $B_X = \{x \in X : \|x\| \leq 1\}$ and the unit sphere is denoted as $S_X = \{x \in X : \|x\| = 1\}$. Let's assume that $\dim X$ is greater than or equal to 2.

2. The Constant $L_{\alpha,\beta}(X)$

Combining the extended isosceles and Pythagorean orthogonalities, we define a new constant to describe the difference between these two orthogonalities.

Definition 2.1. *Let X be a Banach space, the geometric constant of the isosceles orthogonality type is defined as*

$$L_{\alpha,\beta}(X) = \sup\left\{\frac{\|x-\beta y\|^2 + \|\alpha x - y\|^2}{\|x-y\|^2 + \|\alpha x - \beta y\|^2} : x \perp_I y, (x,y) \neq (0,0)\right\}, \text{ where } \alpha, \beta \geq 0, \alpha \neq 1, \beta \neq 1.$$

In this section, we first give bounds on the constant $L_{\alpha,\beta}(X)$ and its value in some particular spaces is obtained. We find that X is a Hilbert space if and only if the constant $L_{\alpha,\beta}(X)$ value is 1. Moreover, the relationship between $L_{\alpha,\beta}(X)$ and uniformly non-square is given. Now, we recall the notion of the uniformly non-square.

Definition 2.2. *([10]) A Banach space X is called uniformly non-square if there exists $\delta \in (0,1)$ such that for any $x, y \in S_X$, then*

$$\frac{\|x+y\|}{2} \leq 1-\delta \text{ or } \frac{\|x-y\|}{2} \leq 1-\delta.$$

Proposition 2.1. *Let X be a Banach space, then*

$$1 \leq L_{\alpha,\beta}(X) \leq \begin{cases} \frac{2}{1+\beta^2}, & 0 \leq \beta \leq \alpha < 1; \\ \frac{2}{1+\alpha^2}, & 0 \leq \alpha \leq \beta < 1; \\ \frac{\alpha^2+\beta^2}{1+\beta^2}, & 1 < \beta \leq \alpha; \\ \frac{\alpha^2+\beta^2}{1+\alpha^2}, & 1 < \alpha \leq \beta. \end{cases}$$

proof. Let $x_0 = 0, y_0 \neq 0,$ satisfy $x_0 \perp_I y_0$, hence

$$L_{\alpha,\beta}(X) \geq \frac{\|x_0 - \beta y_0\|^2 + \|\alpha x_0 - y_0\|^2}{\|x_0 - y_0\|^2 + \|\alpha x_0 - \beta y_0\|^2} = 1.$$

On the other hand, using the triangle inequality, we get:

$$\begin{aligned} &\frac{\|x - \beta y\|^2 + \|\alpha x - y\|^2}{\|x - y\|^2 + \|\alpha x - \beta y\|^2} \\ &\leq \frac{\left(\left|\frac{1-\beta}{2}\right| \|x+y\| + \left|\frac{1+\beta}{2}\right| \|x-y\|\right)^2 + \left(\left|\frac{\alpha-1}{2}\right| \|x+y\| + \left|\frac{\alpha+1}{2}\right| \|x-y\|\right)^2}{\|x-y\|^2 + \left(\left|\frac{\alpha-\beta}{2}\right| \|x+y\| - \left|\frac{\alpha+\beta}{2}\right| \|x-y\|\right)^2}. \end{aligned}$$

If we take the categorical approach, there will be the following four approaches.

Case 1 When $0 \leq \beta \leq \alpha < 1$, we have

$$\frac{\|x - \beta y\|^2 + \|\alpha x - y\|^2}{\|x - y\|^2 + \|\alpha x - \beta y\|^2} \leq \frac{2}{1+\beta^2}.$$

Case 2 When $0 \leq \alpha \leq \beta < 1$, we have

$$\frac{\|x - \beta y\|^2 + \|\alpha x - y\|^2}{\|x - y\|^2 + \|\alpha x - \beta y\|^2} \leq \frac{2}{1+\alpha^2}.$$

Case 3 When $1 < \beta \leq \alpha$, we have

$$\frac{\|x - \beta y\|^2 + \|\alpha x - y\|^2}{\|x - y\|^2 + \|\alpha x - \beta y\|^2} \leq \frac{\alpha^2+\beta^2}{1+\beta^2}.$$

Case 4 When $1 < \alpha \leq \beta$, we have

$$\frac{\|x - \beta y\|^2 + \|\alpha x - y\|^2}{\|x - y\|^2 + \|\alpha x - \beta y\|^2} \leq \frac{\alpha^2+\beta^2}{1+\alpha^2}.$$

□

Next, we will show that there are points in some particular spaces where the value of the constant $L_{\alpha,\beta}(X)$ is an upper bound.

