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Convexity Properties in Non-Newtonian Calculus and Their Applications

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Abstract. The study presented some results on convexity properties in non-Newtonian calculus. Also presented is the Jensen-Steffensen inequality in non-Newtonian calculus and some applications. The research was mainly on positive real numbers.

1. Introduction

Classical calculus was introduced by Newton and Leibnitz which is applied on our present day mathematics [1]. There are different operations with respect to addition and subtraction of numbers under this calculus. However, Grossman and Karts came out with another calculus known as non-Newtonian calculus in the $20th$ century [2]. Non-Newtonian calculus which is also called multiplicative calculus is a multiplicative way of generating positive solutions to mathematical problems [3]. It is a recent approach used to solve mathematical problems with positive real numbers.

It is evidently clear that addition is replaced by multiplication in non-Newtonian calculus, and subtraction by division for example, see authors in [4, 5]. This result has been supported by authors in [6], when they introduced the multiplicative calculus and its applications, which has been established to be applicable to solving mathematical problems [6]. Non-Newtonian calculus has been extended in many directions; fractional derivative, complex derivative, integral transformations, differential equations and applications for science and engineering.

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The authors in [7] stated that, the centre of all analysis in the social science is the derivative. They were expecting another method which may treat realistic growth phenomenon better in our economy than the ordinary approach. The change actually came, which confirmed that variations are more naturally measured in ratios than in differences [7], until 1972 that Grossman and Katz came out with non-Newtonian calculus (see [2]). In their work, they also made it clearly that measuring growth in ratios gives a better variations than measuring it in differences. Non-Newtonian calculus is essential in the development of our scientific world, which enhances production and development. It is applicable in various ways such as finance (used in marketing and determining rates of return), health (used in tumor therapy and chemotherapy in medicine, pathogen counts in treated water), thermostatistics, quantum theory, wave phenomenon, pattern recognition in images (eg. in biomedicine), signal processing, biology-thus the rate at which growth increases or decays.

In the Non-Newtonian calculus, ratios are used in measuring change in values whiles in the classical approach, differences are used in measuring change in values.

2. Preliminaries

In this section, we give an overview of known definitions and theories used in achieving our results.

2.1. **Non-Newtonian Arithmetic.** A system that satisfies the basic assumptions whose domain is a subset of $\mathbb{R}$ is called arithmetic. Exactly one arithmetic result is produced by a generator, which is a one-to-one function with a range of B that is a subset of the domain $\mathbb{R}$ [3,8]. The fundamental arithmetic operations are defined using the generator as follows [3,4]:
Addition, $k\dot{+}r = \alpha\left[\alpha^{-1}(k) + \alpha^{-1}(r)\right]$
Subtraction, $k\dot{-}r = \alpha\left[\alpha^{-1}(k) - \alpha^{-1}(r)\right]$
Multiplication, $k\dot{\times}r = \alpha\left[\alpha^{-1}(k) \times \alpha^{-1}(r)\right]$
Division, $k\dot{/}r = \alpha\left[\alpha^{-1}(k)/\alpha^{-1}(r)\right]$
When we take α-generator as $\alpha(k) = e^k, \alpha^{-1}(k) = \ln(k)$ and $k = \mathbb{R}^+$, then α arithmetic reduces to non-Newtonian arithmetic as follows:
Non-Newtonian - addition,

$$k\dot{+}r = \alpha\left[\alpha^{-1}(k) + \alpha^{-1}(r)\right] = e^{(\ln(k)+\ln(r))} = k \cdot r \tag{1}$$

Non-Newtonian - subtraction,

$$k\dot{-}r = \alpha\left[\alpha^{-1}(k) - \alpha^{-1}(r)\right] = e^{(\ln(k)-\ln(r))} = k/r \tag{2}$$

Non-Newtonian - multiplication,

$$k\dot{\times}r = \alpha\left[\alpha^{-1}(k) \times \alpha^{-1}(r)\right] = e^{(\ln(k)\times\ln(r))} = k^{\ln(r)} \tag{3}$$

Non-Newtonian - division,

$$k \dot{/} r = \alpha\left[\alpha^{-1}(k)/\alpha^{-1}(r)\right] = e^{(\ln(k)/\ln(r))} = k^{\frac{1}{\ln(r)}} \tag{4}$$

[1,4,8,9].

The above non-Newtonian arithmetic are widely accepted in non-Newtonian calculus.

When considering n positive real numbers $x_1, x_2, ..., x_n$ then the α-arithmetic mean is given as [10]:

$A_\alpha = \sum_{i=1}^{n} x_i \dot{/} n = \sum_{i=1}^{n} \alpha\left[\frac{\alpha^{-1}(x_i)}{n}\right]$

$A_\alpha = \alpha\left[\frac{\alpha^{-1}(x_1)+\alpha^{-1}(x_2)+...+\alpha^{-1}(x_n)}{n}\right]$

Considering $\alpha = \exp$, we have $A_{exp} = \left[\prod_{i=1}^{n} x_i\right]^{\frac{1}{n}} = (x_1 \times x_2 ... x_n)^{\frac{1}{n}}$

Also considering $x_1, x_2, ..., x_n \in R^+$ and G_α to be the α-geometric mean then:

$G_\alpha = \left[\prod_{i=1}^{n} x_i\right]^{\frac{1}{n}} = \alpha\left[\prod_{i=1}^{n} \alpha^{-1}(x_i)\right]^{\frac{1}{n}}$

$G_\alpha = \alpha\left[(\alpha^{-1}(x_1) \times \alpha^{-1}(x_2) ... \alpha^{-1}(x_n))^{\frac{1}{n}}\right]$.

In a similar way, we take $\alpha =$ exponent, then the α-geometric mean can be interpreted as [10]:

$G_{exp} = \left[(\ln x_1 \times \ln x_2 ... \ln x_n)^{\frac{1}{n}}\right], (x_n > 1)$.

Definition 2.1. *[11] A set $C = [a_1, b_1] \subseteq \mathbb{R}$ is said to be convex if $x, y \in C$, and*

$$qx + (1-q)y \in C, \tag{5}$$

for $q \in [0, 1]$.

