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On the Stability of Hyers Orthogonality Functional Equations in Non-Archimedean Spaces

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Abstract. In this paper, we investigate the stability of specially orthogonally functional equations deriving from additive and quadratic functions

$$4f(x+y)+4f(x-y)+10f(x)+14f(-x)-3f(y)-3f(-y)=f(2x+y)+f(2x-y)$$

and

$$f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x+y-z}{2}\right)+f\left(\frac{x-y+z}{2}\right)+f\left(\frac{y+z-x}{2}\right)=f(x)+f(y)+f(z)$$

where f is a mapping from Abelian group to a non-Archimedean space. By adopting a new method, we have made an attempt to prove the Hyers-Ulam stability in non-Archimedean spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from Ulam in 1940 when he posed the group homomorphism problem "Given an approximately linear mapping f, when does a linear mapping T exist that approximates f?" In 1941, Hyers [1] explored the scenario of approximately additive mapping $f : X \to Y$where X and Y are Banach spaces and f satisfies

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon$$

for all $x, y \in X$. Then there is a unique mapping additive $L : X \to Y$ satisfying

$$\|f(x)-L(x)\| \leqslant \varepsilon$$

with the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}.$$

Rassias [14] weakened the bounded Cauchy difference proposed by Hyers in the map and extended it to the unbounded Cauchy difference

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p)$$

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where $\varepsilon > 0$ and $p \in [0,1)$, Hyers' theorem was extended to approximately linear maps. R. Ger and J. Sikorska [7] restricted the conditions with $(x,y) = 0$ and investigated the stability of the Cauchy functional

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

Of course it is easy to spot that the function $f(x) = \|x\|^2$ satisfies the functional equations (1.1) by the Pythagorean theorem. They founded that there exists a orthogonality additive mapping $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leqslant \frac{16}{3}\varepsilon$$

for all $x \in X$ with restriction on definition domain (1.1) was denoted as a additive equations. Similarly, the equation was called as a quadratic equation which satisfies

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

During several decades, mathematicians have achieved various fruits in studying the stability of functional equations based one these two equations in the spirit of Hyers-Ulam-Rassias.

Now let us introduce the concept of orthogonality $\perp$ defined by Rätz [16]. Suppose X is a real vector space with $\dim X \geqslant 2$ and $\perp$ is a binary relation on X are characterized by the following properties:

(i) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(ii) homogeneity: if $x, y \in X$, $x \perp y$, then $\lambda x \perp \mu y$ for all $\lambda, \mu \in R$;
(iii) independence: if $x, y \in X \setminus \{0\}$, $x \perp y$, if and only if x, y are linearly independent;
(iv) for any two-dimensional subspace P of X and for every $x \in P$, there exists $\lambda, y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

The pair $(X, \perp)$ is called an orthogonality space, which means an orthogonality space having a normed structure. Various notions of othogonlity on a real normed space such as Roberts, Pythagorean, Isosceles, Birkhoff-James, Carlsson, Hermite–Hadamard (HH) type orthogonalities on the basis of the fundamental properties.

Definition 1.1. *[16] A function $\|\cdot\|$:$X \to [0,\infty)$ on a vector space over X a scalar field K with a non-Archimedean valuation $|\cdot|$, is classified as a non-Archimedean norm if it meets the following conditions:*

(i) *nonnegativity: $\|x\| \geqslant 0$ and $\|x\| = 0$ if and only if $x = 0$;*
(ii) *homogeneity*: $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall \lambda \in K, \forall x, y \in X$;
(iii) *the strong triangle inequality*

$$\|x+y\| \leqslant \max\{\|x\|, \|y\|\} \quad \forall x, y \in X$$

Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space.

