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Finite Difference Method for Solving Second-Order Boundary Value Problems With High-Order Accuracy

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Abstract. When researching and solving practical problems in continuous environments, through modeling methods, the vast majority of problems lead to models described by equations containing differential operators. In case the problem model is not complicated, we often obtain simple partial differential equations, then the solution of the problem can be obtained directly through analytical methods. Most complex problems, through the method of approximating differential operators by difference operators, from which differential problems are approximated by corresponding difference schemes and approximate solutions will be obtained. is achieved through solving systems of difference equations based on the tools of electronic computers. Then, building difference schemes to approximate the differential problem with high-order accuracy will play an important role in the accuracy of the obtained approximate solution. In this paper, we propose two difference schemes with high-order accuracy to solve second-order differential problems with Dirichlet and mixed boundary conditions. Theoretical results and experimental calculations have confirmed the accuracy of the proposed schemes.

1. Introduction

In this paper, we consider the second-order boundary problem with mixed boundary conditions

$$\begin{cases} u''(x) = f(x),\ x \in (a,b) \\ c_0 u(a) - c_1 u'(a) = C \\ d_0 u(b) + d_1 u'(b) = D \\ c_0, c_1, d_0, d_1 \geq 0 \end{cases} \tag{1}$$

In some cases when the function $f(x)$ is a polynomial function, trigonometric function, exponential function or product of the above functional forms, the solution to problem (1) can be found by analytical methods or using Using Green's function method, we must find the approximate solution of problem (1) by numerical methods using electronic computers.

In order to find approximate solutions using numerical methods, we need to build systems of difference equations that approximate the differential problem and then find numerical solutions

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using algebraic methods. Then the accuracy order of approximating differential operators using difference operators will determine the accuracy order of the obtained approximated solution.

In [1], a method for building a difference scheme for problem (1) was introduced by approximating the first and second derivatives of the function $u(x)$ with accuracy $O(h)$, where h is the grid step in grid space, in [2] a method of building a difference scheme with accuracy $O(h^2)$. In addition, some authors [3]- [5] have proposed difference methods with order of accuracy from $O(h^4)$ to $O(h^8)$. However, these methods lead to very complicated systems of difference equations that requiring the use of intermediate grid points. This makes solving these systems of difference equations very complicated. Therefore, it is required to build a difference scheme with order of accuracy for problem (1) so that the resulting system of difference equations is simple and can be solved with an algorithm of linear complexity.

Studying methods to approximate the derivative value at a point of a function is an important research direction that mathematicians are especially interested in. In the literature on numerical methods [1], formulas for approximating the first and second derivatives of the function $u(x)$ with accuracy order $O(h^2)$ have been given, which uses the value of the function at 3 neighboring points of point x and h is denoted as the grid step on the grid space. Based on the interpolation polynomial approximation method, in the document [6], the m-order derivative approximation formulas with accuracy order $O(h^{5-m})$ are given, which uses value of the function at 5 neighboring points of point x. Recently, in the document [7], a set of formulas for approximating derivatives of all order with accuracy order $O(h^{n-m})$ has been introduced, which uses the value of the function at $n+1$ neighboring points of point x. Based on the published formulas, numerical solutions for differential problems with nonlinear differential equations of all order have been improved. In [8]- [12], published algorithms for numerically solving 3rd and 4th order nonlinear differential equations with accuracy order $O(h^6)$. However, the published results of the above derivative approximation formulas are all results obtained by the direct method, results obtained by direct calculation of analytical expressions, not general. Thus, to improve the accuracy of derivative approximation, it is necessary to build a general algorithm to provide formulas for approximating derivatives of any degree with arbitrary order of accuracy based on the support of the computer.

The main content of the article presents research results based on the Taylo expansion formula [1] and the algorithm for determining numerical derivatives with high order accuracy, thereby proposing a method to build two difference scheme for problem (1) in the case of Dirichlet boundary conditions and mixed boundary conditions with arbitrary high-order precision. Difference schemes are very simple and can be solved with the algorithm has $O(N)$ complexity where N is the number of points in the grid space. The experimental results we calculated confirm the accuracy of the proposed schemes.

The structure of the article consists of four parts: In Section 1, we introduce some published results on based on the difference method. Section 2, we present the theoretical basis of the derivative approximation method based on the Taylor formula, then we propose a method to build a difference scheme with high-order accuracy based on the high-order derivative approximation and evaluate schema accuracy. Section 3, we present some experimental calculation results to evaluate the accuracy of the proposed algorithm. Finally, there are conclusions and references.