Definition 2.2. *[11] Let ϕ be defined on a real interval M. The function ϕ is convex if:*

$$\phi(\lambda_1 x + \lambda_2 y) \leq \lambda_1 \phi(x) + \lambda_2 \phi(y), \tag{6}$$

$$\phi(\lambda_1 x + (1-\lambda_1)y) \leq \lambda_1 \phi(x) + (1-\lambda_1)\phi(y), \tag{7}$$

where $\lambda_1 + \lambda_2 = 1$, $\forall\, x, y \in M$ and $\lambda_1, \lambda_2 \in [0, 1]$.

Definition 2.3. *[12] Let ϕ be convex function and $u_1, v_1 \in \mathbb{R}$, then*

$$\phi\left(\frac{u_1 + v_1}{2}\right) \leq \frac{\phi(u_1) + \phi(v_1)}{2}. \tag{8}$$

Proposition 2.1. *[13] Let $x_1 \leq y_1, x_2 \leq y_2$, and ϕ_w be a convex function on an interval of real positive values, then*

$$\frac{\phi_w(y_2 - y_1)}{x_2 - x_1} \leq \frac{\phi_w(y_2) - \phi_w(y_1)}{x_2 - x_1}. \tag{9}$$

Theorem 2.1 (Jensen-Steffensen inequality). *[14] Let ϕ be a convex function defined on an interval of the real line and let x_i, $p_i \in \mathbb{R}$, $i = 1, ...m$. If $x_1, ..., x_m$ and $p_1, ..., p_m, p_m > 0$, then*

$$\phi\left(\frac{1}{p_m}\sum_{i=1}^{m} p_i x_i\right) \leq \frac{1}{p_m}\sum_{i=1}^{m} p_i \phi(x_i). \tag{10}$$

Theorem 2.2. *[14] Let ϕ be a convex function on an interval $S = [a_1, b_1] \subset \mathbb{R}^+$, where $a_1 < b_1$. Let $x = (x_1, x_2, ..., x_n)$ and $p = (p_1, p_2, ..., p_n)$, then*

$$\phi_k(a_1 + b_1 - \frac{1}{p_n}\sum_{i=1}^{n} p_i x_i) \leq \phi_k(a_1) + \phi_k(b_1) - \frac{1}{p_n}\sum_{i=1}^{n} p_i \phi_k(x_i). \tag{11}$$

Let $p_n = \sum_{i=1}^{n} p_i = 1$, then

$$\phi_k(a_1 + b_1 - \sum_{i=1}^{n} p_i x_i) \leq \phi_k(a_1) + \phi_k(b_1) - \sum_{i=1}^{n} p_i \phi_k(x_i). \tag{12}$$

Definition 2.4. *[15] The definition of a p-convex set for an interval C is*

$$(qu^p + (1-q)v^p)^{\frac{1}{p}} \in C, \tag{13}$$

for all $u, v \in C$ and $q \in [0,1]$.

Definition 2.5. *[12, 15, 16] Let $C = [a_1, b_1]$ be an interval on real numbers $\mathbb{R}$. An expression $\phi : C = [a_1, b_1] \mapsto \mathbb{R}$ is p-convex if*

$$\phi(qu^p + (1-q)v^p)^{\frac{1}{p}} \leq q\phi(u) + (1-q)\phi(v), \tag{14}$$

$\forall\ u, v \in C$ and $q \in [0,1]$.

Definition 2.6. *[15] Let ϕ be p-convex function and $u, v \in \mathbb{R}^+$, then*

$$\phi\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}} \leq \frac{\phi(u) + \phi(v)}{2}. \tag{15}$$

Definition 2.7. *[15] Suppose the function $\phi : C = [x, y] \mapsto \mathbb{R}^+$ is strongly convex and $\beta \geq 1$; then*

$$\phi(qa + (1-q)b) \leq q\phi(a) + (1-q)\phi(b) - \beta q(1-q)(b-a)^2 \tag{16}$$

for all $a, b \in C$ and $q \in [0,1]$.

Definition 2.8. *[15] A function $\phi : C = [x, y] \mapsto \mathbb{R}$ is strongly p-convex function, if*

$$\phi(qa^p + (1-q)b^p)^{\frac{1}{p}} \leq q\phi(a) + (1-q)\phi(b) - \beta q(1-q)(b^p - a^p)^2, \tag{17}$$

for all $a, b \in C$ and $q \in [0,1]$.

Definition 2.9. *[15] In the event that an interval C is a harmonic convex set, then*

$$\left(\frac{uv}{qu + (1-q)v}\right) \in C, \tag{18}$$

for all $u, v \in C$ and $q \in [0,1]$.

Definition 2.10. *[15] Let the function $\phi : C = [a_1, b_1] \subseteq \mathbb{R}^+$ and $C = [a_1, b_1]$ be on an interval on set $\mathbb{R}^+$ without zero, then*

$$\phi\left(\frac{uv}{qu + (1-q)v}\right) \leq (1-q)\phi(u) + q\phi(v). \tag{19}$$

Definition 2.11. *[15] Let $C = [a_1, b_1]$ represent an interval on the p-harmonic convex set $\mathbb{R}$ without zero. If a function $\phi : C = [a_1, b_1] \subseteq \mathbb{R}$ is p-harmonic convex, it does not include zero if*

$$\phi\left(\frac{u^p v^p}{qu^p + (1-q)v^p}\right)^{\frac{1}{p}} \le (1-q)\phi(u) + q\phi(v), \tag{20}$$

for all $u, v \in C$ and $q \in [0, 1]$.

In 1984, G. Toader defines m-convex function as follows [17]:

Definition 2.12. *Let the function ϕ be a real $\mathbb{R}^+$ on $[u, v]$ and $m \in [0, 1]$, then m-convex function is given as;*

$$\phi\left[qx_1 + m(1-q)y_1\right] \le q\phi(x_1) + m(1-q)\phi(y_1), \tag{21}$$

for all $x_1, y_1 \in [u, v]$ and $q \in [0, 1]$. Also, ϕ is m-concave if $-\phi$ is $m-$convex.

Definition 2.13. *Let the function ϕ be a positive real value on $S = [u, v]$, then c-convex is represented by Luenberger (1969) as;*

$$\phi\left[(1-q)x_1 + qy_1\right] \le c(1-q)\phi(x_1) + q\phi(y_1), \tag{22}$$

where $c \in [0, 1]$, for all $x_1, y_1 \in S$ and $q \in [0, 1]$.

