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Hardy-Littlewood-Sobolev Theorem for Bourgain-Morrey Spaces and Approximation

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Abstract. In this paper, we establish an extension of the Hardy-Littlewood-Sobolev theorem to the setting of the Bourgain-Morrey space $\mathcal{M}^{\alpha}_{q,p}(\mathbb{R}^d)$ $(1 \leq q, p, \alpha \leq \infty)$, which theory goes back to Bourgain in 1991. We also prove that $\mathcal{M}^{\alpha}_{q,p}(\mathbb{R}^d)$ is included in the closure of the Lebesgue space L^{α} in the Morrey-type space $\mathbb{F}(q, p, \alpha)$, which arises naturally in 2015 in the study of boundedness properties of fractional integral operators. Therefore, we establish in $\mathcal{M}^{\alpha}_{q,p}$ some approximation results by compactly supported and/or regular functions. As an application of these results, we obtain an explicit solution in $[L^p(\mathbb{R}^d)]^d$ of the equation $\operatorname{div} F = f$ whenever f is in $\mathcal{M}^{\alpha}_{q,p}$, with $d \geq 3$, $1 \leq q \leq \alpha < d$ and $\frac{1}{p} = \frac{1}{\alpha} - \frac{1}{d}$.

1. Introduction

Let d be a fixed positive integer. $\mathbb{R}^d$ is equipped with its usual Hilbert space structure and the euclidean norm of any element x of $\mathbb{R}^d$ is denoted by $|x|$.

Recall that the classical Lebesgue space $L^q := L^q(\mathbb{R}^d)$, with $q \in [1, \infty]$, is defined to be the set of all measurable complex functions f on $\mathbb{R}^d$ such that

$$\|f\|_q := \left[\int_{\mathbb{R}^d} |f(x)|^q \, dx\right]^{\frac{1}{q}} < \infty$$

with the usual modification made when $q = \infty$. In what follows, $|E|$ and χ_E denote the Lebesgue measure and the characteristic function of any measurable set $E \subset \mathbb{R}^d$, respectively. L^q_{loc} denotes the set of all measurable complex functions f on $\mathbb{R}^d$ such that $f\chi_K \in L^q$ for any bounded measurable subset K of $\mathbb{R}^d$.

For $1 \leq q, \alpha \leq \infty$, the Morrey space $\mathcal{M}^{\alpha}_q := \mathcal{M}^{\alpha}_q(\mathbb{R}^d)$ is defined as the set of all elements f of L^q_{loc} for which

$$\|f\|_{\mathcal{M}^{\alpha}_q} := \sup_{x \in \mathbb{R}^d,\, r>0} |Q(x,r)|^{\frac{1}{\alpha} - \frac{1}{q}} \left\|f\chi_{Q(x,r)}\right\|_q < \infty,$$

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where

$$Q(x,r)=\prod_{j=1}^{d}\left[x_j-\frac{r}{2},x_j+\frac{r}{2}\right),\qquad x=(x_1,x_2,...,x_d)\in\mathbb{R}^d \text{ and } 0<r<\infty.$$

Morrey spaces were introduced in 1938 by C. Morrey [11] in order to study both the regularity problem of solutions for quasi-linear elliptic partial differential equations and the calculus of variations. Note that, for $1\leq q\leq\alpha\leq\infty$, L^α is included in $\mathcal{M}_q^\alpha$ and the inclusion is proper when $q<\alpha<\infty$. Moreover, Morrey spaces describe local regularity of functions more precisely than Lebesgue spaces. However, some nice and useful properties of L^α are not shared by $\mathcal{M}_q^\alpha$ when $1\leq q<\alpha<\infty$. For example, in this case, the set of all compactly supported and/or regular elements is not dense in $\mathcal{M}_q^\alpha$. Because of this unpleasant issue, several distinguished linear subspaces of Morrey spaces have been considered for their easy use in Harmonic Analysis, specifically in boundedness problem of classical operators.

The present paper focuses on Bourgain-Morrey spaces $\mathcal{M}_{q,p}^\alpha$ with $1\leq q,\alpha,p\leq\infty$. Recall that, a special case of these spaces was first introduced by Bourgain [2] in 1991 in order to study the Stein-Tomas estimate. Later on, Bourgain-Morrey spaces have been used fruitfully in the study of Fourier restriction, multipliers problems and partial differential equations, and in the proof of refinements of Strichartz inequality (see [8–10] and the references therein). They are defined as follows.

Definition 1.1. *Let $1\leq q,\alpha,p\leq\infty$. The Bourgain-Morrey space $\mathcal{M}_{q,p}^\alpha:=\mathcal{M}_{q,p}^\alpha(\mathbb{R}^d)$ is defined as the set of all $f\in L^q_{\mathrm{loc}}$ for which*

$$\|f\|_{\mathcal{M}_{q,p}^\alpha}:=\left\|\left\{|Q_{k,m}|^{\frac{1}{\alpha}-\frac{1}{q}}\left\|f\chi_{Q_{k,m}}\right\|_q\right\}_{(k,m)\in\mathbb{Z}^d\times\mathbb{Z}}\right\|_{\ell^p}<\infty,$$

where the sets

$$Q_{k,m}=\prod_{j=1}^{d}\left[k_j2^m,(k_j+1)\,2^m\right),\quad k=(k_1,k_2,...,k_d)\in\mathbb{Z}^d,\quad m\in\mathbb{Z}$$

are the usual dyadic cubes of $\mathbb{R}^d$ and for any sequence $\{a_i\}_{i\in\mathbb{I}}$ included in $\mathbb{C}$,

$$\|\{a_i\}_{i\in\mathbb{I}}\|_{\ell^p}:=\begin{cases}\left(\sum\limits_{i\in\mathbb{I}}|a_i|^p\right)^{\frac{1}{p}} & \text{if } p<\infty\\ \sup\limits_{i\in\mathbb{I}}|a_i| & \text{if } p=\infty.\end{cases}$$

It is well known that, when $1\leq q<\alpha<p\leq\infty$, L^α is properly included in $\mathcal{M}_{q,p}^\alpha$, which is a linear subspace of $\mathcal{M}_q^\alpha$. Actually we have

$$\begin{cases}L^\alpha\subset\mathcal{M}_{q,p}^\alpha\subset\mathcal{M}_{q,p_1}^\alpha\subset\mathcal{M}_{q,\infty}^\alpha=\mathcal{M}_q^\alpha\ , & 1\leq q<\alpha<p\leq p_1\leq\infty.\\ \mathcal{M}_{q,p}^\alpha\subset\mathcal{M}_{q_1,p}^\alpha\ , & 1\leq q_1\leq q\leq\alpha\leq p\leq\infty.\end{cases}\tag{1}$$

Many useful results, well known for Lebesgue or Morrey spaces, have been extended to the setting of Bourgain-Morrey spaces (see [8, 9]). For instance, the boundedness of some classical operators on these spaces has been investigated by Hatano et al. [8].

Let $0 < \gamma < 1$. Define the fractional integral operator I_γ as

$$I_\gamma f(x) = \int_{\mathbb{R}^d} |x-y|^{d(\gamma-1)} f(y) dy$$

when the above integral makes sense.

Recall that the Hardy-Littlewood-Sobolev theorem of fractional integration is one of the most important tools in the study of partial differential equations. It reads as follows.