2. Proposed method

2.1. Derivative approximation formulas

Consider the function $u(x) \in C^{n+1}[a, b]$, expanding the Taylor series [3], we have the formula

$$u(x+\Delta x) = u(x) + u'(x)\Delta x + \frac{u''(x)}{2!}(\Delta x)^2 + ... + \frac{u^{(n)}(x)}{n!}(\Delta x)^n + R_n(x) \tag{2}$$

where $R_n(x) = \frac{u^{(n+1)}(\theta)}{(n+1)!}(\Delta x)^{n+1}$, $\theta \in (x, x+\Delta x)$ Since formula (2), we can approximate the derivative of the function $u(x)$ in the neighborhood x as follows:

$$u(x+\Delta x) \approx u(x) + u'(x)\Delta x + \frac{u''(x)}{2!}(\Delta x)^2 + ... + \frac{u^{(n)}(x)}{n!}(\Delta x)^n \tag{3}$$

Divide the interval $[a, b]$ by $(n+1)$ grid points $x_i = a + ih$, $i = 0, 1, ..., n$, grid step $h = \frac{b-a}{n}$, since formula (3), we get the following formulas

$$u(x_i + h) = u(x_i) + u'(x_i)h + \frac{u''(x_i)}{2!}h^2 + ... + \frac{u^{(n)}(x_i)}{n!}h^n + O(h^{n+1}),\ i = \overline{0, n-1} \tag{4}$$

$$u(x_i - h) = u(x_i) - u'(x_i)h + \frac{u''(x_i)}{2!}h^2 + ... + (-1)^n\frac{u^{(n)}(x_i)}{n!}h^n + O(h^{n+1}),\ i = \overline{0, n-1} \tag{5}$$

Using (4), we obtain

$$u'(x_i) = \frac{u(x_i+h) - u(x_i)}{h} - \frac{u''(x_i)}{2!}h - ... - \frac{u^{(n)}(x_i)}{n!}h^{n-1} + O(h^n) \tag{6}$$

Formula (6) is a formula that supports approximating the first derivative on a uniform grid with error $O(h^n)$.

Since (4) and (5), we have

$$u''(x_i) = \frac{u(x_i+h) - 2u(x_i) + u(x_i-h)}{h^2} - \frac{u^{(4)}(x_i)}{4!}h^2 - ... - \frac{u^{(2m)}(x_i)}{(2m)!}h^{2m-2} + O(h^{n-1}) \tag{7}$$

Let $v(x)$ be a function defined on $[a, b]$, ω_h be the grid space with grid step $h = (b-a)/N$, $x_i = a + ih$, $(i = 0, 1, 2, \ldots, N)$, $V = (V(x_0), V(x_1), \ldots, V(x_N))$ is the approximated value of the V_d on ω_h, where $V_d = (v(x_0), v(x_1), \ldots, v(x_N))$.

Definition 1: *On grid ω_h, a method for approximating the function $v(x)$ that is said to have n-order accuracy if $\|V - V_d\|_{\omega_h} = O(h^n)$.*

Since (3), (6), (7), we can build derivative approximation formulas with (n+1)- order accuracy for the grid function, and n-order accuracy for first-order derivative, (n-1)- order accuracy for second-order derivative.

Lemma 1: *If the accuracy of the second derivative approximation is of order $(n-2)$, then the accuracy of the function approximation will be of order n and vice versa.*

Let

$$T_i(x) = (x - z_0)(x - z_1)...(x - z_{i-1})(x - z_{i+1})...(x - z_n),$$

$$M_i(x) = (z_i - z_0)(z_i - z_1)...(z_i - z_{i-1})(z_i - z_{i+1})...(z_i - z_n).$$