3. Results and Discussions

In this section, all the results are presented in non-Newtonian form.

Definition 3.1. *Let $x, y \in C$ and $C = [a_1, b_1] \subseteq \mathbb{R}^+$ be a set. C must be convex if*

$$x^{\ln(t)} \cdot y^{\ln(\frac{1}{t})} \in C, \tag{23}$$

for $t \in [1, e]$.

Lemma 3.1. *Let, $x_1, y_1 \in \mathbb{R}^+$ and ϕ_1 be a convex function. Then*

$$\phi_1\left[x_1^{\ln(t_1)} \cdot y_1^{\ln(\frac{1}{t_1})}\right] \le \phi_1(x_1)^{\ln(t_1)} \cdot \phi_1(y_1)^{\ln(\frac{1}{t_1})}, \tag{24}$$

for $t_1 \in [1, e]$.

Proof. Using equation (3), (1) and by convexity, we have

$$\begin{aligned}
\phi_1\left[x_1^{\ln(t_1)}\cdot y_1^{\ln(\frac{1}{t_1})}\right] &= \phi_1\left[x_1\dot{\times}t_1\dot{+}y_1\dot{\times}(\frac{1}{t_1})\right]\\
&\leq \phi_1(x_1)\dot{\times}t_1\dot{+}\phi_1(y_1)\dot{\times}(\frac{1}{t_1})\\
&\leq \alpha\left[\alpha^{-1}\phi_1(x_1)\dot{\times}\alpha^{-1}(t_1)\dot{+}\alpha^{-1}\phi_1(y_1)\dot{\times}\alpha^{-1}(\frac{1}{t_1})\right]\\
&\leq \alpha\left[\ln\phi_1(x_1)\times\ln(t_1)+\ln\phi_1(y_1)\times\ln(\frac{1}{t_1})\right]\\
&\leq e^{\left[\ln\phi_1(x_1)\ln(t_1)+\ln\phi_1(y_1)\ln(\frac{1}{t_1})\right]}\\
&\leq (e^{\ln\phi_1(x_1)})^{\ln(t_1)}\cdot(e^{\ln\phi_1(y_1)})^{\ln(\frac{1}{t_1})}\\
\phi_1\left[x_1^{\ln(t_1)}\cdot y_1^{\ln(\frac{1}{t_1})}\right] &\leq \phi_1(x_1)^{\ln(t_1)}\cdot\phi_1(y_1)^{\ln(\frac{1}{t_1})},
\end{aligned}$$

as required. □

Lemma 3.2. *Let $a_1, b_1 \in \mathbb{R}^+$ and ϕ_1 be a convex function, we have*

$$\phi_1\left[\frac{(b_1)}{(a_1)}\right] \leq \frac{\phi_1(b_1)}{\phi_1(a_1)}. \tag{25}$$

Proof. Using equation (2) and by convexity, we have

$$\begin{aligned}
\phi_1\left[\frac{(b_1)}{(a_1)}\right] &= \phi_1\left[(b_1)\dot{-}(a_1)\right]\\
&\leq \phi_1(b_1)\dot{-}\phi_1(a_1)\\
&\leq \alpha\left[\alpha^{-1}(\phi_1(b_1))\dot{-}\alpha^{-1}(\phi_1(a_1))\right]\\
&\leq e^{\left[\ln\phi_1(b_1)-\ln\phi_1(a_1)\right]}\\
&\leq \frac{e^{\ln\phi_1(b_1)}}{e^{\ln\phi_1(a_1)}}\\
&\leq \frac{\phi_1(b_1)}{\phi_1(a_1)}\\
\phi_1\left[\frac{(b_1)}{(a_1)}\right] &\leq \frac{\phi_1(b_1)}{\phi_1(a_1)},
\end{aligned}$$

as required. □

Lemma 3.3. *Let ϕ_1 be convex and $u_1, v_1 \in \mathbb{R}^+$. Then*

$$\phi_1\left[\frac{u_1\cdot v_1}{2}\right] \leq \frac{\phi_1(u_1)\cdot\phi_1(v_1)}{2}. \tag{26}$$

Proof. Using equation (1) and by convexity, we have

$$
\begin{aligned}
\phi_1\left[\frac{u_1 \cdot v_1}{2}\right] =& \phi_1\left[\frac{1}{2}(u_1 \dot{+} v_1)\right] \\
\leq& \frac{1}{2}\left[\left(\phi_1(u_1) \dot{+} \phi_1(v_1)\right)\right] \\
\leq& \frac{1}{2}\left[\alpha\left(\alpha^{-1}\phi_1(u_1) \dot{+} \alpha^{-1}\phi_1(v_1)\right)\right] \\
\leq& \frac{1}{2}\left[e^{\left(\ln\phi_1(u_1) + \ln\phi_1(v_1)\right)}\right] \\
\leq& \frac{1}{2}\left[e^{\ln\phi_1(u_1)} \cdot e^{\ln\phi_1(v_1)}\right] \\
\leq& \frac{1}{2}\left[\phi_1(u_1) \cdot \phi_1(v_1)\right] \\
\phi_1\left[\frac{u_1 \cdot v_1}{2}\right] \leq& \frac{\phi_1(u_1) \cdot \phi_1(v_1)}{2},
\end{aligned}
$$

as required. □

Lemma 3.4. *Consider an increasing function ϕ. If $\phi(v_y) \geq \phi(v_x)$ and $v_y \geq v_x$, then*

$$\frac{\phi(v_x)^{\ln(v_x)} \cdot \phi(v_y)^{\ln(v_y)}}{\phi(v_y)^{\ln(v_x)} \cdot \phi(v_x)^{\ln(v_y)}} \leq 1, \tag{27}$$