Theorem 1.2. *[14] Let $0 < \gamma < \frac{1}{\alpha} \leq 1$ and $\frac{1}{p} = \frac{1}{\alpha} - \gamma$. Then there is a real number $A_{\alpha,\gamma}$ such that:*

(i) when $1 < \alpha$

$$\|I_\gamma f\|_p \leq A_{\alpha,\gamma} \|f\|_\alpha \ , \qquad f \in L^\alpha \tag{2}$$

(ii) when $\alpha = 1$

$$\|f\|^*_{\frac{1}{1-\gamma},\infty} \leq A_{1,\gamma} \|f\|_1 \ , \qquad f \in L^1. \tag{3}$$

Recall that, for $q \in [1, \infty)$, $\| \cdot \|^*_{q,\infty}$ denotes the quasi-norm of the weak-Lebesgue WL^q defined by

$$WL^q = \left\{ f \in L^1_{\text{loc}} \ : \ \|f\|^*_{q,\infty} = \sup_{\lambda > 0} \lambda \left| \{x \in \mathbb{R}^d : |f(x)| > \lambda\} \right|^{\frac{1}{q}} < \infty \right\}.$$

The first aim of the present paper is to establish an extension of the above useful theorem to the setting of Bourgain-Morrey spaces. Note that, our result refines that of Hatano et al., which states that fractional integral operators map Bourgain-Morrey spaces into the same type spaces (see [8, Theorem 4.4]).

Another Morrey-type space considered in this paper is the space $\mathbb{F}(q, p, \alpha)$ $(1 \leq q, \alpha, p \leq \infty)$, which arises naturally in the study of boundedness properties of fractional integral operators. It has been introduced in 2015 by Fofana et al. [7]. Note that, recently in 2020, the space $\mathbb{F}(q, p, \alpha)$ has been studied also in [15], where it is called the Riesz-Morrey space and denoted by $\text{RM}_{p,q,\frac{1}{p}-\frac{1}{\alpha}}(\mathbb{R}^d)$. It is defined as follows.

Definition 1.3. *Let $1 \leq q, p, \alpha \leq \infty$. The space $\mathbb{F}(q, p, \alpha) := \mathbb{F}(q, p, \alpha)(\mathbb{R}^d)$ is defined as the set of all $f \in L^q_{\text{loc}}$ for which $\|f\|_{\mathbb{F}(q,p,\alpha)}$ is finite, where*

$$\|f\|_{\mathbb{F}(q,p,\alpha)} = \begin{cases} \sup\limits_{\{Q_i\} \in \mathcal{P}} \left\| \left\{ |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \|f\chi_{Q_i}\|_q \right\}_{i \in \mathbb{I}} \right\|_{\ell^p} & \textit{if } p < \infty \\ \sup\limits_{Q \in \mathcal{Q}} |Q|^{\frac{1}{\alpha}-\frac{1}{q}} \|f\chi_{Q_i}\|_q & \textit{if } p = \infty, \end{cases}$$

with

- $\mathcal{Q} = \left\{ Q(x, r) \ : \ (r, x) \in (0, \infty) \times \mathbb{R}^d \right\}$
- $\mathcal{P} = \left\{ \{Q_i\}_{i \in \mathbb{I}} \subset \mathcal{Q} \ : \ \mathbb{I} \textit{ is countable and } Q_i \cap Q_j = \varnothing \ \textit{ if } i \neq j \right\}$.

It is well known that the space $\mathbb{F}(q,p,\alpha)$ is a linear subspace of L^q_{loc} and a Banach space, when endowed with $\|\cdot\|_{\mathbb{F}(q,p,\alpha)}$. $\mathbb{F}(q,p,\alpha)$ is nontrivial if and only if $q \le \alpha \le p$ (see [7]). Moreover, when $1 \le q_1 \le q \le \alpha \le p \le p_1$, the following inclusion and equality relations hold:

$$\left\{ \begin{array}{l} L^\alpha = \mathbb{F}(q,\alpha,\alpha) \subset \mathbb{F}(q,p,\alpha) \subset \mathbb{F}(q,p_1,\alpha) \subset \mathbb{F}(q,\infty,\alpha) = \mathcal{M}^\alpha_q \\ \mathbb{F}(q,p,\alpha) \subset \mathbb{F}(q_1,p,\alpha). \end{array} \right. \tag{4}$$

Note that (4) shows that the spaces $\mathbb{F}(q,p,\alpha)$ provide a bridge connecting both Lebesgue spaces and Morrey spaces. Many results, well known for Lebesgue or Morrey spaces, have been extended in the framework of these spaces (see [5,7]). Furthermore, the relations (1) and (4) point out that the spaces $\mathcal{M}^\alpha_{q,p}$ and $\mathbb{F}(q,p,\alpha)$ satisfy almost the same inclusion relations. We also observe that the norm structures of these two spaces are very similar. Thus, a natural question is that, what is the link between the spaces $\mathcal{M}^\alpha_{q,p}$ and $\mathbb{F}(q,p,\alpha)$?

The second aim of this paper is to study the above mentioned question. We succeeded in proving that $\mathcal{M}^\alpha_{q,p}$ is continuously included in $\mathbb{F}(q,p,\alpha)$ and, when $p<\infty$, $\mathcal{M}^\alpha_{q,p}$ is included in the closure of L^α in $\mathbb{F}(q,p,\alpha)$. Therefore, we also establish in $\mathcal{M}^\alpha_{q,p}$ some approximation results by compactly supported and/or regular functions.

As an application of the above mentioned results, we obtain an explicit solution in $(L^p)^d$ of the equation $\operatorname{div} F = f$ whenever f is in $\mathcal{M}^\alpha_{q,p}$, with $d \ge 3$, $1 \le q \le \alpha < d$ and $\frac{1}{p} = \frac{1}{\alpha} - \frac{1}{d}$.

The remainder of the paper is organized as follows. Section 2 contains a more detailed presentation of our main results. Section 3 deals with some preliminary results on $\mathcal{M}^\alpha_{q,p}$. In Section 4 we prove the inclusion of $\mathcal{M}^\alpha_{q,p}$ in $\mathbb{F}(q,p,\alpha)$ and also approximation results. Section 5 is devoted to prove our main theorem showing the action of fractional integral operators on $\mathcal{M}^\alpha_{q,p}$. Section 6 contains an application to the divergence equation div $F = f$.

Finally, let us make some conventions on notations used in this paper.

• $\mathcal{C}^\infty$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^d$ and $\mathcal{C}^\infty_c$ stands for the set of all elements of $\mathcal{C}^\infty$ with compact support in $\mathbb{R}^d$.

• Let ϕ be a fixed nonnegative element of $\mathcal{C}^\infty$ such that its support is included in the unit cube $[0,1]^d$ and satisfying $\int_{\mathbb{R}^d} \phi(x)dx = 1$. For any integer $n \ge 1$, we denote by ϕ_n the dilation defined by

$$\phi_n(x) = n^d \phi(nx) \ , \quad x \in \mathbb{R}^d.$$

• Let ω be a fixed element of $\mathcal{C}^\infty$ satisfying $\chi_{Q(0,1)} \le \omega \le \chi_{Q(0,2)}$. For any integer $n \ge 1$, ω_n is defined by

$$\omega_n(x) = \omega\left(\frac{x}{n}\right) \ , \quad x \in \mathbb{R}^d.$$

2. Statement of the main results

For $0<\gamma<1$, the fractional integral operator I_γ is known to be closely related to the fractional maximal operator $\mathfrak{M}_\gamma$ defined by

$$\mathfrak{M}_\gamma f(x)=\sup_{Q\ni x}|Q|^{\gamma-1}\int_Q|f(y)|dy, \qquad f\in L^1_{\mathrm{loc}}\,, \quad x\in\mathbb{R}^d,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^d$ containing x.

Our first result reads as follows.

Theorem 2.1. *Let us assume that $0<\gamma<\frac{1}{\alpha}\leq 1$ and $\frac{1}{p}=\frac{1}{\alpha}-\gamma$. Then, for any element f of $\mathcal{M}^\alpha_{1,p}$, we have*

$$\|\mathfrak{M}_\gamma f\|_p\leq 2^{d\left(2-\frac{1}{p}\right)}\,3^{d(2-\gamma)}\,\|f\|_{\mathcal{M}^\alpha_{1,p}}\,. \tag{5}$$

Note that Theorem 2.1 refines [8, Corollary 4.5]. As an immediate consequence of this theorem, we obtain the following result which refines [8, Theorem 4.4] and is our most significant result.