Let $B_n^((m))$ be the coefficient matrix in the m- order derivative approximation formula using $(n+1)$ neighboring points, then the grid derivative of orders will be determined by the formula:

$$\begin{cases} u^{(m)}(x_i) = \frac{1}{h^m}\sum\limits_{k=0}^{n} B_n^{(m)}(i,k)u_k, \ i < n/2 \\ u^{(m)}(x_i) = \frac{1}{h^m}\sum\limits_{k=0}^{n} B_n^{(m)}(i,k)u_{i+k-n/2}, \ n/2 \le i \le N - n/2 \\ u^{(m)}(x_i) = \frac{1}{h^m}\sum\limits_{k=0}^{n} B_n^{(m)}(i,k)u_{N-n+k}, \ i > N - n/2 \end{cases} \tag{8}$$

where,$L_i^{(m)}(x) = m!\frac{T_i^{(m)}(x)}{M_i}$; $T_i^{(m)}(x) = (x - z_m)(x - z_{m+1})...(x - z_n) + (x - z_0)(x - z_{m+1})...(x - z_n) + (x - z_0)(x - z_1)(x - z_{m+2})...(x - z_n) + ... + (x - z_0)(x - z_1)(x - z_2)...(x - z_{n-m})$.

Theorem 1: *The accuracy of approximating the grid derivative of order m using $(n+1)$ neighboring points on the regular grid is of order $(n - m + 1)$.*

Formulas (6), (7), (8) will be used to propose the difference scheme given in section 2.2

2.2. Difference scheme with high order accuracy

Consider the differential problem

$$L_n u(x) = f(x),\ B_a u(a) = G_a,\ B_b u(b) = G_b,\ x \in [a,b], \tag{9}$$

where L_n is the linear differential operator, B_a, B_b are the boundary condition operators.

Let

$$\Lambda_n U = \Phi,\ \Lambda_a U = \Phi_a,\ \Lambda_b U = \Phi_b \tag{10}$$

be the difference scheme for the differential problem (9), where Λ_n is the linear difference operator, Λ_a, Λ_b are the boundary condition difference operators. Let U be the grid function determined based on the difference scheme, let U_d be the value of $u(x)$ on the grid ω_h. Below we give two difference schemes corresponding to two differential problems.

2.2.1. Differential scheme 1

Consider the second-order boundary problem with dirichlet boundary conditions

$$\begin{cases} u''(x) = f(x),\ x \in (a,b) \\ u(a) = C;\ u(b) = D. \end{cases} \tag{11}$$

Considering the grid space ω_h, let $U_i = u(x_i)$, $U = (U_0, U_1, \dots, U_N)$ be the grid function, $F = (F_0, F_1, \dots, F_N)$ is the right-hand grid function, $D^{(m)}F$ is the m-order derivative of the grid function F. Since $f(x)$ satisfies the differential equation (11), we have $f(x_i) = u''(x_i), i = 0, 1, 2, \dots, N$. Using (7) we obtain the approximated formula

$$u''(x_i) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - \frac{h^2}{4!}D^{(2)}F_i - \frac{h^4}{6!}D^{(4)}F_i - \dots - \frac{h^{2m-2}}{(2m)!}D^{(2m-2)}F_i + O(h^{n-1}) \quad (12)$$

Set $\Phi_i = h^2 F_i + \frac{h^4}{4!}D^{(2)}F_i + \frac{h^6}{6!}D^{(4)}F_i + \dots + \frac{h^{2m}}{(2m)!}D^{(2m-2)}F_i$, $1 \le i \le N-1$, we obtain the system of difference equations

$$\begin{cases} U_{i-1} - 2U_i + U_{i+1} = \Phi_i,\ i = 1, 2, ..., N-1 \\ U_0 = C;\ U_N = D \end{cases} \quad (13)$$

(13) is is called a three-diagonal system because the matrix of the system has a three-diagonal form. Solving system (13) is performed using the following pursuit algorithm. In the case of constructing scheme (13), we used formula (7) which is used to approximate the 2nd derivative with $(n-2)$-order accuracy, therefore, since Lemma 1, it is implies solution of system (13) is the approximation of the function with n-order accuracy. Since there we have the following theorem:

Theorem 2: *The system of difference equations (13) is stable and its solution approximates to solution of problem (11) with n-order accuracy.*

The solution of the system is obtained from the pursuit algorithm which has complexity $O(N)$.