Example 2.1. *Let $0 \leq \beta < 1$, $\alpha = \beta$ and X be the space $\mathbb{R}^2$ with l_1 norm defined by*

$$\|(x_1, x_2)\| = |x_1| + |x_2|,$$

then $L_{\alpha,\beta}(X) = \frac{2}{1+\beta^2}$.

Let $x = (1,1), y = (1,-1)$, *satisfy* $x \perp_I y$. *We get* $\|\alpha x - \beta y\| = 2\beta$, $\|x - y\| = \|x - \beta y\| = \|\alpha x - y\| = 2$.

Thus, $L_{\alpha,\beta}(X) = \frac{2}{1+\beta^2}$.

Example 2.2. *Let* $0 \leq \beta < 1$, $\alpha = \beta$ *and* X *be the space* $\mathbb{R}^2$ *with* l_∞ *norm defined by*

$$\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\},$$

then $L_{\alpha,\beta}(X) = \frac{2}{1+\beta^2}$.

Let $x = (1,0), y = (0,-1)$, *satisfy* $x \perp_I y$. *We get* $\|\alpha x - \beta y\| = \beta$, $\|x - y\| = \|x - \beta y\| = \|\alpha x - y\| = 1$.

Thus, $L_{\alpha,\beta}(X) = \frac{2}{1+\beta^2}$.

Proposition 2.2. *Let* X *be a Banach space,* $0 \leq \beta \leq \alpha < 1$, *then* $L_{\alpha,\beta}(X) = 1$ *if and only if* X *is a Hilbert space.*

proof. Since X is a Hilbert space, combined with reference [3], we have $L_{\alpha,\beta}(X) = 1$. Conversely, assuming that $L_{\alpha,\beta}(X) = 1$ and useing the homogeneity of $x \perp y$, we can prove by induction that

$$\|x - y\|^2 + \|\alpha^n x - \beta y\|^2 \geq \|\alpha^n x - y\|^2 + \|x - \beta y\|^2.$$

Since $0 \leq \alpha < 1$, taking the limit $n \to \infty$, we get

$$\|x - y\|^2 \geq \left(1 - \beta^2\right) \|y\|^2 + \|x - \beta y\|^2.$$

If $0 \leq \beta < 1$, a second induction shows that $\|x - y\|^2 = \left(1 - \beta^{2n}\right) \|y\|^2 + \|x - \beta^n y\|^2$. Also taking the limit $n \to \infty$, Therefore, in this case we get $\|x - y\|^2 \geq \|x\|^2 + \|y\|^2$. Then, $x \perp_I y$ implies

$$\|x + y\|^2 + \|x - y\|^2 \geq 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$, hence we can assert that X is a Hilbert space. □

Among the many properties of Banach spaces, we give below a sufficient condition that X is not a uniform non-square space. In the process of proving, we apply the lemma given by James.

Lemma 2.1. *[8, Lemma 4.1] Let* X *be a Banach space and* $x, y \in X$. *If* $x \perp_I y$, *then the following inequality holds.*

(i) $\|x + ky\| \leq |k|\|x \pm y\|$ *and* $\|x \pm y\| \leq \|x + ky\|$, *when* $|k| \geq 1$.

(ii) $\|x + ky\| \leq \|x \pm y\|$ *and* $|k|\|x \pm y\| \leq \|x + ky\|$, *when* $|k| \leq 1$.

Proposition 2.3. *Let* X *be a finite dimensional Banach space, if* $L_{\alpha,\beta}(X) = \frac{2}{1+\beta^2}$ *for some* $0 \leq \beta_0 \leq \alpha_0 < 1$, *then* X *is not uniformly non-square.*

proof. Since $L_{\alpha,\beta}(X) = \frac{2}{1+\beta^2}$, there exist $x_n \in S_X, y_n \in B_X$ that satisfy $x_n \perp_I y_n$ and

$$\lim_{n\to\infty} \frac{\|x_n - \beta y_n\|^2 + \|\alpha x_n - y_n\|^2}{\|x_n - y_n\|^2 + \|\alpha x_n - \beta y_n\|^2} = \frac{2}{1+\beta^2}.$$

At the same time, a Banach space X is finite dimensional, then there exist $x_0, y_0 \in B_X$ that satisfy $x_0 \perp_I y_0$ and

$$\lim_{k\to\infty} \left\|x_{n_k}\right\| = \|x_0\|, \lim_{k\to\infty} \left\|y_{n_k}\right\| = \|y_0\|.$$

Combine lemma 2.1, we have $\|x_n - \beta_0 y_n\| \le \|x_n + y_n\|$ and $\|\alpha_0 x_n - y_n\| \le \|x_n + y_n\|$.