Gordji [9] investigated the stability of the traditionally functional equations

$$D(x,y) = f(x+y) - f(x) - f(y)$$

where $f : X \to Y$ X, Y are both non-Arohimedean Banach spaces. They established the existence of functions $\varphi, \psi : A \times A \to [0, \infty)$ such that

$$\|D(x,y)\| \leqslant \varphi(x,y)$$
$$\|f(xy) - f(x)f(y)\| \leqslant \psi(x,y)$$

for all $x, y \in X$, and they considered the case if there exists a constant $0 < L < 1$ such that

$$\varphi(2x,2y) \leqslant |2|L\varphi(x,y)$$
$$\varphi(2x,2y) \leqslant |2|^2L\psi(x,y)$$

Then there exist a unique ring homomorphis $H : X \to Y$ such that

$$\|f(x) - H(x)\| \leqslant \frac{1}{|2|(1-L)}\varphi(x,x)$$

Kang [10] explored the stability of the orthogonally functional equation(1.3) through the classification of the oddness and evenness of f within the same spaces

$$4f(x+y) + 4f(x-y) + 10f(x) + 14f(-x) - 3f(y) - 3f(-y) = f(2x+y) + f(2x-y) \quad (1.3)$$

Park [12] investigated the stability of the orthogonally additive-additive and orthogonally quadratic-quadratic functional equation(1.4) in non-Archimedean orthogonality spaces using conventional methods

$$f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x+y-z}{2}\right) + f\left(\frac{x-y+z}{2}\right) + f\left(\frac{y+z-x}{2}\right) = f(x) + f(y) + f(z) \quad (1.4)$$

Drawing inspiration from [14], this paper we explore different spaces and employ new methods to investigete the stability of the aforementioned equation(1.4) and (1.3).

2. Stability of the orthogonally additive-quadratic functional equation

In this section, we will use the following symbol

$$\begin{aligned} D_1f(x,y) = f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) \\ -10f(x) - 14f(-x) + 3f(y) + 3f(-y) \end{aligned} \quad (2.1)$$

we deal with the stability problem for the orthogonally additive-quartic functional equation for $D_1f(x,y) = 0$ by referring to the stability proof of [13, 14].

Lemma 2.1. *Assume $f : G \to X$ be a mapping with G be an Abelian group and $(X, \|\cdot\|)$ be a complete non -Archimedean normed space. For all $x, y \in G$ and there is a constant $C > 0$ such that*

$$\left\|f(2x) - \frac{3}{8}f(4x) + \frac{1}{8}f(-4x)\right\| \leqslant C \tag{2.2}$$

Then we define

$$h(x,n) = \left\|f(2x) - \frac{2^n+1}{2\cdot 4^n}f\left(2^{n+1}x\right) + \frac{2^n-1}{2\cdot 4^n}f\left(-2^{n+1}x\right)\right\|$$

and

$$g_n(x) = \frac{2^n+1}{2\cdot 4^n}f\left(2^n x\right) - \frac{2^n-1}{2\cdot 4^n}f\left(-2^n x\right). \quad n \in N$$

(1)Then we have

$$|h(x,n+1) - h(x,n)| \leqslant \frac{2^n+1}{2\cdot 4^n}C \tag{2.3}$$

$$h(x,n) \leqslant C \tag{2.4}$$

(2)and $\{g_n(x)\}$ is a Cauchy sequence, for every $x \in G$. Hence, the mapping $g : G \to X$ can be defined as

$$g(x) = \lim_{n\to\infty} g_n(x)$$

and then we get

$$\|f(2x) - g(2x)\| \leqslant C \tag{2.5}$$

Proof: Adding one and subtracting one with $h(x, n+1)$ for matching and then using the inequality, we obtain