Proof. Using equation (3) and (1), we have

$$
\begin{aligned}
\phi(v_x)^{\ln(v_x)} \cdot \phi(v_y)^{\ln(v_y)} =& \phi(v_x) \dot{\times} v_x \dot{+} \phi(v_y) \dot{\times} v_y \\
=& v_x \dot{\times} \phi(v_x) \dot{+} v_y \dot{\times} \phi(v_y) \\
\leq& v_x \dot{\times} \phi(v_y) \dot{+} v_y \dot{\times} \phi(v_x) \\
\leq& \alpha\left[\alpha^{-1}(v_x) \dot{\times} \alpha^{-1}\phi(v_y) \dot{+} \alpha^{-1}(v_y) \dot{\times} \alpha^{-1}\phi(v_x)\right] \\
\leq& \alpha\left[\ln(v_x) \times \ln\phi(v_y) + \ln(v_y) \times \ln\phi(v_x)\right] \\
\leq& e^{\left[\ln(v_x) \times \ln\phi(v_y) + \ln(v_y) \times \ln\phi(v_x)\right]} \\
\leq& e^{\ln(v_x)\ln\phi(v_y)} \cdot e^{\ln(v_y)\ln\phi(v_x)} \\
\leq& \left(e^{\ln\phi(v_y)}\right)^{\ln(v_x)} \cdot \left(e^{\ln\phi(v_x)}\right)^{\ln(v_y)} \\
\phi(v_x)^{\ln(v_x)} \cdot \phi(v_y)^{\ln(v_y)} \leq& \phi(v_y)^{\ln(v_x)} \cdot \phi(v_x)^{\ln(v_y)},
\end{aligned}
$$

as required. □

Lemma 3.5. *Consider a decreasing function ϕ. If $\phi(v_x) \geq \phi(v_y)$ and $v_x \geq v_y$, then*

$$\frac{\phi(v_x)^{\ln(v_x)} \cdot \phi(v_y)^{\ln(v_y)}}{\phi(v_y)^{\ln(v_x)} \cdot \phi(v_x)^{\ln(v_y)}} \geq 1, \tag{28}$$

or

$$\phi(v_x)^{\ln(v_x)} \cdot \phi(v_y)^{\ln(v_y)} \geq \phi(v_y)^{\ln(v_x)} \cdot \phi(v_x)^{\ln(v_y)}. \tag{29}$$

The proof is similar to inequality (25), with inequality sign reversed.

Theorem 3.1. *Let $v_i, z_i \in \mathbb{R}^+$, $i = 1, ..., k$, and let ϕ be a convex function defined on a range of the real line. If $B_k > 0$, then*

$$\phi\left[\left(\frac{1}{B_k}\right)^{\ln\prod_{i=1}^{k}(v_i)^{\ln(z_i)}}\right] \leq \left(\frac{1}{B_k}\right)^{\ln\prod_{i=1}^{k}\phi(v_i)^{\ln(z_i)}}. \tag{30}$$

Proof. Using equation (3) and by convexity, we have

$$\begin{aligned}
\phi\left[\left(\frac{1}{B_k}\right)^{\ln\prod_{i=1}^{k}(v_i)^{\ln(z_i)}}\right] &= \phi\left[\frac{1}{B_k}\dot{\times}\prod_{i=1}^{k}(z_i)\dot{\times}(v_i)\right] \\
&\leq \frac{1}{B_k}\dot{\times}\prod_{i=1}^{k}(z_i)\dot{\times}\phi(v_i) \\
&\leq \alpha\left(\alpha^{-1}(\frac{1}{B_k})\dot{\times}\alpha^{-1}\prod_{i=1}^{k}(z_i)\dot{\times}\alpha^{-1}\phi(v_i)\right) \\
&\leq e^{\left(\ln(\frac{1}{B_k})\ln\prod_{i=1}^{k}(z_i)\times\ln\phi(v_i)\right)} \\
&\leq \left(e^{\ln(\frac{1}{B_k})}\right)^{\ln\prod_{i=1}^{k}(z_i)\times\ln\phi(v_i)} \\
&\leq \left(\frac{1}{B_k}\right)^{\ln\prod_{i=1}^{k}(z_i)\times\ln\phi(v_i)} \\
\phi\left[\left(\frac{1}{B_k}\right)^{\ln\prod_{i=1}^{k}(v_i)^{\ln(z_i)}}\right] &= \left(\frac{1}{B_k}\right)^{\ln\prod_{i=1}^{k}\phi(v_i)^{\ln(z_i)}},
\end{aligned}$$

as required. ☐

Definition 3.2. *A set $M = [a_1, b_1] \subseteq \mathbb{R}^+$ is a p-convex set, if*

$$\left[[(x_1)^p]^{\ln(j)} \cdot [(y_1)^p]^{\ln(\frac{1}{j})}\right]^{\frac{1}{p}} \in M, \tag{31}$$

$\forall\ x_1, y_1 \in M$ *and* $j \in [1, e]$.

Lemma 3.6. *Let $x, y \in M$ where $M \subseteq \mathbb{R}^+$ and $j \in [1, e]$. For a p-convex function ϕ, we have*

$$\phi\left[[(x_1)^p]^{\ln(j)} \cdot [(y_1)^p]^{\ln(\frac{1}{j})}\right]^{\frac{1}{p}} \leq \left[\phi(x_1)\right]^{\ln(j)} \cdot \left[\phi(y_1)\right]^{\ln(\frac{1}{j})}. \tag{32}$$

Proof. Using equation (3), (1) and by p-convexity, we have

$$\begin{aligned}
\phi\left[[(x_1)^p]^{\ln(j)}\cdot[(y_1)^p]^{\ln(\frac{1}{j})}\right]^{\frac{1}{p}} =&\phi\left[(x_1)^p\dot{\times}j\dot{+}(y_1)^p\dot{\times}(\frac{1}{j})\right]^{\frac{1}{p}}\\
\leq&\phi(x_1)\dot{\times}j\dot{+}\phi(y_1)\dot{\times}(\frac{1}{j})\\
\leq&\left[\alpha[\alpha^{-1}\phi(x_1)\dot{\times}\alpha^{-1}(j)\dot{+}\alpha^{-1}\phi(y_1)\dot{\times}\alpha^{-1}(\frac{1}{j})]\right]\\
\leq&\left[\alpha[\ln\phi(x_1)\dot{\times}\ln(j)\dot{+}\ln\phi(y_1)\dot{\times}\ln(\frac{1}{j})]\right]\\
\leq&e^{\left[\ln\phi(x_1)\ln(j)+\ln\phi(y_1)\ln(\frac{1}{j})\right]}\\
\leq&e^{\ln\phi(x_1)\ln(j)}\cdot e^{\ln\phi(y_1)\ln(\frac{1}{j})}\\
\leq&\left(e^{\ln\phi(x_1)}\right)^{\ln(j)}\cdot\left(e^{\ln\phi(y_1)}\right)^{\ln(\frac{1}{j})}\\
\leq&\left[\phi(x_1)\right]^{\ln(j)}\cdot\left[\phi(y_1)\right]^{\ln(\frac{1}{j})}\\
\phi\left[[(x_1)^p]^{\ln(j)}\cdot[(y_1)^p]^{\ln(\frac{1}{j})}\right]^{\frac{1}{p}}\leq&\left[\phi(x_1)\right]^{\ln(j)}\cdot\left[\phi(y_1)\right]^{\ln(\frac{1}{j})},
\end{aligned}$$

