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On the Dirichlet Boundary Value Problem for the Cauchy–Riemann Equations in the Half Disc

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Abstract. In this article, we investigate the Dirichlet boundary value problem for the Cauchy–Riemann equations in the half disc. First, using the technique of parqueting–reflection and the Cauchy–Pompeiu representation formula for a half disc, we obtain an integral representation formula in the half disc. In other words, we construct a unique solution for the Dirichlet boundary value problem. Finally, we solve the Dirichlet boundary value problem for both the homogeneous and the inhomogeneous Cauchy–Riemann equations. In particular, the boundary behaviors at the corner points are considered.

1. Introduction and Preliminaries

Boundary value problems are an essential concept in the field of mathematical analysis and partial differential equations. They arise when seeking solutions partial differential equations subject to specific conditions on different parts of the boundary of the domain. The Dirichlet boundary value problem is a fundamental concept in mathematical analysis, particularly in the field of partial differential equations. It deals with finding a solution to a partial differential equation that satisfies certain prescribed conditions on the boundary of a given domain.

One of the most powerful tools for constructing solutions to the Dirichlet problem is the integral representation formula. It provides a way to express the solution in term of an integral over the boundary of domain, which can often simplify the problem and lead to explicit solutions. This formula allows for the efficient and accurate computation of solutions wide range of partial differential equations, making it an essential tool in the field of partial differential equations.

The parqueting–reflection principle is a technique used in constructing integral representation formulas for the Dirichlet boundary value problem. For specific regions of complex plane whose boundary consists of sub-arcs of circles or straight lines, the parqueting–reflection method for

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constructing integral representation formula to solve the Dirichlet boundary value problem for the Cauchy–Riemann equation is used. If the boundary of the region has the mentioned characteristics, it is possible to create a new region by reflecting the main region with respect to its boundary and by reflecting this new region with respect to its boundary, another region is obtained. By continuing this process, we reach pieces of the plane that the union of these pieces provides a cover for the complex plane. This cover can be achieved through a single reflection or multiple consecutive reflections or infinite repetitive reflections. Therefore, the reflection of the main region at its boundary is repeated to achieve a cover for complex plane.

Many results have been obtained for boundary value problems of complex partial differential equations in some particular domains, see, e.g. [1–16]. In the year 2009, Harmonic boundary value problem for the Poisson equation in a half disc was presented by H. Begehr and T. Vaitekhovich [2]. In 2012, Y. Wang introduced Schwarz-type boundary value problems for the polyanalytic equation in the half unit disc [16].

In this article, in addition to introducing the domain of half disc, we want to express the reflections, covering and points that we obtain at each stage. we also construct the integral representation formula using the parqueting–reflection method and investigate the Dirichlet problem. In particular, we study the explicit solvability of the Dirichlet boundary value problem for both the homogeneous and the inhomogeneous Cauchy–Riemann equations in the half disc.

In this article, Let M be the half disc domain in the complex plane $\mathbb{C}$ defined by

$$M = \{z \in \mathbb{C} : |z| < 1,\ Imz > 0\}$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ and the boundary of M is denoted by ∂M. It is formed by arc of the circle $\mathbb{D}$ and a line segment on the real axis from point -1 to 1. See figure 1.

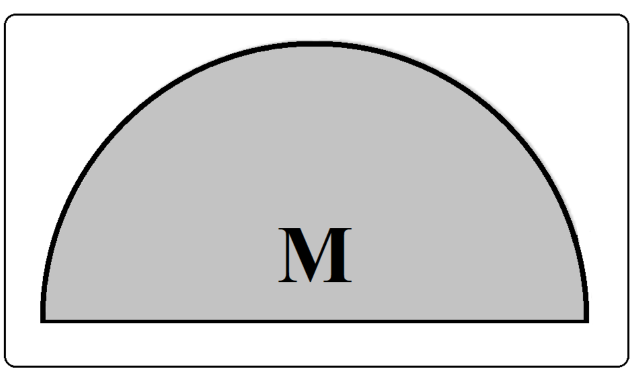

Figure 1. Half Disc M

Now, we introduce some important definitions and properties of complex analytic functions, results which will be required in subsequent sections.