Theorem 2.2. *Let us assume that $0<\gamma<\frac{1}{\alpha}\leq 1$ and $\frac{1}{p}=\frac{1}{\alpha}-\gamma$. Then there exists a real constant $C>0$ such that, for any element f of $\mathcal{M}^\alpha_{1,p}$, we have*

$$\|I_\gamma f\|_p\leq\; C\;\|f\|_{\mathcal{M}^\alpha_{1,p}}\,. \tag{6}$$

Since $L^\alpha\subset\mathcal{M}^\alpha_{q,p}\subset\mathcal{M}^\alpha_{1,p}$ when $1\leq q<\alpha<p$, Theorem 2.2 provides an extension of the Hardy-Littlewood-Sobolev theorem (Theorem 1.2) to the setting of Bourgain-Morrey spaces.

From (1) and (6), we have

$$L^\alpha\subset\mathcal{M}^\alpha_{1,p}\subset\mathbb{B}(\gamma,p)\,,\quad 0<\gamma<\frac{1}{\alpha}<1 \text{ and } \frac{1}{p}=\frac{1}{\alpha}-\gamma, \tag{7}$$

where

$$\mathbb{B}(\gamma,p)=\left\{f\in L^1_{\mathrm{loc}}\;:\;I_\gamma(|f|)\in L^p\right\}\,,\quad 0<\gamma<1\leq p\leq\infty.$$

We recall that, for $1\leq q,p,\alpha\leq\infty$, the space $\mathbb{F}(q,p,\alpha)$ arises naturally in the search of a characterization of the set $\mathbb{B}(\gamma,p)$ in [7], where it is established that

$$\mathbb{B}(\gamma,p)\subset\mathbb{F}(1,p,\alpha)_c\subset\mathbb{F}(1,p,\alpha)\subset W\mathbb{B}(\gamma,p)\,,\;\; 0<\gamma<\frac{1}{\alpha}\leq 1 \text{ and } \frac{1}{p}=\frac{1}{\alpha}-\gamma, \tag{8}$$

with

$$\mathbb{F}(q,p,\alpha)_c=\left\{f\in\mathbb{F}(q,p,\alpha)\;:\;\lim_{y\to 0}\|f-f(\cdot-y)\|_{\mathbb{F}(q,p,\alpha)}=0\right\}$$

and

$$W\mathbb{B}(\gamma,p)=\left\{f\in L^1_{\mathrm{loc}}\;:\;I_\gamma(|f|)\in WL^p\right\}\,,\quad 0<\gamma<1\leq p\leq\infty.$$

Note that, it is proved in [7] that $\mathbb{F}(q,p,\alpha)_{c}$ is the closure of L^{α} in $\mathbb{F}(q,p,\alpha)$ if $p<\infty$. It is clear that the inclusion relations (7) and (8) yield what follows

$$\mathcal{M}^{\alpha}_{1,p}\subset\mathbb{F}(1,p,\alpha)_{c}\ ,\quad 0<\gamma<\frac{1}{\alpha}<1 \text{ and } \frac{1}{p}=\frac{1}{\alpha}-\gamma. \tag{9}$$

In the present paper we prove, without the use of fractional integral operators, the following extension of the relation (9).

Theorem 2.3. *Let us assume that $1\leq q\leq\alpha\leq p\leq\infty$. Then*

$$\|f\|_{\mathbb{F}(q,p,\alpha)}\leq 3^{d\left(1+\frac{1}{\alpha}-\frac{1}{q}+\frac{1}{p}\right)}\,2^{d\left(\frac{1}{q}-\frac{1}{p}\right)}\,\|f\|_{\mathcal{M}^{\alpha}_{q,p}}\ ,\qquad f\in L^{1}_{\text{loc}} \tag{10}$$

and therefore $\mathcal{M}^{\alpha}_{q,p}$ is continuously included in $\mathbb{F}(q,p,\alpha)$. Moreover, if $p<\infty$ then $\mathcal{M}^{\alpha}_{q,p}$ is included in $\mathbb{F}(q,p,\alpha)_{c}$.

As done in [4, 6] for some special subspaces of the Morrey-type space $(L^{q},l^{p})^{\alpha}$, usually called the Fofana space and closely related to $\mathbb{F}(q,p,\alpha)$, we investigate in Bourgain-Morrey spaces approximation by smooth functions. We shall prove what follows.

Theorem 2.4. *Let $1\leq q\leq\alpha\leq p<\infty$ and f be an element of L^{1}_{loc}. Then the following assertions are equivalent :*

(i) f belongs to $\mathcal{M}^{\alpha}_{q,p}$,

(ii) $\lim\limits_{n\to\infty}\|f-f*\phi_{n}\|_{\mathcal{M}^{\alpha}_{q,p}}=0$, where $f*\phi_{n}$ is the convolution product of f and ϕ_{n},

(iii) f belongs to the closure in $\mathcal{M}^{\alpha}_{q,p}$ of the set

$$\mathcal{C}^{\infty}_{\mathcal{M}^{\alpha}_{q,p}}=\left\{g\in\mathcal{C}^{\infty}\ :\ \partial^{\beta}g\in\mathcal{M}^{\alpha}_{q,p}\quad\textit{for any }\beta\textit{ in }\mathbb{N}^{d}\ \right\},$$

where $\partial^{\beta}g$ stands for the derivative of order β of g.

Note that Theorem 2.4 implies that both $\mathcal{C}^{\infty}\cap\mathcal{M}^{\alpha}_{q,p}$ and $\mathcal{C}^{\infty}_{\mathcal{M}^{\alpha}_{q,p}}$ are dense in $\mathcal{M}^{\alpha}_{q,p}$ if $p<\infty$. As a consequence of this theorem, we obtain the following approximation result.

Theorem 2.5. *Let $1\leq q\leq\alpha\leq p<\infty$ and f be any element of $\mathcal{M}^{\alpha}_{q,p}$. Then*

$$\lim_{n\to\infty}\|f-(f\omega_{n})*\phi_{n}\|_{\mathcal{M}^{\alpha}_{q,p}}=0.$$

It is easy to see that, for any integer $n\geq1$, $(f\omega_{n})*\phi_{n}$ belongs to $\mathcal{C}^{\infty}_{c}$. Consequently, Theorem 2.5 implies that $\mathcal{C}^{\infty}_{c}\cap\mathcal{M}^{\alpha}_{q,p}$ is dense in $\mathcal{M}^{\alpha}_{q,p}$ if $p<\infty$.

Let us consider the divergence equation

$$\operatorname{div}F=f\ ,\quad f\in L^{1}_{\text{loc}}. \tag{11}$$

To our knowledge, for a given p in $[1,\infty)$, the characterization of the class of functions f for which the equation (11) has a solution $F=(F_{j})_{1\leq j\leq d}$ in $(L^{p})^{d}$ is still an open problem. However, Phuc and Torres proved that (see [13, Theorem 3.2]), for $\frac{d}{d-1}<p<\infty$, the equation (11) has a solution in $(L^p)^d$ if and only if f belongs to the set $\mathbb{B}(\frac{1}{d},p)$. This result combined with (7) shows that, for $1<\alpha<d$ and $\frac{1}{p}=\frac{1}{\alpha}-\frac{1}{d}$, a sufficient condition for the solvability in $(L^p)^d$ of the equation (11) is that f belongs to $\mathcal{M}^{\alpha}_{1,p}$. Moreover, an application of Theorem 2.2 and Theorem 2.5 allows us to obtain an explicit solution of the equation (11) in $(L^p)^d$, as shown below.

Theorem 2.6. *Let us assume that $d\geq 3$, $1\leq q\leq\alpha<d$, $\frac{1}{p}=\frac{1}{\alpha}-\frac{1}{d}$ and f is an element of $\mathcal{M}^{\alpha}_{q,p}$. Then there exists a real constant c_d such that $F=\left(c_d\, R_j\left(I_{\frac{1}{d}}f\right)\right)_{1\leq j\leq d}$ is a solution in $(L^p)^d$ of the equation (11), where R_j $(1\leq j\leq d)$ stands for the Riesz transform defined by*

$$R_j\varphi(x)=\frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}\lim_{\epsilon\to 0^+}\int_{|x-y|\geq\epsilon}\frac{x_j-y_j}{|x-y|^{d+1}}\varphi(y)dy\ ,\quad x\in\mathbb{R}^d,\quad \varphi\in L^p.$$

3. Preliminaries

This section is devoted to prove some preliminary results.