as required. □

Remark 1. *The inequality* (24) *is obtained when* $p = 1$.

Lemma 3.7. *Let* ϕ *be convex and* $x_1, y_1 \in \mathbb{R}^+$. *For a* p*-convex function, we have*

$$\phi\left[\frac{(x_1)^p\cdot(y_1)^p}{2}\right]^{\frac{1}{p}}\leq\frac{\phi(x_1)\cdot\phi(y_1)}{2}. \tag{33}$$

Proof. Using equation (1) and by p-convexity:

$$\begin{aligned}
\phi\left[\frac{(x_1)^p\cdot(y_1)^p}{2}\right]^{\frac{1}{p}}=&\phi\left[\frac{1}{2}((x_1)^p\dot{+}(y_1)^p)\right]^{\frac{1}{p}}\\
\leq&\frac{1}{2}\left[\phi(x_1)\dot{+}\phi(y_1)\right]\\
\leq&\frac{1}{2}\left[\alpha\left(\alpha^{-1}\phi(x_1)\dot{+}\alpha^{-1}\phi(y_1)\right)\right]\\
\leq&\frac{1}{2}\left[\alpha\left(\ln\phi(x_1)+\ln\phi(y_1)\right)\right]\\
\leq&\frac{1}{2}\left[e^{\left(\ln\phi(x_1)+\ln\phi(y_1)\right)}\right]\\
\leq&\frac{1}{2}\left[e^{\ln\phi(x_1)}\cdot e^{\ln\phi(y_1)}\right]\\
\leq&\frac{1}{2}\left[\phi(x_1)\cdot\phi(y_1)\right]\\
\phi\left[\frac{(x_1)^p\cdot(y_1)^p}{2}\right]^{\frac{1}{p}}\leq&\frac{\phi(x_1)\cdot\phi(y_1)}{2},
\end{aligned}$$

as required. $\square$

Remark 2. *The inequality* (26) *is obtained when* $p = 1$.

Definition 3.3. *A harmonic convex set of an interval* M *is described, if*

$$\frac{x^{\ln(y)}}{x^{\ln(j)} \cdot y^{\ln(\frac{1}{j})}} \in M, \tag{34}$$

$\forall\ x, y \in M$ *and* $j \in [1, e]$.

Lemma 3.8. *Let the set* $W \subseteq \mathbb{R}^+$ *be a harmonic set. For harmonic convex function* ϕ, *we have*

$$\phi\left[\frac{x_1^{\ln(y_1)}}{x^{\ln(q_1)} \cdot y_1^{\ln(\frac{1}{q_1})}}\right] \leq \frac{\phi(x_1)^{\ln\phi(y_1)}}{\phi(x_1)^{\ln(q_1)} \cdot \phi(y_1)^{\ln(\frac{1}{q_1})}}, \tag{35}$$

$\forall\ x_1, y_1 \in W$ *and* $q_1 \in [1, e]$.

Proof. Using equation (3), (1) and by convexity, we have

$$\begin{aligned}
\phi\left[\frac{x_1^{\ln(y_1)}}{x_1^{\ln(q_1)} \cdot y_1^{\ln(\frac{1}{q_1})}}\right] =& \phi\left[x_1 \dot{\times} y_1 \dot{-} (q_1 \dot{\times} x_1 \dot{+} (\frac{1}{q_1}) \dot{\times} y_1)\right] \\
\leq& \phi(x_1) \dot{\times} \phi(y_1) \dot{-} (q_1 \dot{\times} \phi(x_1) \dot{+} (\frac{1}{q_1}) \dot{\times} \phi(y_1)) \\
\leq& \alpha\left[\alpha^{-1}\phi(x_1) \dot{\times} \alpha^{-1}\phi(y_1) \dot{-} (\alpha^{-1}(q_1) \dot{\times} \alpha^{-1}\phi(x_1) \dot{+} \alpha^{-1}(\frac{1}{q_1}) \dot{\times} \alpha^{-1}\phi(y_1))\right] \\
\leq& \alpha\left[\ln\phi(x_1) \times \ln\phi(y_1) - (\ln(q_1) \times \ln\phi(x_1) + \ln(\frac{1}{q_1}) \times \ln\phi(y_1))\right] \\
\leq& e^{\left[\ln\phi(x_1) \times \ln\phi(y_1) - (\ln(q_1) \times \ln\phi(x_1) + \ln(\frac{1}{q_1}) \times \ln\phi(y_1))\right]} \\
\leq& e^{\left[\ln\phi(x_1) \times \ln\phi(y_1) - (\ln\phi(x_1) \times \ln(q_1) + \ln\phi(y_1) \times \ln(\frac{1}{q_1}))\right]} \\
\leq& \frac{(e^{\ln\phi(x_1)})^{\ln\phi(y_1)}}{(e^{\ln\phi(x_1)})^{\ln(q_1)} \cdot (e^{\ln\phi(y_1)})^{\ln(\frac{1}{q_1})}} \\
\phi\left[\frac{x_1^{\ln(y_1)}}{x_1^{\ln(q_1)} \cdot y_1^{\ln(\frac{1}{q_1})}}\right] \leq& \frac{\phi(x_1)^{\ln\phi(y_1)}}{\phi x_1^{\ln(q_1)} \cdot \phi(y_1)^{\ln(\frac{1}{q_1})}},
\end{aligned}$$

proved. $\square$

Definition 3.4. *Let* W *be a subset on* $\mathbb{R}^+$, *then* W *is* p*-harmonic convex set if*

$$\left[\frac{[x^p]^{\ln(y)^p}}{[x^p]^{\ln(j)} \cdot [y^p]^{\ln(\frac{1}{j})}}\right]^{\frac{1}{p}} \in W, \tag{36}$$

$\forall\ x, y \in W$ *and* $j \in [1, e]$.