Defining the complex partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ by

$$\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}),\ \ \frac{\partial}{\partial \bar{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}).$$

Let the complex-valued function ω be defined in M and let u and v denote its real and imaginary parts: $\omega=u+iv$, where $u(x,y)$ and $v(x,y)$ are real-valued functions. The two partial differential equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \tag{1.1}$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}, \tag{1.2}$$

are called the Cauchy–Riemann equations for the pair of functions u and v. Multiplying the both sides of the equality (1.2) by i,

$$i(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})=0. \tag{1.3}$$

Adding (1.1) and (1.3) leads to

$$\frac{\partial}{\partial x}(u+iv)+i\frac{\partial}{\partial y}(u-\frac{1}{i}v)=0.$$

Thus

$$\frac{1}{2}(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y})=0.$$

The Cauchy–Riemann equation can be written as $\frac{\partial \omega}{\partial \bar{z}}=\omega_{\bar{z}}=0$ and this is the condition for ω to be analytic function. Recall the definition of the Pompeiu integral operator

$$Tf(z)=-\frac{1}{\pi}\int_M \frac{f(t)}{t-z}d\xi d\eta,$$

$Tf(z)$ is weakly differentiable with $\omega_{\bar{z}}=f$ when $f\in L_p(M;\mathbb{C}), p>2$ and $t=\xi+i\eta$, [14].

2. An Integral Representation Formula for M

In this section, the integral representation formula for the half disc domain is constructed. A convenient technique for constructing the integral representation formula for domain with boundaries consisting of arcs and straight lines is given by the parqueting–reflection method. Now using the parqueting–reflection technique for the introduced domain, we obtain a cover for the entire complex plane. This coverage is obtained from three consecutive reflections.

Reflecting any z at circle gives

$$|z-a|=r\Rightarrow(z-a)(\bar{z}-\bar{a})=r^2\Rightarrow z_r=\frac{a\bar{z}-a\bar{a}+r^2}{\bar{z}-\bar{a}}.$$

Reflecting any $z \in M$ at the $\mathbb{D}$, gives

$$|z| = 1 \Rightarrow z\bar{z} = 1 \Rightarrow z^* = \frac{1}{\bar{z}}.$$

Reflecting any z at the real axis gives $\bar{z}$. Therefore, we can obtain the following points

$$z_1^* = \frac{1}{\bar{z}},\ z_2^* = \frac{1}{z},\ z_3^* = \bar{z}.$$

Those reflections produce a parqueting of the entire complex plane. To solve the Dirichlet boundary value problems for analytic functions the integral representation formula is important. Now, using the Cauchy–Pompeiu representation formula, we construct the integral representation formula for the half disc domain.

Theorem 2.1. *Any $\omega \in C^1(M;\mathbb{C}) \bigcap C(\overline{M};\mathbb{C})$ can be represented as*

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial M} \omega(t)\left[\frac{1}{t-z} + \frac{z}{tz-1}\right]dt - \frac{1}{\pi}\int_M \omega_{\bar{t}}(t)\left[\frac{1}{t-z} + \frac{z}{tz-1}\right]d\xi d\eta, \tag{2.1}$$

where $t = \xi + i\eta$.

Proof. Applying the Cauchy–Pompeiu formula [3]

$$\frac{1}{2\pi i}\int_{\partial M} \omega(t)\frac{dt}{t-z} - \frac{1}{\pi}\int_M \omega_{\bar{t}}(t)\frac{d\xi d\eta}{t-z} = \begin{cases} \omega(z) & z \in M, \\ 0 & z \notin \overline{M}. \end{cases} \tag{2.2}$$

for $z \in M$ and $z_2^* \notin \overline{M}$. Substitute the points into the Cauchy–Pompeiu formula (2.2)

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial M} \omega(t)\frac{dt}{t-z} - \frac{1}{\pi}\int_M \omega_{\bar{t}}(t)\frac{d\xi d\eta}{t-z}. \tag{2.3}$$

Since point z_2^* is outside the domain M, therefore, the Cauchy–Pompeiu formula is equal to zero at this point.

$$0 = \frac{1}{2\pi i}\int_{\partial M} \omega(t)\frac{z dt}{tz-1} - \frac{1}{\pi}\int_M \omega_{\bar{t}}(t)\frac{z d\xi d\eta}{tz-1}, \tag{2.4}$$

By combining the obtained equations, the integral representation formula is obtained. □

The integral representation formula (2.1), serves to solve the related the Dirichlet problem for the Cauchy–Riemann equations in M.