Thus,

$$\frac{\|x_n + y_n\|^2 + \|x_n + y_n\|^2}{\|x_n + y_n\|^2 + (\left|\frac{\alpha-\beta}{2}\right| - \left|\frac{\alpha+\beta}{2}\right|)^2 \|x_n + y_n\|^2} \le \frac{2}{1+\beta^2},$$

$$\frac{\|x_n + y_n\|^2 + \|x_n + y_n\|^2}{(1+\beta^2)\|x_n + y_n\|^2} \le \frac{2}{1+\beta^2}.$$

We can obtain $\|x_0 - \beta_0 y_0\| = \|x_0 + y_0\|$ and $\|\alpha_0 x_0 - y_0\| = \|x_0 + y_0\|$. Since $\|x_0 - \beta_0 y_0\| \le (1-\beta_0)\|x_0\| + \beta_0 \|x_0 + y_0\|$, then $\|x_0 + y_0\| \le \|x_0\|$. Moreover, we can prove that $\|x_0 + y_0\| \le \|y_0\|$, then

$$\max\{\|x_0 + y_0\|, \|x_0 - y_0\|\} = \|x_0 + y_0\| \le \min\{\|x_0\|, \|y_0\|\} \le 1 < 1 + \delta$$

for any $\delta \in (0,1)$, this means that X is not uniformly non-square. $\square$

3. The Constant $L'_{\alpha,\beta}(X)$

In this section, if x and y satisfy the isosceles orthogonality condition and restrict $x, y \in S_X$, then we define the new constant:

Definition 3.1. *Let X be a Banach space, another geometric constant of the isosceles orthogonal type is defined as*

$$L'_{\alpha,\beta}(X) = \sup\left\{\frac{\|x - \beta y\|^2 + \|\alpha x - y\|^2}{\|x - y\|^2 + \|\alpha x - \beta y\|^2} : x, y \in S_X, x \perp_I y\right\}, \text{ where } \alpha, \beta \ge 0, \alpha \ne 1, \beta \ne 1.$$

We give bounds on the constant $L'_{\alpha,\beta}(X)$ and calculate its value in $\ell_\infty - \ell_1$ normed linear space when $\alpha = \beta = \frac{1}{2}$.

Proposition 3.1. *Let X be a Banach space, then*

$$\begin{cases} \frac{\alpha^2+\beta^2}{1+\alpha^2} \\ \frac{\alpha^2+\beta^2}{1+\beta^2} \\ \frac{2}{1+\alpha^2} \\ \frac{2}{1+\beta^2} \end{cases} \le L'_{\alpha,\beta}(X) \le \begin{cases} \frac{2}{1+\beta^2}, & 0 \le \beta \le \alpha < 1; \\ \frac{2}{1+\alpha^2}, & 0 \le \alpha \le \beta < 1; \\ \frac{\alpha^2+\beta^2}{1+\beta^2}, & 1 < \beta \le \alpha; \\ \frac{\alpha^2+\beta^2}{1+\alpha^2}, & 1 < \alpha \le \beta. \end{cases}$$

proof. Combined with the idea of Proposition 2.1, we have

$$\begin{aligned}&\frac{\|x-\beta y\|^2+\|\alpha x-y\|^2}{\|x-y\|^2+\|\alpha x-\beta y\|^2}\\ &\geq \frac{\left(\left|\frac{1-\beta}{2}\right|\|x+y\|-\left|\frac{1+\beta}{2}\right|\|x-y\|\right)^2+\left(\left|\frac{\alpha-1}{2}\right|\|x+y\|-\left|\frac{\alpha+1}{2}\right|\|x-y\|\right)^2}{\|x-y\|^2+\left(\left|\frac{\alpha-\beta}{2}\right|\|x+y\|+\left|\frac{\alpha+\beta}{2}\right|\|x-y\|\right)^2}.\end{aligned}$$

Case 1 When $0 \leq \beta \leq \alpha < 1$, we have

$$\frac{\|x-\beta y\|^2+\|\alpha x-y\|^2}{\|x-y\|^2+\|\alpha x-\beta y\|^2} \geq \frac{\alpha^2+\beta^2}{1+\alpha^2}.$$