2.2.2. Differential scheme 2

Consider the second-order boundary problem with mixed boundary conditions

$$\begin{cases} u''(x) = f(x),\ x \in (a, b) \\ c_0 u(a) - c_1 u'(a) = C;\ d_0 u(b) + d_1 u'(b) = D \end{cases} \quad (14)$$

Consider the grid space ω_h, set $U_i = u(x_i)$, $F_i = f(x_i)$, $U = (U_0, U_1, \dots, U_N)$ be grid function. $F = (F_0, F_1, \dots, F_N)$ is the right-hand grid function, $D_n^{(m)}F$ is the m-order derivative of the grid function F with n-order accuracy. Since (6), we obtain the difference formula

$$u'(x_i) = \frac{U_{i+1} - U_i}{h} - \frac{h}{2!}F_i - \frac{h^2}{3!}D^{(1)}F_i - \dots - \frac{h^{n-1}}{n!}D^{(n-2)}F_i + O(h^n) \quad (15)$$

Consider two points $x_0 = a$, $x_N = b$, we obtain

$$u'(a) = \frac{U_1 - U_0}{h} - \frac{h}{2!}F_0 - \frac{h^2}{3!}D^{(1)}F_0 - \dots - \frac{h^{n-1}}{n!}D^{(n-2)}F_0 + O(h^n) \quad (16)$$

$$u'(b) = \frac{U_N - U_{N-1}}{h} + \frac{h}{2!}F_N - \frac{h^2}{3!}D^{(1)}F_N + \dots + (-1)^n\frac{h^{n-1}}{n!}D^{(n-2)}F_N + O(h^n) \quad (17)$$

Substituting formulas (16) and (17) into the boundary condition system, combined with scheme number 1, we obtain the difference scheme

$$\begin{cases} c_0U_0 - c_1\left(\frac{U_1-U_0}{h} - \frac{h}{2!}F_0 - \frac{h^2}{3!}D^{(1)}_{n-2}F_0 - \dots - \frac{h^{n-1}}{n!}D^{(n-2)}_1F_0\right) = C \\ U_{i-1} - 2U_i + U_{i+1} = \Phi_i,\ i = 1, 2, \dots, N-1 \\ d_0U_N + d_1\left(\frac{U_N-U_{N-1}}{h} + \frac{h}{2!}F_N - \frac{h^2}{3!}D^{(1)}_{n-2}F_N + \dots + (-1)^n\frac{h^{n-1}}{n!}D^{(n-2)}_1F_N\right) = D \end{cases} \tag{18}$$

The solution of the system is obtained from the pursuit algorithm which has complexity $O(N)$. Since there we have the following theorem:

Theorem 3: *The system of difference equations (18) is stable and the solution approximates the solution of problem (14) with n-order accuracy.*

The following are some experimental calculation results for the proposed theory.

3. Experimental results and discussions

In this section, we verify the accuracy of the proposed schemes, we let the function $u_d(x)$ that satisfies the boundary problem, from there, we determine the right-hand side function $f(x)$, $x \in [a, b]$ and boundary conditions. On grid space ω_h, we define the grid function value $U_d = (u_d(x_0), u_d(x_1), \dots, u_d(x_n))$ is the exact solution value on the grid ω_h. Then, we use the proposed schemes to find approximate solutions $U = (U_0, U_1, \dots, U_N)$, from there, we evaluate the accuracy of the scheme through error $\varepsilon = \|U - U_d\|_{\omega_h}$.

3.1. Evaluate the accuracy of scheme 1

Example 1: Consider the problem

$$\begin{cases} u''(x) = -sinx,\ x \in (0, 1) \\ u(0) = 0;\ u(1) = sin(1) \end{cases}$$

Exact solution of the problem is $u(x) = sinx$.

Using scheme 1 combined with the pursuit algorithm, we obtain the grid solution U. The results of evaluating the accuracy of scheme 1 are given in table 1

Table 1. *Results of evaluating the accuracy of scheme 1 - Example 1*

N	h^8	$\|U - U_d\|_{\omega_h}$	h^{10}	$\|U - U_d\|_{\omega_h}$
10	1.0000e-008	2.9675e-012	1.0000e-010	1.6467e-014
20	3.9063e-011	4.6426e-014	9.7656e-014	1.1343e-017
30	1.5242e-012	4.0825e-015	1.6935e-015	3.6987e-019
40	1.5259e-013	7.2653e-016	9.5367e-017	2.9933e-020
50	2.5600e-014	1.9039e-016	1.0240e-017	4.1543e-021
60	5.9537e-015	6.3781e-017	1.6538e-018	8.1987e-022
70	1.7347e-015	2.5295e-017	3.5401e-019	2.0679e-022

N	h^8	$\|U - U_d\|_{\omega_h}$	h^{10}	$\|U - U_d\|_{\omega_h}$
80	5.9605e-016	1.1351e-017	9.3132e-020	6.2629e-023
90	2.3231e-016	5.5993e-018	2.8680e-020	2.1680e-023
100	1.0000e-016	2.9759e-018	1.0000e-020	8.4722e-024