3. Dirichlet Problem for the Cauchy–Riemann Equation in M

In this section, we study the Dirichlet boundary value problem for the homogeneous and the inhomogeneous Cauchy–Riemann equations. In the following, we solve the Dirichlet boundary value problem for the homogeneous Cauchy–Riemann equation.

Theorem 3.1. *The Dirichlet problem for the homogeneous Cauchy–Riemann equation*

$$\begin{cases} \omega_{\bar{z}} = 0, & in \quad M, \\ \omega = \gamma, & on \quad \partial M, \ \gamma \in C(\partial M; \mathbb{C}) \end{cases} \tag{3.1}$$

with given $\gamma \in C(\partial M; \mathbb{C})$, $\gamma(\pm 1) = 0$, *is solvable, if and only if*

$$\frac{1}{2\pi i} \int_{\partial M} \gamma(t) \left[\frac{1}{t - \bar{z}} + \frac{\bar{z}}{t\bar{z} - 1} \right] dt = 0, \tag{3.2}$$

and the unique solution can be presented as

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial M} \gamma(t) \left[\frac{1}{t - z} + \frac{z}{tz - 1} \right] dt, \ \ z \in M. \tag{3.3}$$

Proof. Let ω defined by (3.3) be a solution to the Dirichlet problem. Then the equality

$$\omega(z) = \gamma(t), \quad t \in \partial M, \tag{3.4}$$

holds. We consider the following function

$$h(z) = \frac{1}{2\pi i} \int_{\partial M} \gamma(t) \left[\frac{1}{t - \bar{z}} + \frac{\bar{z}}{t\bar{z} - 1} \right] dt, \tag{3.5}$$

and take the difference

$$\begin{aligned} \omega(z) - h(z) &= \frac{1}{2\pi i} \int_{\partial M} \gamma(t) \left[\frac{1}{\zeta - z} + \frac{z}{tz - 1} \right] dt - \frac{1}{2\pi i} \int_{\partial M} \gamma(t) \left[\frac{1}{t - \bar{z}} + \frac{\bar{z}}{t\bar{z} - 1} \right] dt \\ &= \frac{1}{2\pi i} \int_{\partial M} \gamma(t) \left[\frac{1}{t - z} - \frac{1}{t - \bar{z}} + \frac{z}{tz - 1} - \frac{\bar{z}}{t\bar{z} - 1} \right] dt \\ &= \frac{1}{2\pi i} \int_{\partial M \cap \mathbb{D}} \gamma(t) \left[\frac{t}{t - z} + \frac{\bar{t}}{\bar{t} - \bar{z}} - \frac{\bar{t}}{\bar{t} - z} - \frac{t}{t - \bar{z}} \right] \frac{dt}{t} \\ &+ \frac{1}{2\pi i} \int_{-1}^{1} \gamma(s) \left[\frac{1}{s - z} - \frac{1}{s - \bar{z}} + \frac{z}{sz - 1} - \frac{\bar{z}}{s\bar{z} - 1} \right] ds \\ &= \frac{1}{2\pi i} \int_{\partial M \cap \mathbb{D}} \gamma(t) \left[\frac{1 - |z|^2}{|t - z|^2} - \frac{1 - |z|^2}{|\bar{t} - z|^2} \right] \frac{dt}{t} \\ &+ \frac{1}{2\pi i} \int_{-1}^{1} \gamma(s) \left[\frac{z - \bar{z}}{|s - z|^2} - \frac{z - \bar{z}}{|1 - zs|^2} \right] ds. \end{aligned}$$