$$\begin{aligned}
&\left\|f(2x) - \frac{2^{n+1}+1}{2\cdot 4^{n+1}}f\left(2^{n+2}x\right) + \frac{2^{n+1}-1}{2\cdot 4^{n+1}}f\left(-2^{n+2}x\right)\right\| \\
\leqslant &\left\|f(2x) - \frac{2^n+1}{2\cdot 4^n}f\left(2^{n+1}x\right) + \frac{2^n-1}{2\cdot 4^n}f\left(-2^{n+1}x\right)\right\| \\
&+ \frac{2^n+1}{2\cdot 4^n}\left\|f\left(2^{n+1}x\right) - \frac{3}{8}f\left(2^{n+2}x\right) + \frac{1}{8}f\left(-2^{n+2}x\right)\right\| \\
&+ \frac{2^n-1}{2\cdot 4^n}\left\|f\left(-2^{n+1}\cdot x\right) + \frac{1}{8}f\left(2^{n+2}x\right) - \frac{3}{8}f\left(-2^{n+2}x\right)\right\| \\
\leqslant &\left\|f(2x) - \frac{2^n+1}{2\cdot 4^n}f\left(2^{n+1}x\right) + \frac{2^n-1}{2\cdot 4^n}f\left(-2^{n+1}x\right)\right\| + C\cdot\max\left\{\frac{2^n+1}{2\cdot 4^n}, \frac{2^n-1}{2\cdot 4^n}\right\}
\end{aligned}$$

Next, it is easy to get

$$|h(x,n+1) - h(x,n)| \leqslant \frac{2^n+1}{2\cdot 4^n}C$$

Then

$$\begin{aligned} h(x,n) &= \left\| \left(\sum_{i=2}^{n} h(x,i) - h(x,i-1) \right) + h(x,1) \right\| \\ &\leqslant C \cdot \max\left\{ \frac{2+1}{2\cdot 4}, \frac{2^2+1}{2\cdot 4^2}, \cdots, \frac{2^n+1}{2\cdot 4^n}, 1 \right\} \\ &= C \end{aligned}$$

Next, we have to prove that for every $x \in G$, the sequence

$$g_n(x) = \frac{2^n+1}{2\cdot 4^n} f\left(2^n x\right) - \frac{2^n-1}{2\cdot 4^n} f\left(-2^n x\right) \quad n \in N$$

is convergent in G. Since X is complete, it is sufficient to show that $(g_n(x))_{n\in N}$ is a Cauchy sequence for all $x \in G$. By matching $\|g_{n+1}(x) - g_n(x)\|$twice then we have

$$\begin{aligned} \|g_{n+1}(x) - g_n(x)\| \leqslant & \frac{2^n+1}{2\cdot 4^n} \left\| f\left(2^n x\right) - \frac{3}{8} f\left(2^{n+1}x\right) + \frac{1}{8} f\left(-2^{n+1}x\right) \right\| \\ & + \frac{2^n-1}{2\cdot 4^n} \left\| f\left(-2^n x\right) - \frac{3}{8} f\left(-2^{n+1}x\right) + \frac{1}{8} f\left(2^{n+1}x\right) \right\| \\ \leqslant & C \cdot \max\left\{ \frac{2^n+1}{2\cdot 4^n}, \frac{2^n-1}{2\cdot 4^n} \right\} \\ = & \frac{2^n+1}{2\cdot 4^n} C \end{aligned}$$

for each $n \in N$. This easily implies that $\{g_n(x)\}$ is a Cauchy sequence. The mapping $g : G \to X$ can be defined as

$$g(x) = \lim_{n\to\infty} g_n(x)$$

Through the above results, we can obtain

$$\|f(2x) - g(2x)\| = \|h(x,n) + g_n(2x) - g(2x)\| \leqslant C$$

In this section, let G be an Abelian group and let $\perp$ be a binary relation defined on G with the properties:

(i) $x \perp 0, 0 \perp x$, for all $x \in X$;

(ii) if $x, y \in X$ and $x \perp y$, then $\frac{x}{2} \perp \frac{y}{2}$, $2x \perp 2y, 4x \perp 4y$ and $-x \perp -y$.

Theorem 2.1. *Suppose $f : G \to X$ where f is a mapping from an Abelian group to a complete non-Archimedean normed space. For $\varepsilon > 0$, when $x \perp y$ for all $x, y \in G$,we obtain*

$$\|D_1 f(x,y)\| \leqslant \varepsilon \tag{2.6}$$

and

$$\|f(x) + f(-x)\| \leqslant \varepsilon \tag{2.7}$$

Then there exists a unique mapping $g : G \to X$ such that $x \perp y$ implies

$$4g(x+y) + 4g(x-y) + 10g(x) + 14g(-x) - 3g(y) - 3g(-y) = g(2x+y) + g(2x-y) \tag{2.8}$$

and

$$\|f(x)-g(x)\| \leqslant \frac{7}{2}\varepsilon \tag{2.9}$$

for all $x \in 2G = \{2x : x \in G\}$.