3.1. Equivalent norms on $\mathcal{M}^{\alpha}_{q,p}$.

We begin this subsection by recalling the definition of classical dyadic grids.

Definition 3.1. *A dyadic grid is a countable collection $\mathcal{D}$ of cubes of $\mathbb{R}^d$ which are dyadic translates and dilations of the unit cube $[0,1)^d$. More precisely, $\mathcal{D}$ may be characterized as follows :*
(i) if $Q\in\mathcal{D}$ then its side-length $\ell(Q)=2^m$ for some $m\in\mathbb{Z}$
(ii) if $Q,P\in\mathcal{D}$ then $Q\cap P\in\{\varnothing,Q,P\}$
(iii) for each $m\in\mathbb{Z}$, the family $\mathcal{D}_m=\{Q\in\mathcal{D}\ /\ \ell(Q)=2^m\}$ form a partition of $\mathbb{R}^d$.

Example 3.2. • *The standard dyadic grid $\mathcal{D}^0$ is defined by*

$$\mathcal{D}^0=\left\{2^m\left([0,1)^d+k\right)\ /\ m\in\mathbb{Z},\ \ k\in\mathbb{Z}^d\right\}.$$

• *Each of the following 3^d collections of cubes in $\mathbb{R}^d$*

$$\mathcal{D}^t=\left\{2^m\left([0,1)^d+k+t\right)\ /\ m\in\mathbb{Z},\ \ k\in\mathbb{Z}^d\right\},\quad t\in\{-1/3,0,1/3\}^d$$

is a dyadic grid.

The following property holds (see [3, Theorem 3.1] and its proof).

Proposition 3.3. *For every cube Q of $\mathbb{R}^d$, there exists an element t of $\{-1/3,0,1/3\}^d$ and a cube Q_t of $\mathcal{D}^t$ such that Q is included in Q_t and $\ell(Q_t)\leq 3\,\ell(Q)$.*

Let us introduce the following definition.

Definition 3.4. *Let $1 \le q \le \alpha \le p \le \infty$. For any dyadic grid $\mathcal{D}$ and any element f of L^q_{loc}, we define*

$$\|f\|_{\mathcal{M}^\alpha_{q,p}(\mathcal{D})} = \left\| \left\{ |Q|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_Q |f(x)|^q dx \right)^{\frac{1}{q}} \right\}_{Q \in \mathcal{D}} \right\|_{\ell^p}.$$

We shall prove that, for any $t \in \{-1/3, 0, 1/3\}^d$, the norms $\|\cdot\|_{\mathcal{M}^\alpha_{q,p}(\mathcal{D}^t)}$ and $\|\cdot\|_{\mathcal{M}^\alpha_{q,p}}$ are equivalent. In order to do this, we establish the following preparatory lemma.

Lemma 3.5. *Let $1 \le q \le \alpha \le p \le \infty$, $\mathcal{D}$ and $\mathcal{D}'$ are two dyadic grids. Then for any element f of L^q_{loc}, we have*

$$\begin{cases} \|f\|_{\mathcal{M}^\alpha_{q,p}(\mathcal{D})} \le 2^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{\mathcal{M}^\alpha_{q,p}(\mathcal{D}')} & \text{if } p < \infty \\ \|f\|_{\mathcal{M}^\alpha_{q,\infty}(\mathcal{D})} \le 2^d \|f\|_{\mathcal{M}^\alpha_{q,\infty}(\mathcal{D}')}. \end{cases}$$

Proof. Let f be any element of L^q_{loc} and fix $m \in \mathbb{Z}$.

We recall that both the families $\mathcal{D}_m = \{Q \in \mathcal{D} \ / \ \ell(Q) = 2^m\}$ and $\mathcal{D}'_m = \{Q' \in \mathcal{D}' \ / \ \ell(Q') = 2^m\}$ form partitions of $\mathbb{R}^d$. Moreover, it is easy to see that, for any element Q of $\mathcal{D}_m$, the subset $\{Q' \in \mathcal{D}' \ / \ Q \cap Q' \neq \varnothing\}$ of $\mathcal{D}'$ has at most 2^d elements.

a) Suppose that $p < \infty$. We have

$$\begin{aligned} \left(\int_Q |f(x)|^q dx \right)^{\frac{p}{q}} &= \left(\sum_{Q' \in \mathcal{D}'_m} \int_{Q \cap Q'} |f(x)|^q dx \right)^{\frac{p}{q}} \\ &\le 2^{d\left(1-\frac{q}{p}\right)\frac{p}{q}} \sum_{Q' \in \mathcal{D}'_m} \left(\int_{Q \cap Q'} |f(x)|^q dx \right)^{\frac{p}{q}}. \end{aligned}$$

Consequently

$$\begin{aligned} &\sum_{Q \in \mathcal{D}_m} \left[|Q|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_Q |f(x)|^q dx \right)^{\frac{1}{q}} \right]^p = 2^{d\,m\left(\frac{1}{\alpha}-\frac{1}{q}\right)p} \sum_{Q \in \mathcal{D}_m} \left(\int_Q |f(x)|^q dx \right)^{\frac{p}{q}} \\ &\le 2^{d\left(\frac{p}{q}-1\right)} 2^{d\,m\left(\frac{1}{\alpha}-\frac{1}{q}\right)p} \sum_{Q \in \mathcal{D}_m} \sum_{Q' \in \mathcal{D}'_m,\ Q \cap Q' \neq \varnothing} \left(\int_{Q \cap Q'} |f(x)|^q dx \right)^{\frac{p}{q}} \\ &= 2^{d\left(\frac{p}{q}-1\right)} 2^{d\,m\left(\frac{1}{\alpha}-\frac{1}{q}\right)p} \sum_{Q' \in \mathcal{D}'_m} \sum_{Q \in \mathcal{D}_m} \left(\int_{Q \cap Q'} |f(x)|^q dx \right)^{\frac{p}{q}} \\ &\le 2^{d\left(\frac{p}{q}-1\right)} 2^{d\,m\left(\frac{1}{\alpha}-\frac{1}{q}\right)p} \sum_{Q' \in \mathcal{D}'_m} \left(\int_{Q'} |f(x)|^q dx \right)^{\frac{p}{q}} \\ &= 2^{d\left(\frac{p}{q}-1\right)} \sum_{Q' \in \mathcal{D}'_m} \left[|Q'|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{Q'} |f(x)|^q dx \right)^{\frac{1}{q}} \right]^p \end{aligned}$$

and so

$$\|f\|_{\mathcal{M}^\alpha_{q,p}(\mathcal{D})} \le 2^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{\mathcal{M}^\alpha_{q,p}(\mathcal{D}')}.$$

b) Suppose that $p = \infty$. Then, for any $Q \in \mathcal{D}_m$, we have

$$\begin{aligned} |Q|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_Q |f(x)|^q dx \right)^{\frac{1}{q}} &= |Q|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\sum_{Q' \in \mathcal{D}'_m} \int_{Q\cap Q'} |f(x)|^q dx \right)^{\frac{1}{q}} \\ \leq |Q|^{\frac{1}{\alpha}-\frac{1}{q}} \sum_{Q' \in \mathcal{D}'_m} \left(\int_{Q\cap Q'} |f(x)|^q dx \right)^{\frac{1}{q}} &= \sum_{Q' \in \mathcal{D}'_m} |Q'|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{Q\cap Q'} |f(x)|^q dx \right)^{\frac{1}{q}} \\ \leq \sum_{Q' \in \mathcal{D}'_m,\, Q\cap Q' \neq \varnothing} |Q'|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{Q'} |f(x)|^q dx \right)^{\frac{1}{q}} &\leq 2^d \|f\|_{\mathcal{M}^{\alpha}_{q,\infty}(\mathcal{D}')} \end{aligned}$$

and so

$$\|f\|_{\mathcal{M}^{\alpha}_{q,\infty}(\mathcal{D})} \leq 2^d \|f\|_{\mathcal{M}^{\alpha}_{q,\infty}(\mathcal{D}')}.$$

The proof is complete. □

The above lemma leads to the following corollary.