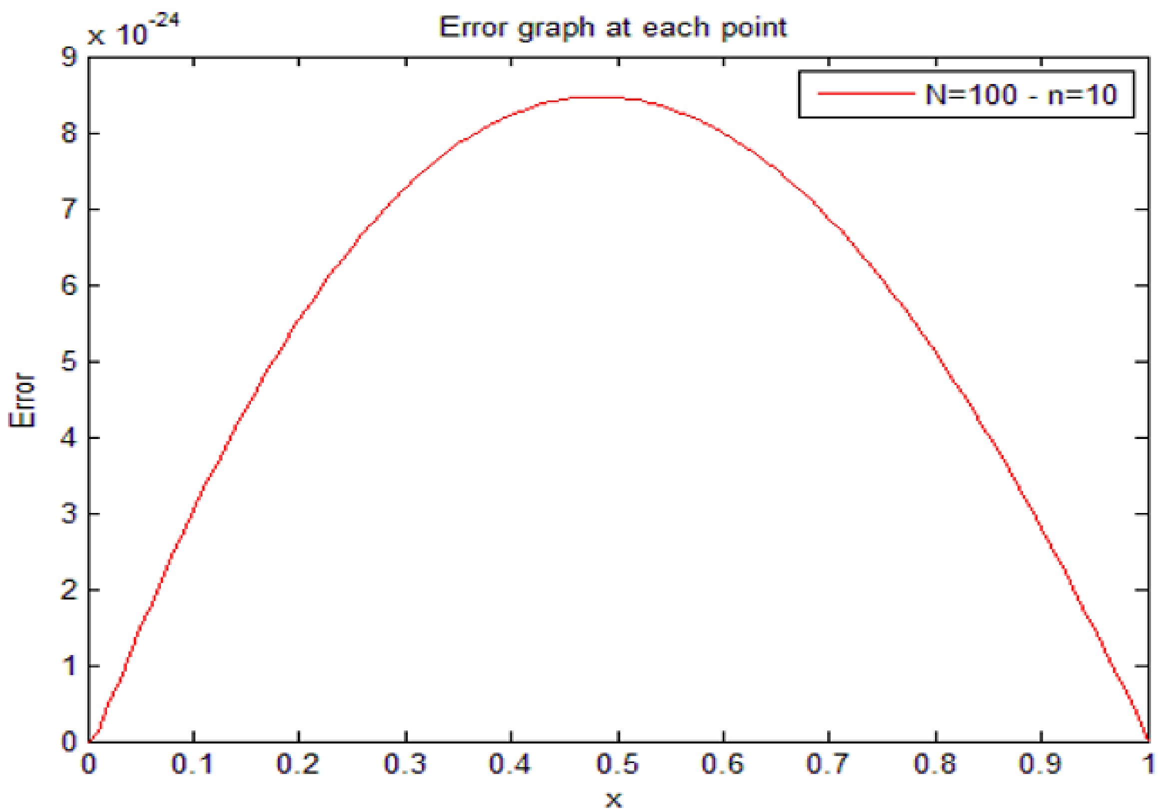

Figure 1. *Error at each point between exact solution and approximated solution - Example 1*

Example 2: Consider the problem

$$\begin{cases} u''(x) = -sinh(x) + \frac{1}{9}e^{-x/3}, \; x \in (0,1) \\ u(0) = 1; \; u(1) = sinh(1) + e^{-1/3} \end{cases}$$

Exact solution of the problem is $u(x) = sinh(x) + e^{-x/3}$.

Table 2. *Results of evaluating the accuracy of scheme 1 - Example 2*

N	h^8	$\|U - U_d\|_{\omega_h}$	h^{10}	$\|U - U_d\|_{\omega_h}$
10	1.0000e-008	3.3899e-012	1.0000e-010	2.0506e-014
20	3.9063e-011	5.3052e-014	9.7656e-014	1.7488e-017
30	1.5242e-012	4.6589e-015	1.6935e-015	5.2038e-019
40	1.5259e-013	8.3000e-016	9.5367e-017	4.0990e-020
50	2.5600e-014	2.1765e-016	1.0240e-017	5.6274e-021