Proof. For all $x \in X$, since $0 \perp x$, $x \perp 0$ and $0 \perp 0$, setting $x = 0, y = 0$ in (2.6), we obtain $\|24f(0)\| \leqslant \varepsilon$, respectively, setting $y = 0$ in (2.6),we obtain the following inequality:

$$\|2f(2x)-18f(x)-14f(-x)+6f(0)\| \leqslant \varepsilon \tag{2.10}$$

By using the strong triangle inequality, we obtain

$$\|2f(2x)-18f(x)-14f(-x)\| \leqslant \max\{\|2f(2x)-18f(x)-14f(-x)+6f(0)\|, \|6f(0)\|\} \leqslant \varepsilon \tag{2.11}$$

By replacing x with $4x$ in (2.7) and applying the triangle inequality twice, we obtain

$$\|2f(2x)-4f(x)\| \leqslant max\{\|2f(2x)-18f(x)-14f(-x)\|, 14\|f(x)+f(-x)\|\} = 14\varepsilon \tag{2.12}$$

Applying (2.7) and (2.12) to $\|3f(4x)-8f(2x)-f(-4x)\|$, we can conclude that

$$\begin{aligned} &\|3f(4x)-8f(2x)-f(-4x)\| \\ &= \|4[f(4x)-2f(2x)]-[f(4x)+f(-4x)]\| \\ &\leqslant \max\{28\varepsilon, \varepsilon\} = 28\varepsilon \end{aligned} \tag{2.13}$$

This means that

$$\left\|f(2x)-\frac{3}{8}f(4x)+\frac{1}{8}f(-4x)\right\| \leqslant \frac{7}{2}\varepsilon \tag{2.14}$$

The next step resembles Lemma2.1, let

$$g_n(x) = \frac{2^n+1}{2\cdot 4^n} f\left(2^n x\right) - \frac{2^n-1}{2\cdot 4^n} f\left(-2^n x\right) \tag{2.15}$$

then we can define a mapping g

$$g : G \to X \quad g(x) = \lim_{n\to\infty} g_n(x).$$

According to Lemma2.1, we obtain

$$\|f(2x)-g(2x)\| \leqslant \frac{7}{2}\varepsilon \tag{2.16}$$

we consider the following inequality

$$\begin{aligned} &\|D_1 g_n(x,y)\| \\ &\leqslant \left\|\frac{2^n+1}{2\cdot 4^n} D_1 f\left(2^n x, 2^n y\right) + \frac{2^n-1}{2\cdot 4^n} D_1 f\left(2^n x, 2^n y\right)\right\| \\ &\leqslant \frac{2^n+1}{2\cdot 4^n}\varepsilon \end{aligned} \tag{2.17}$$

for all $x, y \in G$. Then we let $n \to \infty$, we get (2.8). Now, in order to prove g is unique, we assume g' as another mapping satisfying (2.8) and (2.9) that

$$\begin{aligned}\left\|g(x)-g'(x)\right\| &= \left\|g(x)-f(x)+f(x)-g'(x)\right\| \\ &\leqslant \max\left\{\|g(x)-f(x)\|, \left\|f(x)-g'(x)\right\|\right\} \\ &= \varepsilon \end{aligned} \tag{2.18}$$

for all $x \in 2G = \{2x : x \in G\}$

On the other hand, the mapping $g - g'$ satisfy (2.6) and(2.8)

$$g(2x)-g'(2x) = \frac{2^n+1}{2\cdot 4^n}\left[g\left(2^{n+1}x\right)-g'\left(2^{n+1}x\right)\right] - \frac{2^n-1}{2\cdot 4^n}\left[g\left(-2^{n+1}x\right)-g'\left(-2^{n+1}x\right)\right] \tag{2.19}$$