Case 2 When $0 \leq \alpha \leq \beta < 1$, we have

$$\frac{\|x-\beta y\|^2+\|\alpha x-y\|^2}{\|x-y\|^2+\|\alpha x-\beta y\|^2} \geq \frac{\alpha^2+\beta^2}{1+\beta^2}.$$

Case 3 When $1 < \beta \leq \alpha$, we have

$$\frac{\|x-\beta y\|^2+\|\alpha x-y\|^2}{\|x-y\|^2+\|\alpha x-\beta y\|^2} \geq \frac{2}{1+\alpha^2}.$$

Case 4 When $1 < \alpha \leq \beta$, we have

$$\frac{\|x-\beta y\|^2+\|\alpha x-y\|^2}{\|x-y\|^2+\|\alpha x-\beta y\|^2} \geq \frac{2}{1+\beta^2}.$$

On the other hand, the upper bound on the constant $L'_{\alpha,\beta}(X)$ is the same as the upper bound on the constant $L_{\alpha,\beta}(X)$. □

Example 3.1. *Let $\alpha = \beta = \frac{1}{2}$ and X be the space $\mathbb{R}^2$ with $\ell_\infty - \ell_1$ norm defined by*

$$\|x\| = \begin{cases} \|x\|_1, x_1x_2 \leq 0, \\ \|x\|_\infty, x_1x_2 \geq 0. \end{cases}$$

Then

$$L'_{\frac{1}{2},\frac{1}{2}}(X) = 0.91.$$

proof. If $x = (y_1, 1+y_1), y = (y_2, 1+y_2)$, where $-1 \leq y_1 \leq y_2 \leq 0$; $x = (y_1, y_1-1), y = (y_2, y_2-1)$, where $0 \leq y_1 \leq y_2 \leq 1$. The two cases above, which are determined by $x \perp_I y$, we have $|y_1 - y_2| = 2$, are contradictory. To estimate this constant value, it is only necessary to consider the following two cases.

Case 1: Assuming that $x = (x_1, 1), y = (1, y_2)$, $0 \leq x_1 \leq y_2 \leq 1$. Since $x \perp_I y$, we have

$$1 + y_2 = (1 - x_1) + (1 - y_2),$$

hence $x_1 + 2y_2 = 1$, $y_2 \in \left[\frac{1}{3}, \frac{1}{2}\right]$. Then $\|\alpha x - y\| = 1$, $\|x - \beta y\| = \frac{1}{2} + \frac{3}{2}y_2$, $\|\alpha x - \beta y\| = \frac{1}{2} + \frac{1}{2}y_2$ and $\|x - y\| = 1 + y_2$.

In fact,

$$L'_{\frac{1}{2},\frac{1}{2}}(X) = \max_{\frac{1}{3}\le y_2\le\frac{1}{2}} \frac{9y_2^2 + 6y_2 + 5}{5y_2^2 + 10y_2 + 5}.$$

By simple calculation, we find that $L'_{\frac{1}{2},\frac{1}{2}}(X) = 0.91$ is obtained at the point $(1, \frac{1}{2})$.

Case 2: Assuming that $x = (x_1, 1)$, $y = (y_1, 1 + y_1)$ satisfy $-1 \le y_1 \le 0 \le x_1 \le 1$. Since $x \perp_I y$, we have $\|(x_1 + y_1, 2 + y_1)\| = \|(x_1 - y_1, -y_1)\|$.

If $-x_1 \le y_1$, then $2 + y_1 = x_1 - y_1$ is true, hence $x_1 - 2y_1 = 2$, $y_1 \in \left[-\frac{2}{3}, -\frac{1}{2}\right]$. We have $\|\alpha x - y\| = 1$, $\|x - \beta y\| = 2 + \frac{3}{2}y_1$, $\|\alpha x - \beta y\| = 1 + \frac{1}{2}y_1$ and $\|x - y\| = 2 + y_1$.

In the same way,

$$L'_{\frac{1}{2},\frac{1}{2}}(X) = \max_{-\frac{2}{3}\le y_1\le-\frac{1}{2}} \frac{9y_1^2 + 24y_1 + 20}{5y_1^2 + 20y_1 + 20}.$$

By simple calculation, we find that $L'_{\frac{1}{2},\frac{1}{2}}(X) = 0.91$ is obtained at the point $(1, -\frac{1}{2})$.

Similarly, if $y_1 \le -x_1$, such as **Case 2**, prove omission.

Combined with all of the above, we get

$$L'_{\frac{1}{2},\frac{1}{2}}(X) = 0.91.$$

□

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