N	h^8	$\|U-U_d\|_{\omega_h}$	h^{10}	$\|U-U_d\|_{\omega_h}$
60	5.9537e-015	7.2895e-017	1.6538e-018	1.1046e-021
70	1.7347e-015	2.8908e-017	3.5401e-019	2.7774e-022
80	5.9605e-016	1.2973e-017	9.3132e-020	8.3970e-023
90	2.3231e-016	6.3989e-018	2.8680e-020	2.9033e-023
100	1.0000e-016	3.4009e-018	1.0000e-020	1.1337e-023

The results in Table 1 and Table 2 confirm that our proposed scheme 1 has determined the approximated solution of the second-order boundary problem with Dirichlet boundary conditions with n-order accuracy, where n is the number of neighboring points.

3.2. Evaluate the accuracy of scheme 2

Example 3: Consider the problem

$$\begin{cases} u''(x) = 6x + \frac{1}{8}sinh(1+x/2),\ x \in (0,1) \\ u(0) - u'(0) = sinh(1) - \frac{1}{2}cosh(1);\ u(1) + u'(1) = 4 + sinh(3/2) + \frac{1}{2}cosh(3/2) \end{cases}$$

Exact solution of the problem is $u(x) = x^3 + sinh(1 + x/2)$

Using scheme 2 combined with the pursuit algorithm, we obtain the grid solution U. The results of evaluating the accuracy of scheme 2 are given in table 3

Table 3. *Results of evaluating the accuracy of scheme 2 - Example 3*

N	h^8	$\|U-U_d\|_{\omega_h}$	h^{10}	$\|U-U_d\|_{\omega_h}$
10	1.0000e-008	2.0382e-013	1.0000e-010	5.3121e-016
20	3.9063e-011	3.3597e-015	9.7656e-014	9.1360e-019
30	1.5242e-012	2.9998e-016	1.6935e-015	2.5005e-020
40	1.5259e-013	5.3835e-017	9.5367e-017	2.0884e-021
50	2.5600e-014	1.4183e-017	1.0240e-017	3.1494e-022
60	5.9537e-015	4.7654e-018	1.6538e-018	7.0444e-023
70	1.7347e-015	1.8943e-018	3.5401e-019	1.6852e-023
80	5.9605e-016	8.5167e-019	9.3132e-020	5.3577e-024
90	2.3231e-016	4.2068e-019	2.8680e-020	1.8066e-024
100	1.0000e-016	2.2381e-019	1.0000e-020	3.0015e-025