and therefore

$$\begin{aligned}&\left\|g(2x)-g'(2x)\right\| \\ &\leqslant \max\left\{\left(\frac{2^n+1}{2\cdot 4^n}\right)\left\|g\left(2^{n+x}x\right)-g'\left(2^{n+1}\cdot x\right)\right\|, \left(\frac{2^n-1}{2\cdot 4^n}\right)\left\|g\left(-2^{n+1}x\right)-g'\left(-2^{n+1}x\right)\right\|\right. \\ &\leqslant \max\left\{\left(\frac{2^n+1}{2\cdot 4^n}\right)\varepsilon, \left(\frac{2^n-1}{2\cdot 4^n}\right)\varepsilon\right\} \\ &= \frac{2^n+1}{2\cdot 4^n}\varepsilon \end{aligned} \tag{2.20}$$

for $x \in G$. By using the nonnegativity of norm and the forced convergence we can get that the mapping g is unique on the set $2G$.

□

3. Stability of additive-additive and orthogonally quadratic-quadratic functional equation

In this section, we substituted the equations with the orthogonally additive-additive and orthogonally quadratic-quadratic functional equation concerning [12] in the same method and by referring to the stability proof of [13, 14], we define $D_2(x, y, z)$ as the followig

$$\begin{aligned}D_2 f(x,y,z) = f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x+y-z}{2}\right) + f\left(\frac{x-y+z}{2}\right) \\ + f\left(\frac{y+z-x}{2}\right) - f(x) - f(y) - f(z)\end{aligned}$$

Theorem 3.1. *Suppose $f : G \to X$ where f is a mapping from an Abelian group to a complete non-Archimedean normed space. For $\varepsilon > 0$, when $x \perp y$ for all $x, y, z \in G$,we obtain*

$$\|D_2 f(x,y,z)\| \leqslant \varepsilon \tag{3.1}$$

and

$$\|f(x)+f(-x)\| \leqslant \varepsilon. \tag{3.2}$$

Then there exists a unique mapping $g : X \to Y$ *such that* $x \perp y$ *implies*

$$g\left(\frac{x+y+z}{2}\right)+g\left(\frac{x+y-z}{2}\right)+g\left(\frac{x-y+z}{2}\right)+g\left(\frac{y+z-x}{2}\right)=g(x)+g(y)+g(z) \quad (3.3)$$

and

$$\|f(x)-g(x)\| \leqslant \varepsilon \quad (3.4)$$

for all $x \in 2G = \{2x : x \in G\}$.

Proof. For all $x \in G$, since $0 \perp x$, $x \perp 0$, and $0 \perp 0$, setting $x = 0, y = 0, z = 0$ in inquality (3.1), we obtain $\|f(0)\| \leqslant \varepsilon$, then similarily setting $y = 0, z = 0$ in inequality (3.1), we obtain

$$\left\|3f\left(\frac{x}{2}\right)+f\left(\frac{-x}{2}\right)-f(x)-2f(0)\right\| \leqslant \varepsilon \quad (3.5)$$

Then, by using the strong triangle inequality, we obtain

$$\begin{aligned}&\left\|3f\left(\frac{x}{2}\right)+f\left(\frac{-x}{2}\right)-f(x)\right\|\\ &\leqslant \max\left\{\|2f(0)\|,\left\|3f\left(\frac{x}{2}\right)+f\left(-\frac{x}{2}\right)-f(x)-2f(0)\right\|\right\}\\ &=2\varepsilon\end{aligned} \quad (3.6)$$

By replacing x witn $2x$ in (3.6), we obtain

$$\|3f(x)+f(-x)-f(2x)\| \leqslant 2\varepsilon \quad (3.7)$$

By using the strong triangle inequality twice, we can easily obtain

$$\begin{aligned}&\|2f(x)-f(2x)\|\\ &\leqslant \max\{\|f(x)+f(-x)\|,\|3f(x)+f(-x)-f(2x)\|\}=2\varepsilon\end{aligned} \quad (3.8)$$