Corollary 3.6. *Let $1 \leq q \leq \alpha \leq p \leq \infty$ and t be in $\{-1/3, 0, 1/3\}^d$. Then for any element f of L^q_{loc}, we have*

$$\begin{cases} 2^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{\mathcal{M}^{\alpha}_{q,p}} \leq \|f\|_{\mathcal{M}^{\alpha}_{q,p}(\mathcal{D}^t)} \leq 2^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{\mathcal{M}^{\alpha}_{q,p}} & \text{if } p < \infty \\ 2^{-d} \|f\|_{\mathcal{M}^{\alpha}_{q,\infty}} \leq \|f\|_{\mathcal{M}^{\alpha}_{q,\infty}(\mathcal{D}^t)} \leq 2^d \|f\|_{\mathcal{M}^{\alpha}_{q,\infty}}. \end{cases}$$

3.2. **Continuity of the translation operator in $\mathcal{M}^{\alpha}_{q,p}$.** This subsection deals with the continuity of the translation operator in Bourgain-Morrey spaces. We shall use in the sequel the following properties.

Proposition 3.7. *[8] Let us assume that $1 \leq q \leq \alpha \leq p \leq \infty$.*
1) If $\alpha < \infty$, then there exists $C_1 > 0$ such that for all $y \in \mathbb{R}^d$ and $f \in \mathcal{M}^{\alpha}_{q,p}$, we have

$$\|f(\cdot - y)\|_{\mathcal{M}^{\alpha}_{q,p}} \leq C_1 \|f\|_{\mathcal{M}^{\alpha}_{q,p}}.$$

2) If $q < \alpha < p < \infty$ then the set L^{∞}_{c} of all compactly supported bounded functions is dense in $\mathcal{M}^{\alpha}_{q,p}$.
3) If $q < \alpha < p < \infty$ or $p = \infty$ then there exists $C_2 > 0$ such that for any element f of $\mathcal{M}^{\alpha}_{q,p}$, we have

$$\|f\|_{\mathcal{M}^{\alpha}_{q,p}} \leq C_2 \|f\|_{\alpha}.$$

A classical property of Lebesgue spaces reads as follows.

Lemma 3.8. *If $1 \leq \alpha < \infty$ and f is in L^{α} then we have*

$$\lim_{y \to 0} \|f - f(\cdot - y)\|_{\alpha} = 0.$$

This result can be extended to the setting of Bourgain-Morrey spaces and this extension will play a key role in the proofs of our results.

Proposition 3.9. *Let* $1 \leq q \leq \alpha \leq p < \infty$ *and* f *be any element of* $\mathcal{M}^{\alpha}_{q,p}$*. Then*

$$\lim_{y\to 0} \|f - f(\cdot - y)\|_{\mathcal{M}^{\alpha}_{q,p}} = 0.$$

Proof. If $q = \alpha$ or $\alpha = p$ then $\mathcal{M}^{\alpha}_{q,p} = \{0\}$ and therefore, we have nothing to prove. Thus we suppose that $q < \alpha < p$. By Point 2) of Proposition 3.7, there exists a sequence $(f_n)_{n\geq 1}$ of elements of L^{∞}_{c} such that $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{M}^{\alpha}_{q,p}} = 0$. Moreover, according to Point 1) of Proposition 3.7, there exists $C_1 > 0$ such that

$$\|(f_n - f)(\cdot - y)\|_{\mathcal{M}^{\alpha}_{q,p}} \leq C_1 \|f_n - f\|_{\mathcal{M}^{\alpha}_{q,p}}, \quad y \in \mathbb{R}^d, \quad n \geq 1. \qquad (*)$$

Let $\epsilon > 0$ be a fixed real number. There exists an integer N_ϵ such that

$$\|f_{N_\epsilon} - f\|_{\mathcal{M}^{\alpha}_{q,p}} < \frac{\epsilon}{2(1 + C_1)}. \qquad (**)$$

From $(*)$, $(**)$, Point 1) and Point 3) of Proposition 3.7 we have

$$\begin{aligned}
\|f - f(\cdot - y)\|_{\mathcal{M}^{\alpha}_{q,p}} &\leq \|f - f_{N_\epsilon}\|_{\mathcal{M}^{\alpha}_{q,p}} + \|f_{N_\epsilon} - f_{N_\epsilon}(\cdot - y)\|_{\mathcal{M}^{\alpha}_{q,p}} + \|(f_{N_\epsilon} - f)(\cdot - y)\|_{\mathcal{M}^{\alpha}_{q,p}} \\
&\leq \|f - f_{N_\epsilon}\|_{\mathcal{M}^{\alpha}_{q,p}} + \|f_{N_\epsilon}(\cdot - y) - f_{N_\epsilon}\|_{\mathcal{M}^{\alpha}_{q,p}} + C_1 \|f_{N_\epsilon} - f\|_{\mathcal{M}^{\alpha}_{q,p}} \\
&\leq (1 + C_1) \|f - f_{N_\epsilon}\|_{\mathcal{M}^{\alpha}_{q,p}} + \|f_{N_\epsilon}(\cdot - y) - f_{N_\epsilon}\|_{\mathcal{M}^{\alpha}_{q,p}} \\
&< \frac{\epsilon}{2} + C_2 \|f_{N_\epsilon}(\cdot - y) - f_{N_\epsilon}\|_{\alpha}.
\end{aligned}$$

According to Lemma 3.8, for any $y \in \mathbb{R}^d$ such that $0 < |y| < 1$, we have

$$\|f_{N_\epsilon}(\cdot - y) - f_{N_\epsilon}\|_{\alpha} < \frac{\epsilon}{2\,C_2}$$

and therefore we obtain

$$\|f - f(\cdot - y)\|_{\mathcal{M}^{\alpha}_{q,p}} < \epsilon.$$

This ends the proof. □

4. Inclusion and approximation results

4.1. **Inclusion of $\mathcal{M}^{\alpha}_{q,p}$ in $\mathbb{F}(q, p, \alpha)$.** This subsection is devoted to prove exclusively Theorem 2.3.

Proof of Theorem 2.3

1) • We recall that $\mathcal{M}^{\alpha}_{q,\infty} = \mathcal{M}^{\alpha}_{q} = \mathbb{F}(q, \infty, \alpha)$. Therefore, we have nothing to prove if $p = \infty$.

• If $p < \infty$ and $\alpha \in \{q, p\}$ then $\mathcal{M}^{\alpha}_{q,p} = \{0\}$. Thus the result is obvious.

• Assume that $1 \leq q < \alpha < p < \infty$.