Lemma 3.9. *Consider the p-harmonic convex set $M = [a, b] \subseteq \mathbb{R}^+$. If ϕ is p-harmonic convex function, we have*

$$\phi\left[\frac{[(x_1)^p]^{\ln(y_1)^p}}{[(x_1)^p]^{\ln(j_1)}\cdot[(y_1)^p]^{\ln(\frac{1}{j_1})}}\right]^{\frac{1}{p}} \leq \frac{[\phi(x_1)]^{\ln\phi(y_1)}}{[\phi(x_1)]^{\ln(j)}\cdot[\phi(y_1)]^{\ln(\frac{1}{j_1})}}, \tag{37}$$

$\forall\ x_1, y_1 \in M$ *and* $j_1 \in [1, e]$.

Proof. Using equation (3), (2) and by p-convexity, we have

$$\begin{aligned}
\phi\left[\frac{[(x_1)^p]^{\ln(y_1)^p}}{[(x_1)^p]^{\ln(j_1)}\cdot[(y_1)^p]^{\ln(\frac{1}{j_1})}}\right]^{\frac{1}{p}} &= \phi\left[(x_1)^p\dot\times(y_1)^p\dot-(j_1\dot\times(x_1)^p\dot+(\frac{1}{j_1})\dot\times(y_1)^p)\right]^{\frac{1}{p}}\\
&\leq \phi(x_1)\dot\times\phi(y_1)\dot-(j_1\dot\times\phi(x_1)\dot+(\frac{1}{j_1})\dot\times\phi(y_1))\\
&\leq \alpha\left[\ln\phi(x_1)\times\ln\phi(y_1)-(\ln(j_1)\times\ln\phi(x_1)+\ln(\frac{1}{j_1})\times\ln\phi(y_1)\right]\\
&\leq e^{\left[\ln\phi(x_1)\times\ln\phi(y_1)-(\ln(j_1)\times\ln\phi(x_1)+\ln(\frac{1}{j_1})\times\ln\phi(y_1)\right]}\\
&\leq e^{\left[\ln\phi(x_1)\times\ln\phi(y_1)-(\ln\phi(x_1)\times\ln(j_1)+\ln\phi(y_1)\times\ln(\frac{1}{j_1})\right]}\\
&\leq \frac{e^{[\ln\phi(x_1)\,\ln\phi(y_1)]}}{e^{[\ln\phi(x_1)\,\ln(j_1)]}\cdot e^{[\ln\phi(y_1)\,\ln(\frac{1}{j_1})]}}\\
&\leq \frac{(\phi(x_1))^{\ln\phi(y_1)}}{(\phi(x_1))^{\ln(j_1)}\cdot(\phi(y_1))^{\ln(\frac{1}{j_1})}}\\
\phi\left[\frac{[(x_1)^p]^{\ln(y_1)^p}}{[(x_1)^p]^{\ln(j_1)}\cdot[(y_1)^p]^{\ln(\frac{1}{j_1})}}\right]^{\frac{1}{p}} &\leq \frac{[\phi(x_1)]^{\ln\phi(y_1)}}{[\phi(x_1)]^{\ln(j_1)}\cdot[\phi(y_1)]^{\ln(\frac{1}{j_1})}},
\end{aligned}$$

as required. □

Remark 3. *When $p = 1$, the inequality (35) is obtained.*

Definition 3.5. *M is referred to as p-Jensen-Steffensen's set if, it is a subset of $\mathbb{R}^+$, assuming that*

$$\left[\left(\frac{1}{B_n}\right)^{\ln\prod_{i=1}^{n}((x_i)^p)^{\ln(z_i)}}\right]^{\frac{1}{p}} \in M. \tag{38}$$

Lemma 3.10. *Let $M \subseteq \mathbb{R}^+$ be p-Jensen-Steffensen set. For p-Jensen-Steffensen's inequality, we have*

$$\phi\left[\left(\frac{1}{B_n}\right)^{\ln\prod_{r=1}^{n}((x_i)^p)^{\ln(z_i)}}\right]^{\frac{1}{p}} \leq \left(\frac{1}{B_n}\right)^{\ln\prod_{r=1}^{n}\phi(x_i)^{\ln(z_i)}}. \tag{39}$$

Proof. Using equation (3) and by p-convexity, we have

$$
\begin{aligned}
\phi\left[\left(\frac{1}{B_n}\right)^{\ln\prod_{r=1}^{n}((x_i)^p)^{\ln(z_i)}}\right]^{\frac{1}{p}} &= \phi\left[\frac{1}{B_n}\dot{\times}\prod_{r=1}^{n}(z_i)\dot{\times}(x_i)^p\right]^{\frac{1}{p}} \\
&\leq \frac{1}{B_n}\dot{\times}\prod_{r=1}^{n}(z_i)\dot{\times}\phi(x_i) \\
&\leq \alpha\left(\alpha^{-1}(\frac{1}{B_n})\dot{\times}\alpha^{-1}\prod_{r=1}^{n}(z_i)\dot{\times}\alpha^{-1}\phi(x_i)\right) \\
&\leq e^{\left(\ln(\frac{1}{B_n})\ln\prod_{r=1}^{n}(z_i)\times\ln\phi(x_i)\right)} \\
&\leq \left(e^{\ln(\frac{1}{B_n})}\right)^{\ln\prod_{r=1}^{n}(z_i)\times\ln\phi(x_i)} \\
&\leq \left(\frac{1}{B_n}\right)^{\ln\prod_{r=1}^{n}(z_i)\times\ln\phi(x_i)} \\
\phi\left[\left(\frac{1}{B_n}\right)^{\ln\prod_{r=1}^{n}((x_i))^{\ln(z_i)}}\right]^{\frac{1}{p}} &\leq \left(\frac{1}{B_n}\right)^{\ln\prod_{r=1}^{n}\phi(x_i)^{\ln(z_i)}},
\end{aligned}
$$

as required. □

Definition 3.6. *Let the set W be a subset on $\mathbb{R}^+$, then W is strongly convex set if*

$$\left[\left[\mu^{\ln(j)}\right]^{\ln(\frac{1}{j})}\right]^{\ln(\frac{y}{x})^2} \in W, \tag{40}$$

where $\mu \geq 1$, $\forall\ x, y \in W$ and $j \in [1, e]$.