By replacing x witn $4x$ in (3.2),then combining the following with (3.2)and(3.8), we can conclude that

$$\begin{aligned}&\|3f(4x)-8f(2x)-f(-4x)\|\\ &=\|4[f(4x)-2f(2x)]-[f(4\dot{x})+f(-4x)]\|\\ &\leqslant \max\{8\varepsilon,\varepsilon\}=8\varepsilon\end{aligned} \quad (3.9)$$

Then dividing both side of the inequality by 8,we obtain

$$\left\|f(2x)-\frac{3}{8}f(4x)+\frac{1}{8}f(-4x)\right\| \leqslant \varepsilon \quad (3.10)$$

The next step resembles Lemma2.1,

$$g_n(x)=\frac{2^n+1}{2\cdot 4^n}f\left(2^n x\right)-\frac{2^n-1}{2\cdot 4^n}f\left(-2^n x\right) \quad n \in N.$$

Let

$$g : G \to X \quad g(x)=\lim_{n\to\infty} g_n(x).$$

According to Lemma2.1, we obtain

$$\|f(2x)-g(2x)\| = \|h(x,n)+g_n(2x)-g(2x)\| \leqslant \varepsilon \tag{3.11}$$

For the purpose of proving that g is orthogonally additive, firstiy, we apply the strong triangle inequality and the nonnegativity property for the following , we obtain

$$\begin{aligned} &\|D_2 g(x,y,z)\| \\ =&\left\|\frac{2^n+1}{2\cdot 4^n}D_2 f(2^n x,2^n y,2^n z)+\frac{2^n-1}{2\cdot 4^n}D_2 f(2^n x,2^n y,2^n z)\right\| \\ \leqslant &\max\left\{\frac{2^n+1}{2\cdot 4^n}\varepsilon,\frac{2^n-1}{2\cdot 4^n}\varepsilon\right\}=\frac{2^n+1}{2\cdot 4^n}\varepsilon \end{aligned} \tag{3.12}$$

for all $x,y,z \in G$ with $x \perp y$ and $n \in N, n > 1$. When we let $n \to \infty$, we get (3.3). The rest of proof resembles Theorem 2.1, according to (2.18)to (2.20) , we can get the mapping g is unique on the set $2G$ similarly. □

Theorem 3.2. *Suppose $f : G \to X$ where f is a mapping from an Abelian group to a complete non-Archimedean normed space. For $\varepsilon > 0$, when $x \perp y$ for all $x,y,z \in G$,we obtain*

$$\|D_2 f(x,y,z)\| \leqslant \varepsilon \tag{3.13}$$

and

$$\|f(x)-f(-x)\| \leqslant \varepsilon. \tag{3.14}$$

Then there exists a unique mapping $g : X \to Y$ such that $x \perp y$ implies

$$g\left(\frac{x+y+z}{2}\right)+g\left(\frac{x+y-z}{2}\right)+g\left(\frac{x-y+z}{2}\right)+g\left(\frac{y+z-x}{2}\right)=g(x)+g(y)+g(z) \tag{3.15}$$

and

$$\|f(x)-g(x)\| \leqslant \frac{1}{2}\varepsilon \tag{3.16}$$

for all $x \in 2G = \{2x : x \in G\}$.

Proof. Our proof resembles Theorem3.1, the same step from (3.5) to (3.7), we get that

$$\|3f(x)+f(-x)-f(2x)\| \leqslant 2\varepsilon \tag{3.17}$$

Adding (3.14) to (3.17) and using the triangle inequality, we can obtain

$$\begin{aligned} &\|f(2x)-4f(x)\| \\ &\leqslant max\{\|3f(x)+f(-x)-f(2x)\|,\|f(x)-f(-x)\|\}=2\varepsilon \end{aligned} \tag{3.18}$$

Hence, by using the result, there is

$$\begin{aligned} &\|3f(4x) - 8f(2x) - f(-4x)\| \\ &= \|2[f(4x) - 4f(2x)] + f(4x) - f(-4x)]\| \\ &\leqslant \max\{4\varepsilon, \varepsilon\} = 4\varepsilon \end{aligned} \tag{3.19}$$

The rest of proof is similar to the Theorem 3.1. □

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