Let f be in L^1_{loc} and $\{Q_i : i \in \mathbf{I}\}$ be a disjoint family of cubes of $\mathbb{R}^d$.

a) Let us consider an element i of $\mathbf{I}$. We can associate to i an element t of $\{-1/3, 0, 1/3\}^d$ and an element $R(i,t)$ of $\mathcal{D}^t$ such that $Q_i \subset R(i,t)$ and $\ell(R(i,t)) \leq 3\,\ell(Q_i)$ (see Proposition 3.3). We have

$$|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{Q_i} |f(x)|^q dx\right)^{\frac{1}{q}} \leq \left(\frac{|Q_i|}{|R(i,t)|}\right)^{\frac{1}{\alpha}-\frac{1}{q}} |R(i,t)|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{R(i,t)} |f(x)|^q dx\right)^{\frac{1}{q}}$$
$$\leq 3^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} |R(i,t)|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{R(i,t)} |f(x)|^q dx\right)^{\frac{1}{q}}.$$

b) Let us fix t in $\{-1/3, 0, 1/3\}^d$ and set

$$\mathcal{R}^t = \left\{R \in \mathcal{D}^t : \exists\, i \in \mathbf{I} \text{ such that } R(i,t) = R\right\}.$$

Note that, for all $R \in \mathcal{R}^t$, we have

$$\begin{cases} \forall\, i \in \mathbf{I},\ R = R(i,t) \Longrightarrow \ell(Q_i) \leq \ell(R) \leq 3\ell(Q_i) \Longrightarrow |Q_i| \leq |R| \leq 3^d |Q_i| \\ \displaystyle\sum_{i \in \mathbf{I},\, R = R(i,t)} |Q_i| \leq |R|. \end{cases}$$

This shows that the cardinality of the set $\{i \in \mathbf{I} : R = R(i,t)\}$ does not exceed 3^d.

c) We have

$$\sum_{i\in \mathbf{I}} \left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{Q_i} |f(x)|^q dx\right)^{\frac{1}{q}}\right)^p$$
$$= \sum_{t\in\{-1/3,0,1/3\}^d} \sum_{R\in\mathcal{R}^t} \sum_{i:\, R=R(i,t)} \left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{Q_i} |f(x)|^q dx\right)^{\frac{1}{q}}\right)^p$$
$$\leq \sum_{t\in\{-1/3,0,1/3\}^d} \sum_{R\in\mathcal{R}^t} \sum_{i:\, R=R(i,t)} \left(3^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)p} |R|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{R} |f(x)|^q dx\right)^{\frac{1}{q}}\right)^p \quad \text{(by Point a))}$$
$$\leq 3^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)p}\, 3^d \sum_{t\in\{-1/3,0,1/3\}^d} \sum_{R\in\mathcal{R}^t} \left(|R|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{R} |f(x)|^q dx\right)^{\frac{1}{q}}\right)^p \quad \text{(by Point b))}$$
$$\leq 3^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)p}\, 3^d \sum_{t\in\{-1/3,0,1/3\}^d} \sum_{R\in\mathcal{D}^t} \left(|R|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{R} |f(x)|^q dx\right)^{\frac{1}{q}}\right)^p \quad \text{(because of } \mathcal{R}^t \subset \mathcal{D}^t\text{)}.$$

Therefore

$$\left[\sum_{i\in \mathbf{I}} \left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left(\int_{Q_i} |f(x)|^q dx\right)^{\frac{1}{q}}\right)^p\right]^{\frac{1}{p}} \leq 3^{d\left(\frac{1}{\alpha}-\frac{1}{q}+\frac{1}{p}\right)} \sum_{t\in\{-1/3,0,1/3\}^d} \|f\|_{\mathcal{M}^{\alpha}_{q,p}(\mathcal{D}^t)}.$$

Since the above inequality is true for all disjoint family $\{Q_i : i \in \mathbf{I}\}$ of cubes of $\mathbb{R}^d$, we have

$$\|f\|_{\mathbb{F}(q,p,\alpha)} \leq 3^{d\left(\frac{1}{\alpha}-\frac{1}{q}+\frac{1}{p}\right)} \sum_{t\in\{-1/3,0,1/3\}^d} \|f\|_{\mathcal{M}^{\alpha}_{q,p}(\mathcal{D}^t)}.$$

Therefore, Corollary 3.6 implies that

$$\|f\|_{\mathbb{F}(q,p,\alpha)} \leq 3^{d\left(1+\frac{1}{\alpha}-\frac{1}{q}+\frac{1}{p}\right)} 2^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{\mathcal{M}^{\alpha}_{q,p}}$$

and consequently $\mathcal{M}^{\alpha}_{q,p}$ is continuously included in $\mathbb{F}(q,p,\alpha)$.

2) Assume that $p<\infty$ and let f be any element of $\mathcal{M}^{\alpha}_{q,p}$. By Point 1), f is in $\mathbb{F}(q,p,\alpha)$ and, for any $y\in\mathbb{R}^d$, we have

$$\|f-f(\cdot-y)\|_{\mathbb{F}(q,p,\alpha)} \leq 3^{d\left(1+\frac{1}{\alpha}-\frac{1}{q}+\frac{1}{p}\right)} 2^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \|f-f(\cdot-y)\|_{\mathcal{M}^{\alpha}_{q,p}}.$$

Therefore, Proposition 3.9 implies that

$$\lim_{y\to 0}\|f-f(\cdot-y)\|_{\mathbb{F}(q,p,\alpha)}=0$$

and consequently f belongs to $\mathbb{F}(q,p,\alpha)_c$. Thus, we obtain the desired result. □

4.2. **Approximation in $\mathcal{M}^{\alpha}_{q,p}$.** In this subsection, we investigate approximation of elements of Bourgain-Morrey spaces by smooth functions. We shall use the following result.

Proposition 4.1. *[8] Let us assume that $1\leq q\leq\alpha\leq p\leq\infty$ with $\alpha<\infty$. Then there exists $C>0$ such that for all $g\in L^1$ and $f\in\mathcal{M}^{\alpha}_{q,p}$, we have*

$$\|g*f\|_{\mathcal{M}^{\alpha}_{q,p}} \leq C\,\|g\|_1\,\|f\|_{\mathcal{M}^{\alpha}_{q,p}}.$$

Propositions 3.7, 3.9 and 4.1 allow us to prove Theorem 2.4.

Proof of Theorem 2.4

- $(i)\Rightarrow(ii)$ Assume that $f\in\mathcal{M}^{\alpha}_{q,p}$ and n is a nonegative integer. For almost every $x\in\mathbb{R}^d$,

$$\begin{aligned} f(x)-f*\phi_n(x) &= \int_{\mathbb{R}^d} f(x)\phi(u)du - \int_{\mathbb{R}^d} f(x-y)n^d\phi(ny)dy \\ &= \int_{\mathbb{R}^d} f(x)\phi(u)du - \int_{\mathbb{R}^d} f\left(x-\frac{u}{n}\right)\phi(u)du \\ &= \int_{\mathbb{R}^d}\left[f(x)-f\left(x-\frac{u}{n}\right)\right]\phi(u)du. \end{aligned}$$

Therefore, for any dyadic cube $Q_{k,m}$ $\left((k,m)\in\mathbb{Z}^d\times\mathbb{Z}\right)$, the Minkowski inequality implies that,

$$\left\|(f-f*\phi_n)\chi_{Q_{k,m}}\right\|_q \leq \int_{\mathbb{R}^d}\left\|\left[f-f\left(\cdot-\frac{u}{n}\right)\right]\chi_{Q_{k,m}}\right\|_q \phi(u)du$$

and so

$$\|f-f*\phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} \leq \int_{\mathbb{R}^d}\left\|f-f\left(\cdot-\frac{u}{n}\right)\right\|_{\mathcal{M}^{\alpha}_{q,p}}\phi(u)du.$$

According to Proposition 3.9, we have

$$\lim_{n\to\infty}\left\|f-f\left(\cdot-\frac{u}{n}\right)\right\|_{\mathcal{M}^{\alpha}_{q,p}}\phi(u)=0\,, \qquad u\in\mathbb{R}^d.$$

Furthermore, by Minkowski's inequality and Point 1) of Proposition 3.7, we have

$$\left\|f - f\left(\cdot - \frac{u}{n}\right)\right\|_{\mathcal{M}^{\alpha}_{q,p}} \phi(u) \leq (1 + C_1)\ \|f\|_{\mathcal{M}^{\alpha}_{q,p}} \phi(u)\ , \qquad u \in \mathbb{R}^d\ , \quad n \geq 1.$$

Thus, an application of the dominated convergence theorem gives

$$\lim_{n\to\infty} \|f - f * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} = 0.$$

• $(ii) \Rightarrow (iii)$ Assume that the assertion (ii) holds.