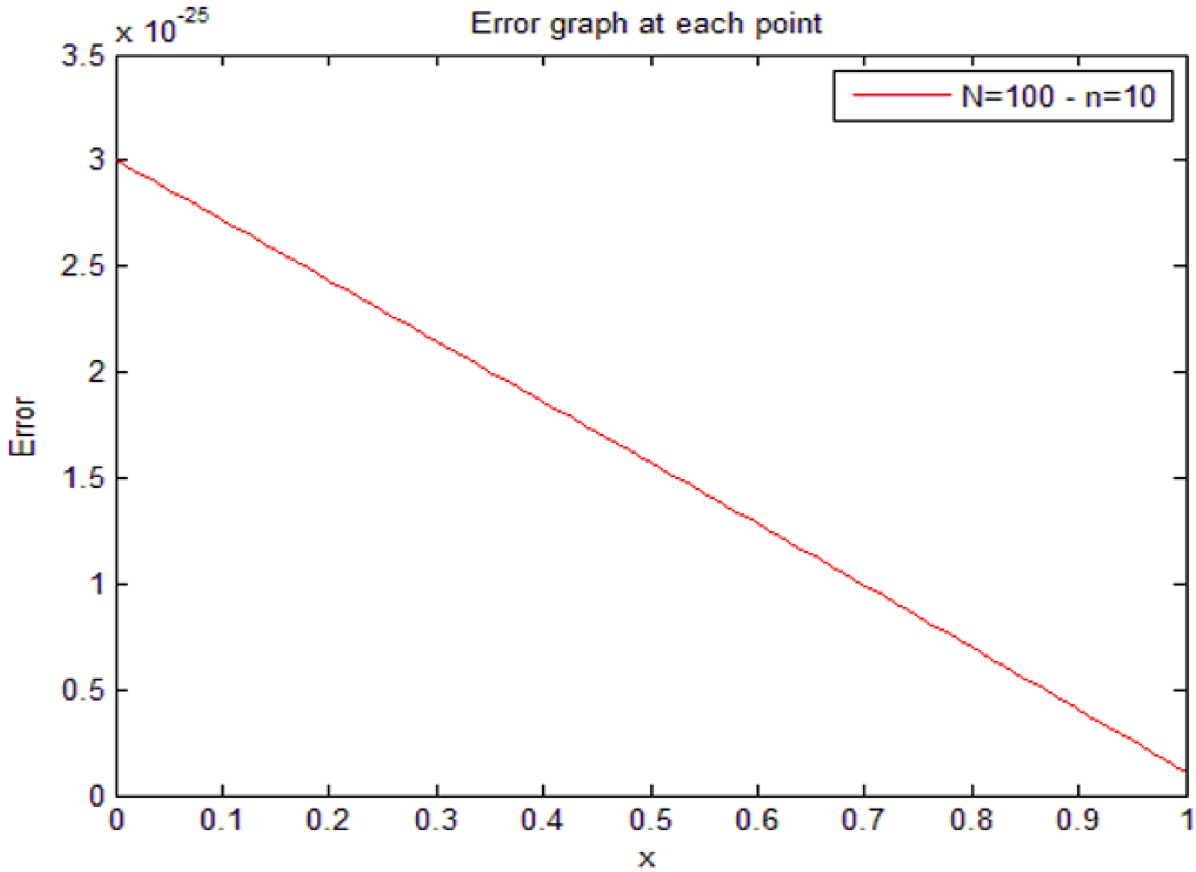

Figure 2. *Error at each point between exact solution and approximated solution - Example 3*

Example 4: Consider the problem

$$\begin{cases} u''(x) = cosh(x) + \frac{1}{9}e^{x/3}, \ x \in (0,1) \\ u(0) - \frac{1}{2}u'(0) = \frac{1}{2}; \ \frac{1}{3}u(1) + u'(1) = \frac{1}{3}\left(cosh(1) + e^{1/3}\right) - \left(sinh(1) + \frac{1}{3}e^{1/3}\right) \end{cases}$$

Exact solution of the problem is $u(x) = cosh(x) + e^{x/3}$

Using scheme 2 combined with the pursuit algorithm, we obtain the grid solution U. The results of evaluating the accuracy of scheme 2 are given in table 4.

Table 4. *Results of evaluating the accuracy of scheme 2 - Example 4*

N	h^8	$\|U - U_d\|_{\omega_h}$	h^{10}	$\|U - U_d\|_{\omega_h}$
10	1.0000e-008	3.7083e-011	1.0000e-010	3.5960e-013
20	3.9063e-011	6.1634e-013	9.7656e-014	5.6152e-016
30	1.5242e-012	5.5099e-014	1.6935e-015	1.1085e-017
40	1.5259e-013	9.8929e-015	9.5367e-017	6.5992e-019
50	2.5600e-014	2.6068e-015	1.0240e-017	7.2379e-020
60	5.9537e-015	8.7603e-016	1.6538e-018	1.1673e-020

N	h^8	$\|U - U_d\|_{\omega_h}$	h^{10}	$\|U - U_d\|_{\omega_h}$
70	1.7347e-015	3.4826e-016	3.5401e-019	2.4481e-021
80	5.9605e-016	1.5659e-016	9.3132e-020	6.1971e-022
90	2.3231e-016	7.7351e-017	2.8680e-020	1.7954e-022
100	1.0000e-016	4.1155e-017	1.0000e-020	5.7442e-023

The results in Table 3 and Table 4 also confirm that our proposed Scheme 2 has determined the approximate solution of the second-order boundary problem with mixed boundary conditions with n-order accuracy, where n is the number of neighboring points.

4. Conclusion

The main content of article has proposed two difference schemes with high-order accuracy to solve second-order boundary problems with the Dirichlet boundary condition system and the mixed boundary condition system. The highlight of these two schemes compared to other schemes [3]- [5] is that the two proposed schemes have a simple structure, high accuracy and finding solutions is done using a pursuit algorithm with the computational complexity is $O(N)$, where N is the number of grid points. Through the calculation results, it has been confirmed that the two schemes both provide approximated solutions with n-level accuracy compared to the grid step with $n+1$ being the number of neighboring points used to determine approximate derivatives of levels. These difference schemes will allow to improve the accuracy of the solution for the class of nonlinear boundaries with mixed boundary conditions.

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