Lemma 3.11. *Consider a convex set $W \subseteq \mathbb{R}^+$. For a strongly convex function ϕ, we have*

$$\phi\left[x^{\ln(q)}\cdot y^{\ln(\frac{1}{q})}\right] \leq \frac{\phi(x)^{\ln(q)}\cdot\phi(y)^{\ln(\frac{1}{q})}}{\left[\left[\mu^{\ln(q)}\right]^{\ln(\frac{1}{q})}\right]^{\ln(\frac{y}{x})^2}}, \tag{41}$$

where $\mu \geq 1$, $\forall\ x, y \in W$ and $q \in [1, e]$.

Proof. Using equation (3), (1) and by convexity, we have

$$
\begin{aligned}
\phi\left[x^{\ln(q)}\cdot y^{\ln(\frac{1}{q})}\right] &= \phi\left[x\dot{\times}q\dot{+}y\dot{\times}(\frac{1}{q})\right] \\
&\le q\dot{\times}\phi(x)\dot{+}(\frac{1}{q})\dot{\times}\phi(y)\dot{-}\mu\dot{\times}j\dot{\times}(\frac{1}{q})\dot{\times}(\frac{y}{x})^2 \\
&\le \alpha\left[\alpha^{-1}q\dot{\times}\alpha^{-1}\phi(x)\dot{+}\alpha^{-1}(\frac{1}{q})\dot{\times}\alpha^{-1}\phi(y)\dot{-}\alpha^{-1}(\mu)\dot{\times}\alpha^{-1}(q)\dot{\times}\alpha^{-1}(\frac{1}{q})\dot{\times}(\frac{y}{x})^2\right] \\
&\le \frac{e^{\ln(q)\ln\phi(x)}\cdot e^{\ln(\frac{1}{q})\ln\phi(y)}}{e^{\ln(\mu)\ln(q)\ln(\frac{1}{q})\ln(\frac{y}{x})^2)}} \\
&\le \frac{\left(e^{\ln\phi(x)}\right)^{\ln(q)}\cdot\left(e^{\ln\phi(y)}\right)^{\ln(\frac{1}{q})}}{\left(e^{\ln(\mu)}\right)^{\ln(q)\ln(\frac{1}{q})\ln(\frac{y}{x})^2}} \\
\phi\left[x^{\ln(q)}\cdot y^{\ln(\frac{1}{q})}\right] &\le \frac{\phi(x)^{\ln(q)}\cdot\phi(y)^{\ln(\frac{1}{q})}}{\left[\left[\mu^{\ln(q)}\right]^{\ln(\frac{1}{q})}\right]^{\ln(\frac{y}{x})^2}},
\end{aligned}
$$

proved. □

Lemma 3.12. *Consider the convex set $W=[a,b]\subseteq\mathbb{R}^+$. If ϕ is highly p-convex function, we have*

$$
\phi\left[(x^p)^{\ln(j)}\cdot(y^p)^{\ln(\frac{1}{j})}\right]^{\frac{1}{p}} \le \frac{\phi(x)^{\ln(j)}\cdot\phi(y)^{\ln(\frac{1}{j})}}{\left[\left[\mu^{\ln(j)}\right]^{\ln(\frac{1}{j})}\right]^{\ln(\frac{y}{x})^2}}, \tag{42}
$$

where $\mu\ge 1$, $\forall\ x,y\in W$ and $j\in[1,e]$.

Remark 4. *When $p=1$, the strongly p-convex function returns to strongly convex function. Thus the inequality (41) is obtained.*

Proof. Using equation (3), (1) and by p-convexity, we have

$$
\begin{aligned}
\phi\left[(x^p)^{\ln(j)}\cdot(y^p)^{\ln(\frac{1}{j})}\right]^{\frac{1}{p}} &\le j\dot{\times}\phi(x)\dot{+}(\frac{1}{j})\dot{\times}\phi(y)\dot{-}\mu\dot{\times}j\dot{\times}(\frac{1}{j})\dot{\times}(\frac{y}{x})^2 \\
&\le \alpha\left[\alpha^{-1}j\dot{\times}\alpha^{-1}\phi(x)\dot{+}\alpha^{-1}(\frac{1}{j})\dot{\times}\alpha^{-1}\phi(y)\dot{-}\alpha^{-1}(\mu)\dot{\times}\alpha^{-1}(j)\dot{\times}\alpha^{-1}(\frac{1}{j})\dot{\times}(\frac{y}{x})^2\right] \\
&\le e^{\left(\ln(j)\ln\phi(x)+\ln(\frac{1}{j})\ln\phi(y)-\ln(\mu)\ln(j)\ln(\frac{1}{j})\ln(\frac{y}{x})^2\right)} \\
&\le \frac{e^{\ln(j)\ln\phi(x)}\cdot e^{\ln(\frac{1}{j})\ln\phi(y)}}{e^{\ln(\mu)\ln(j)\ln(\frac{1}{j})\ln(\frac{y}{x})^2)}} \\
&\le \frac{\left(e^{\ln\phi(x)}\right)^{\ln(j)}\cdot\left(e^{\ln\phi(y)}\right)^{\ln(\frac{1}{j})}}{\left(e^{\ln(\mu)}\right)^{\ln(j)\ln(\frac{1}{j})\ln(\frac{y}{x})^2}}
\end{aligned}
$$

$$\phi\left[(x^p)^{\ln(j)}\cdot(y^p)^{\ln(\frac{1}{j})}\right]^{\frac{1}{p}}\leq\frac{\phi(x)^{\ln(j)}\cdot\phi(y)^{\ln(\frac{1}{j})}}{\left[\left[\mu^{\ln(j)}\right]^{\ln(\frac{1}{j})}\right]^{\ln(\frac{y}{x})^2}},$$

proved. □

Definition 3.7. *Let the set W be a subset on $\mathbb{R}^+$, then W is m-convex set if*

$$x^{\ln(t)}\cdot\left[(y)^{\ln(m)}\right]^{\ln(\frac{1}{t})}\in W, \tag{43}$$

where $m\in[1,e]$, $\forall\ x,y\in W$ and $j\in[1,e]$.