Let us fix an integer $n \geq 1$ and $\beta \in \mathbb{N}^d$. Since $\phi_n \in \mathcal{C}^{\infty}_c$, $f * \phi_n$ belongs to $\mathcal{C}^{\infty}$ and by Proposition 4.1, $f * \phi_n$ is in $\mathcal{M}^{\alpha}_{q,p}$. Furthermore, it is well known that $\partial^{\beta}(f * \phi_n) = f * \partial^{\beta}\phi_n$ and by noting that $\partial^{\beta}\phi_n \in L^1$, Proposition 4.1 implies that $\partial^{\beta}(f * \phi_n)$ belongs to $\mathcal{M}^{\alpha}_{q,p}$. Thus $f * \phi_n$ belongs to $\mathcal{C}^{\infty}_{\mathcal{M}^{\alpha}_{q,p}}$ and since, by hypothesis,

$$\lim_{n\to\infty} \|f - f * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} = 0,$$

we can conclude that f belongs to the closure in $\mathcal{M}^{\alpha}_{q,p}$ of $\mathcal{C}^{\infty}_{\mathcal{M}^{\alpha}_{q,p}}$.

• $(iii) \Rightarrow (i)$ Since $\mathcal{C}^{\infty}_{\mathcal{M}^{\alpha}_{q,p}}$ is a subset of $\mathcal{M}^{\alpha}_{q,p}$, it is obvious that its closure in $\mathcal{M}^{\alpha}_{q,p}$ is included in $\mathcal{M}^{\alpha}_{q,p}$ and therefore the claim follows. The proof is complete. □

We recall the following well known result in Lebesgue spaces.

Lemma 4.2. *[1] If $1 \leq \alpha < \infty$ and f is in L^{α} then we have*

$$\lim_{n\to\infty} \|f\chi_{E_n}\|_{\alpha} = 0,$$

where $(E_n)_{n\geq 1}$ is a nonincreasing sequence of measurable subsets of $\mathbb{R}^d$ satisfying $\left|\bigcap_{n\geq 1} E_n\right| = 0.$

The next proposition shows that an analogous result holds for Bourgain-Morrey spaces.

Proposition 4.3. *Let $1 \leq q \leq \alpha \leq p < \infty$, f be any element of $\mathcal{M}^{\alpha}_{q,p}$ and $(E_n)_{n\geq 1}$ be a nonincreasing sequence of measurable subsets of $\mathbb{R}^d$ satisfying $\left|\bigcap_{n\geq 1} E_n\right| = 0$. Then*

$$\lim_{n\to\infty} \|f\chi_{E_n}\|_{\mathcal{M}^{\alpha}_{q,p}} = 0.$$

Proof. If $q = \alpha$ or $\alpha = p$ then $\mathcal{M}^{\alpha}_{q,p} = \{0\}$ and therefore we have nothing to prove. Hence we suppose that $1 \leq q < \alpha < p < \infty$. By Point 2) of Proposition 3.7, there exists a sequence $(f_n)_{n\geq 1}$ of elements of L^{∞}_c such that

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{M}^{\alpha}_{q,p}} = 0.$$

Let $\epsilon > 0$ be a fixed real number. From what precedes, there exists an integer N_{ϵ} such that

$$\|f_{N_{\epsilon}} - f\|_{\mathcal{M}^{\alpha}_{q,p}} < \frac{\epsilon}{2}.$$

This and Point 3) of Proposition 3.7 imply that, for any $n \geq 1$,

$$\begin{aligned} \|f\chi_{E_n}\|_{\mathcal{M}^{\alpha}_{q,p}} &\leq \|(f - f_{N_\epsilon})\chi_{E_n}\|_{\mathcal{M}^{\alpha}_{q,p}} + \|f_{N_\epsilon}\chi_{E_n}\|_{\mathcal{M}^{\alpha}_{q,p}} \leq \|f - f_{N_\epsilon}\|_{\mathcal{M}^{\alpha}_{q,p}} + \|f_{N_\epsilon}\chi_{E_n}\|_{\mathcal{M}^{\alpha}_{q,p}} \\ &\leq \frac{\epsilon}{2} + C_2 \|f_{N_\epsilon}\chi_{E_n}\|_{\alpha}. \end{aligned}$$

Since $f_{N_\epsilon} \in L^{\alpha}$, Lemma 4.2 implies that there exists an integer $N_0 \geq 1$ such that

$$n \geq N_0 \Longrightarrow \|f_{N_\epsilon}\chi_{E_n}\|_{\alpha} < \frac{\epsilon}{2\,C_2}.$$

Therefore

$$n \geq N_0 \Longrightarrow \|f\chi_{E_n}\|_{\mathcal{M}^{\alpha}_{q,p}} < \epsilon.$$

This provides the desired result. □

[1, Proposition 3.6] asserts that Proposition 4.3 is equivalent to the following dominated convergence theorem.

Proposition 4.4. *Let $1 \leq q \leq \alpha \leq p < \infty$ and f be any element of $\mathcal{M}^{\alpha}_{q,p}$. If $(f_n)_{n\geq 1}$ is a sequence of measurable functions satisfying $|f_n| \leq |f|$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} f_n = g$ almost everywhere, for some measurable function g, then*

$$\lim_{n\to\infty} \|f_n - g\|_{\mathcal{M}^{\alpha}_{q,p}} = 0.$$

Proposition 4.4 yields obviously what follows.

Lemma 4.5. *Let $1 \leq q \leq \alpha \leq p < \infty$. Then for any element f of $\mathcal{M}^{\alpha}_{q,p}$, we have*

$$\lim_{n\to\infty} \left\|f - f\chi_{Q(0,n)}\right\|_{\mathcal{M}^{\alpha}_{q,p}} = 0.$$

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5

For any integer $n \geq 1$, we have, by Proposition 4.1,

$$\begin{aligned} \|f - (f\omega_n) * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} &\leq \|f - f * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} + \|(f - f\omega_n) * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} \\ &\leq \|f - f * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} + C\,\|f - f\omega_n\|_{\mathcal{M}^{\alpha}_{q,p}} \|\phi_n\|_1 \\ &\leq \|f - f * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} + C\,\|f - f\omega_n\|_{\mathcal{M}^{\alpha}_{q,p}}. \end{aligned}$$

Notice that, for any integer $n \geq 1$, $|f - f\omega_n| \leq \left|f - f\chi_{Q(0,n)}\right|$ and therefore we obtain

$$\|f - (f\omega_n) * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} \leq \|f - f * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} + C\left\|f - f\chi_{Q(0,n)}\right\|_{\mathcal{M}^{\alpha}_{q,p}}.$$

Thus, it follows from Theorem 2.4 and Lemma 4.5 that

$$\lim_{n\to\infty} \|f - (f\omega_n) * \phi_n\|_{\mathcal{M}^{\alpha}_{q,p}} = 0.$$

This finishes the proof. □

5. Fractional operators in $\mathcal{M}^{\alpha}_{q,p}$

This section is devoted to prove Theorem 2.1 and Theorem 2.2. In order to do this we need some preparatory lemmas.

Let $0 < \gamma < 1$ and $\mathcal{D}$ be a dyadic grid. The dyadic fractional maximal operator $\mathfrak{M}^{\mathcal{D}}_{\gamma}$ is defined by

$$\mathfrak{M}^{\mathcal{D}}_{\gamma} f(x) = \sup \left\{ |Q|^{\gamma-1} \int_Q |f(y)| dy \ / \ Q \in \mathcal{D}, \ x \in Q \right\}, \qquad f \in L^1_{\text{loc}}, \ \ x \in \mathbb{R}^d.$$

The following lemma is a consequence of Proposition 3.3.