Studying the boundary behavior of the boundary integral implies computations on the different parts of the boundary ∂M. For $|t_0| = 1, Imt_0 > 0$. As $|\bar{t} - t_0|^2 \neq 0$, $1 - |t_0|^2 = 0$ and $|s - t_0|^2 = |1 - t_0 s|^2$. Thus,

$$\lim_{z \to t} (\omega(z) - h(z)) = \gamma(t).$$

For $|t_0| < 1$, $Imt_0 = 0$. Since $|t - t_0| = |\bar{t} - t_0|$, $|1 - t_0 s|^2 \neq 0$, $t - \bar{t}_0 = 0$. Thus,

$$\lim_{z \to t} (\omega(z) - h(z)) = \gamma(t).$$

Now, we consider the boundary behavior at the corner points. Let

$$\omega(z) - h(z) = \frac{1}{2\pi i} \int_{\partial M \cap \mathbb{D}} \gamma(t) \left[\frac{t + z}{t - z} - \frac{\bar{t} + z}{\bar{t} - z} \right] \frac{dt}{t}$$

we can write

$$\begin{aligned}\omega(z)-h(z)&=\frac{1}{2\pi i}\int_{\partial M\cap\mathbb{D}}\gamma(t)\frac{t+z}{t-z}\frac{dt}{t}+\frac{1}{2\pi i}\int_{\partial M\cap\mathbb{D}}\gamma(t)\frac{\bar{t}+z}{\bar{t}-z}\frac{\overline{dt}}{\bar{t}}\\&=\frac{1}{2\pi i}\int_{\partial M\cap\mathbb{D}}\gamma(t)\frac{t+z}{t-z}\frac{dt}{t}-\frac{1}{2\pi i}\int_{\overline{\partial M\cap\mathbb{D}}}\gamma(\bar{t})\frac{t+z}{t-z}\frac{dt}{t}\\&=\frac{1}{2\pi i}\int_{\partial M\cap\mathbb{D}}\Upsilon(t)\frac{t+z}{t-z}\frac{dt}{t}\end{aligned}$$

where

$$\Upsilon(t)=\begin{cases}\gamma(t), & Imz\geq 0,\\ -\gamma(\bar{t}), & Imz<0.\end{cases}$$

From the properties of the Poisson kernel for unit disc [2,3], we have

$$\lim_{z\to t}(\omega(z)-h(z))=\Upsilon(t).$$

In particular

$$\lim_{z\to\pm 1}(\omega(z)-h(z))=\gamma(\pm 1)=0,$$

is seen because of the continuity of Υ at ± 1.

Similar to what was done above, from the properties of the Poisson kernel for half plane [2], we have

$$\lim_{z\to t}(\omega(z)-h(z))=\gamma(t). \tag{3.6}$$

By (3.4) and (3.6), we have

$$\lim_{z\to t}h(z)=0,\quad t\in\partial M.$$

Then, from the maximum principle for analytic functions $h(z)=0$ for $z\in M$, which is given as condition (3.2).

Conversely, If the condition (3.2) is is satisfied, then, the analytic function ω can be expressed as

$$\begin{aligned}\omega(z)&=\omega(z)-h(z)\\&=\frac{1}{2\pi i}\int_{\partial M}\gamma(t)\left[\frac{1}{t-z}+\frac{z}{tz-1}\right]dt-\frac{1}{2\pi i}\int_{\partial M}\gamma(t)\left[\frac{1}{t-\bar{z}}+\frac{\bar{z}}{t\bar{z}-1}\right]dt\\&=\frac{1}{2\pi i}\int_{\partial M}\gamma(t)\left[\frac{1}{t-z}-\frac{1}{t-\bar{z}}+\frac{z}{tz-1}-\frac{\bar{z}}{t\bar{z}-1}\right]dt.\end{aligned}$$

Hence,

$$\lim_{z\to t}\ \omega(z)=\gamma(t),\quad t\in\partial M. \tag{3.7}$$

Follows again from the properties of the Poisson kernel. □

In the next stage, we investigate the Dirichlet problem for the inhomogeneous Cauchy–Riemann equation. To solve this problem, we reduce the inhomogeneous problem into a homogeneous one, using definition and properties of the Pompeiu operator, and then find a solution for it using the previous theorem.