Lemma 3.13. *Let the function ϕ_1 be m-convex and $m_1\in[1,e]$, then*

$$\phi_1\left[x_1^{\ln(t_1)}\cdot\left[y_1^{\ln(m)}\right]^{\ln(\frac{1}{t_1})}\right]\leq\phi_1(x_1)^{\ln(t_1)}\cdot\left[\phi_1(y_1)^{\ln(m)}\right]^{\ln(\frac{1}{t_1})}, \tag{44}$$

for all $x_1,y_1\in\mathbb{R}^+$ and $t\in[1,e]$.

Proof. Using equation (3), (1) and by convexity, we have

$$\begin{aligned}
\phi_1\left[x_1^{\ln(t_1)}\cdot\left[(y_1)^{\ln(m_1)}\right]^{\ln(\frac{1}{t_1})}\right]=&\phi_1\left[x_1\dot{\times}t\dot{+}m_1\dot{\times}(y_1)\dot{\times}(\frac{1}{t_1})\right]\\
\leq&\phi_1(x_1)\dot{\times}t_1\dot{+}m_1\dot{\times}\phi_1(y_1)\dot{\times}(\frac{1}{t_1})\\
\leq&\left[\alpha[\alpha^{-1}\phi_1(x_1)\dot{\times}\alpha^{-1}(t_1)\dot{+}\alpha^{-1}(m_1)\dot{\times}\alpha^{-1}\phi_1(y_1)\dot{\times}\alpha^{-1}(\frac{1}{t_1})]\right]\\
\leq&\left[\alpha[\ln\phi_1(x_1)\times\ln(t_1)+\ln(m_1)\dot{\times}\ln\phi_1(y_1)\times\ln(\frac{1}{t_1})]\right]\\
\leq&\left[e^{[\ln\phi_1(x_1)\times\ln(t_1)+\ln(m_1)\times\ln\phi_1(y_1)\times\ln(\frac{1}{t_1})]}\right]\\
\leq&\left[e^{[\ln\phi_1(x_1)\ln(t_1)+\ln m_1\ln\phi_1(y_1)\ln(\frac{1}{t_1})]}\right]\\
\leq&(e^{\ln\phi_1(x_1)})^{\ln(t_1)}\cdot((e^{\ln\phi_1(y_1)})^{\ln(m_1)})^{\ln(\frac{1}{t_1})}\\
\phi_1\left[x_1^{\ln(t_1)}\cdot\left[(y_1)^{\ln(m_1)}\right]^{\ln(\frac{1}{t})}\right]\leq&\phi_1(x_1)^{\ln(t_1)}\cdot\left[\phi_1(y_1)^{\ln(m_1)}\right]^{\ln(\frac{1}{t_1})},
\end{aligned}$$

as required. □

Definition 3.8. *Let $W\subseteq\mathbb{R}^+$, then W is said to be c-convex set if*

$$x_1^{\ln(\frac{1}{t})}\cdot y_1^{\ln(t)}\in W, \tag{45}$$

where $c\in[1,e]$, $\forall\ x_1,y_1\in W$ and $t\in[1,e]$.

Lemma 3.14. *Let the function ϕ_1 be c-convex on $W = [a, a_1]$ and $c \in [1, e]$, then*

$$\phi_1 \left[(x_1^{\ln(\frac{1}{t_1})} \cdot y_1^{\ln(t_1)} \right] \leq \left[\phi_1(x_1)^{\ln(c)} \right]^{\ln(\frac{1}{t_1})} \cdot \phi_1(y_1)^{\ln(t_1)}, \tag{46}$$

$\forall\ x_1, y_1 \in W$ and $t_1 \in [1, e]$.

Proof. Using equation (3), (1) and by convexity, we have

$$\begin{aligned}
\phi_1 \left[x_1^{\ln(\frac{1}{t_1})} \cdot (y_1)^{\ln(t_1)} \right] =& \phi_1 \left[x_1 \dot{\times} (\frac{1}{t_1}) \dot{+} (y_1) \dot{\times} t_1 \right] \\
\leq& c \dot{\times} \phi_1(x_1) \dot{\times} (\frac{1}{t_1}) \dot{+} \phi_1(y_1) \dot{\times} t_1 \\
\leq& \left[\alpha[\alpha^{-1}(c) \dot{\times} \alpha^{-1} \phi_1(x_1) \dot{\times} \alpha^{-1}(\frac{1}{t_1}) \dot{+} \alpha^{-1} \phi_1(y_1) \dot{\times} \alpha^{-1}(t_1)] \right] \\
\leq& \left[\alpha[\ln(c) \times \ln \phi_1(x_1) \times \ln(\frac{1}{t_1}) + \ln \phi_1(y_1) \times \ln(t_1)] \right] \\
\leq& \left[e^{[\ln \phi_1(x_1) \times \ln(c) \times \ln(\frac{1}{t_1}) + \ln \phi_1(y_1) \times \ln(t_1)]} \right] \\
\leq& \left[e^{[\ln \phi_1(x_1) \ln(c) \ln(\frac{1}{t_1}) + \ln \phi_1(y_1) \ln(t_1)]} \right] \\
\leq& \left[(e^{\ln \phi_1(x_1)})^{\ln(c)} \right]^{\ln(\frac{1}{t_1})} \cdot (e^{\ln \phi_1(y_1)})^{\ln(t_1)} \\
\phi_1 \left[x_1^{\ln(\frac{1}{t_1})} \cdot (y_1)^{\ln(t_1)} \right] \leq& \left[\phi_1(x_1)^{\ln(c)} \right]^{\ln(\frac{1}{t_1})} \cdot \phi_1(y_1)^{\ln(t_1)},
\end{aligned}$$

as required. □

4. Conclusion

In this paper, some classes of convex functions have been identified and presented. The paper established some convexity properties and inequalities in non-Newtonian calculus and their applications.

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