Theorem 3.2. *The Dirichlet boundary value problem for the inhomogeneous Cauchy–Riemann equation*

$$\omega_{\bar{z}} = f(z), \;\; z \in M, f \in L_p(M;\mathbb{C}), p > 2,$$

$$\omega = \gamma, \;\; on\ \partial M, \;\; \gamma \in C(\partial M;\mathbb{C}), \tag{3.8}$$

is solvable if and only if for $z \in M$,

$$\frac{1}{2\pi i}\int_{\partial M}\gamma(t)\left[\frac{1}{t-\bar{z}}+\frac{\bar{z}}{t\bar{z}-1}\right]dt = \frac{1}{\pi}\int_M f(t)\left[\frac{1}{\zeta-\bar{z}}+\frac{\bar{z}}{t\bar{z}-1}\right]d\xi d\eta, \tag{3.9}$$

and its solution can be uniquely expressed as

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial M}\gamma(t)\left[\frac{1}{t-z}+\frac{z}{tz-1}\right]dt - \frac{1}{\pi}\int_M f(t)\left[\frac{1}{t-z}+\frac{z}{tz-1}\right]d\xi d\eta. \tag{3.10}$$

where $t = \xi + i\eta$.

Proof. By the Theorem 2.1, If the Dirichlet problem (3.8) is solvable, its can be expressed in the form of (3.10). Let $\varphi(z) = \omega(z) - Tf(z)$, by applying the $\partial_{\bar{z}}$ operator to the function ' we have,

$$\partial_{\bar{z}}\varphi = \partial_{\bar{z}}\omega - \partial_{\bar{z}}Tf \Rightarrow \partial_{\bar{z}}\varphi = f - f = 0,$$

$$\varphi = \omega - Tf \Rightarrow \varphi = \gamma - Tf.$$

Then consider the homogeneous Dirichlet problem

$$\varphi_{\bar{z}} = 0, \;\; in\ z \in M, \;\; \varphi = \gamma - Tf, \;\; on\ \partial M. \tag{3.11}$$

which is equivalent to equation (3.11) By the Theorem 3.2 the solvability condition for equation (3.14) is

$$\frac{1}{2\pi i}\int_{\partial M}(\gamma(t) - Tf(t))\left[\frac{1}{t-\bar{z}}+\frac{\bar{z}}{t\bar{z}-1}\right]dt = 0,$$

Using the properties of the integral operator

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial M}Tf(t)\left[\frac{1}{t-\bar{z}}+\frac{\bar{z}}{t\bar{z}-1}\right]dt &= \frac{1}{\pi}\int_M f(\tilde{t})\frac{1}{2\pi i}\int_{\partial M}\left[\frac{1}{t-\bar{z}}+\frac{\bar{z}}{t\bar{z}-1}\right]\frac{dt}{t-\tilde{t}}d\tilde{\xi}d\tilde{\eta}\\ &= \frac{1}{\pi}\int_M f(\tilde{t})\left[\frac{1}{\tilde{t}-\bar{z}}+\frac{\bar{z}}{\tilde{t}\bar{z}-1}\right]d\tilde{\xi}d\tilde{\eta}.\end{aligned}$$

which is just condition (3.9).

On the other hand, if the condition of solvability (3.9) is satisfied, then (3.10) can be expressed as follows.

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial M}\gamma(z)\left[\frac{1}{t-z}+\frac{z}{tz-1}-\frac{1}{t-\bar{z}}-\frac{\bar{z}}{t\bar{z}-1}\right]dt.$$
$$-\frac{1}{\pi}\int_{M} f(z)\left[\frac{1}{t-z}+\frac{z}{tz-1}-\frac{1}{t-\bar{z}}-\frac{\bar{z}}{t\bar{z}-1}\right]d\xi d\eta. \quad (3.12)$$

Since the area integral tends to 0 as $z \to t \in \partial M$, by the proof of Theorem (3.2), (3.12) implies that

$$\lim_{z\to t}\omega(z) = \gamma(t), \quad t \in \partial M.$$

Now, we are going to investigate the uniqueness of the Dirichlet problem solution. Assume that ω and w are two solutions to the Dirichlet problem, therefore we have

$$\omega_{\bar{z}} = f, \ \ in\ z \in M, \ \ \omega = \gamma \ \ on\ \partial M,$$

$$w_{\bar{z}} = f, \ \ in\ z \in M, \ \ w = \gamma, \ \ on\ \partial M.$$

By subtracting the above two relations, we conclude that

$$(\omega - w)_{\bar{z}} = 0, \ \ in\ M$$

$$\omega - w = 0. \ \ in\ M,$$

this completes the proof. □

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