Lemma 5.1. *Let $0 < \gamma < 1$. For any element f of L^1_{loc} we have :*

$$\mathfrak{M}_{\gamma} f(x) \leq 3^{d(1-\gamma)} \max_{t \in \{-1/3, 0, 1/3\}^d} \mathfrak{M}^{\mathcal{D}^t}_{\gamma} f(x) \, , \quad x \in \mathbb{R}^d.$$

Proof. Let us consider an element (f, x) of $L^1_{\text{loc}} \times \mathbb{R}^d$ and a cube Q of $\mathbb{R}^d$ containing x. By Proposition 3.3, there exist an element t of $\{-1/3, 0, 1/3\}^d$ and a cube Q_t of $\mathcal{D}^t$ such that Q is included in Q_t and $\ell(Q_t) \leq 3\, \ell(Q)$. Thus, we have

$$\begin{aligned} |Q|^{\gamma-1} \int_Q |f(y)| dy = \ell(Q)^{d(\gamma-1)} \int_Q |f(y)| dy &\leq \left[\frac{1}{3} \ell(Q_t) \right]^{d(\gamma-1)} \int_Q |f(y)| dy \\ &\leq 3^{d(1-\gamma)} |Q_t|^{\gamma-1} \int_{Q_t} |f(y)| dy \leq 3^{d(1-\gamma)} \mathfrak{M}^{\mathcal{D}^t}_{\gamma} f(x). \end{aligned}$$

Consequently

$$\mathfrak{M}_{\gamma} f(x) \leq 3^{d(1-\gamma)} \max_{t \in \{-1/3, 0, 1/3\}^d} \mathfrak{M}^{\mathcal{D}^t}_{\gamma} f(x).$$

The proof is complete. □

Recall that the density of L^{∞}_{c} in $\mathcal{M}^{\alpha}_{q,p}$ (see Point 2) of Proposition 3.7) has been proved in [8]. Here, we improve this result which will play a key role in the proof of Lemma 5.3.

Lemma 5.2. *Let $1 \leq q \leq \alpha \leq p < \infty$ and f be any element of $\mathcal{M}^{\alpha}_{q,p}$. Then there exists a sequence $(f_n)_{n \geq 1}$ of elements of $L^{\infty}_{c} \cap \mathcal{M}^{\alpha}_{q,p}$ such that $(|f_n|)_{n \geq 1} \uparrow |f|$ almost everywhere and*

$$\lim_{n \to \infty} \|f - f_n\|_{\mathcal{M}^{\alpha}_{q,p}} = 0.$$

Proof. Let us set, for any integer $n \geq 1$,

$$f_n = \text{sgn}(f) \, \min \left(|f|, n\chi_{Q(0,2n)} \right),$$

where, for any $x \in \mathbb{R}^d$,

$$\text{sgn}(f)(x) = \begin{cases} \frac{f(x)}{|f(x)|} & \text{if } \ f(x) \neq 0 \\ 0 & \text{if } \ f(x) = 0. \end{cases}$$

It is easy to see that $(f_n)_{n\geq 1}$ is a sequence of elements of $L^\infty_c \cap \mathcal{M}^\alpha_{q,p}$ satisfying $(|f_n|)_{n\geq 1} \uparrow |f|$ almost everywhere and

$$\begin{cases} |f_n| \leq |f|\,, \quad n \geq 1 \\ \lim\limits_{n\to\infty} f_n = f \text{ almost everywhere.} \end{cases}$$

Therefore, an application of Proposition 4.4 leads to

$$\lim_{n\to\infty} \|f - f_n\|_{\mathcal{M}^\alpha_{q,p}} = 0.$$

This ends the proof. □

As a consequence of Lemma 5.2, the following result holds true.

Lemma 5.3. *Let us assume that $0 < \gamma < \frac{1}{\alpha} \leq 1$ and $\frac{1}{p} = \frac{1}{\alpha} - \gamma$. Then for any dyadic grid $\mathcal{D}$ and any element f of $\mathcal{M}^\alpha_{1,p}$, we have*

$$\|\mathfrak{M}^{\mathcal{D}}_\gamma f\|_p \leq 2\, \|f\|_{\mathcal{M}^\alpha_{1,p}}.$$

Proof. If $\alpha = 1$ then $\mathcal{M}^\alpha_{q,p} = \{0\}$ and so the result is obvious. Thus, we assume that $\alpha > 1$.

Let f be any element of $\mathcal{M}^\alpha_{1,p}$ and $\mathcal{D}$ be a dyadic grid.

1) Assume that f also belongs to L^∞.

a) We have, for all cube Q of $\mathbb{R}^d$,

$$|Q|^{\gamma-1}\int_Q |f(y)|dy \leq |Q|^\gamma \|f\|_\infty$$

and

$$|Q|^{\gamma-1}\int_Q |f(y)|dy = |Q|^{\gamma-\frac{1}{\alpha}}|Q|^{\frac{1}{\alpha}-1}\int_Q |f(y)|dy = |Q|^{\gamma-\frac{1}{\alpha}}\,\|f\|_{\mathcal{M}^\alpha_1}.$$

Consequently,

$$\lim_{\ell(Q)\to\infty} |Q|^{\gamma-1}\int_Q |f(y)|dy = 0 \qquad (*)$$

and, for all $Q \in \mathcal{Q}$,

$$|Q|^{\gamma-1}\int_Q |f(y)|dy \leq \begin{cases} \|f\|_\infty & \text{if } \ell(Q) \geq 1 \\ \|f\|_{\mathcal{M}^\alpha_1} & \text{if } \ell(Q) \leq 1. \end{cases}$$

Thus, for all $x \in \mathbb{R}^d$,

$$\mathfrak{M}_\gamma f(x) \leq M \quad \text{with } M = \max\left(\|f\|_\infty, \|f\|_{\mathcal{M}^\alpha_1}\right).$$

Since $\mathfrak{M}^{\mathcal{D}}_\gamma f \leq \mathfrak{M}_\gamma f$, we have, for any $x \in \mathbb{R}^d$,

$$\mathfrak{M}^{\mathcal{D}}_\gamma f(x) \leq M. \qquad (**)$$

b) Assume that $f \neq 0$. By $(**)$, we have, for all $x \in \mathbb{R}^d$,

$$\mathfrak{M}^{\mathcal{D}}_\gamma f(x) \in (0, M]. \qquad (***)$$

(i) Let us consider an integer $j \geq 0$ and set

$$\begin{cases} \mathbb{E}_j = \left\{x \in \mathbb{R}^d : \mathfrak{M}_\gamma^{\mathcal{D}} f(x) \in \left(2^{-j-1}M, 2^{-j}M\right]\right\} \\ \mathbb{D}_j = \left\{Q \in \mathcal{D} : |Q|^{\gamma-1} \int_Q |f(y)|dy \in \left(2^{-j-1}M, 2^{-j}M\right]\right\}. \end{cases}$$

We have, for all $x \in \mathbb{R}^d$,

$$x \in \mathbb{E}_j \Longleftrightarrow \exists Q_x \in \mathbb{D}_j : x \in Q_x \Longleftrightarrow x \in \bigcup_{Q \in \mathbb{D}_j} Q.$$

Thus $\mathbb{E}_j = \bigcup\limits_{Q \in \mathbb{D}_j} Q$.

Note that, by $(*)$, $\sup \left\{\ell(Q) : Q \in \mathbb{D}_j\right\} < \infty$ and let us denote by Δ_j the set of maximal elements (for the inclusion) of $\mathbb{D}_j$. It is easy to see that

$$\begin{cases} \bigcup\limits_{Q \in \Delta_j} Q = \bigcup\limits_{Q \in \mathbb{D}_j} Q = \mathbb{E}_j \\ \forall\, Q', Q'' \in \Delta_j, Q' \neq Q'' \Longrightarrow |Q' \cap Q''| = 0. \end{cases}$$

Moreover, by the definition of $\mathbb{E}_j$, we have, for all $x \in Q \in \Delta_j$,

$$|Q|^{\gamma-1} \int_Q |f(y)|dy \leq \mathfrak{M}_\gamma^{\mathcal{D}} f(x) \leq 2^{-j}M < 2\, |Q|^{\gamma-1} \int_Q |f(y)|dy$$

and therefore, for all $Q \in \Delta_j$,

$$\int_Q \left[\mathfrak{M}_\gamma^{\mathcal{D}} f(x)\right]^p dx \leq 2^p\, |Q| \left[|Q|^{\gamma-1} \int_Q |f(y)|dy\right]^p = 2^p \left[|Q|^{\frac{1}{\alpha}-1} \int_Q |f(y)|dy\right]^p.$$

So we obtain

$$\begin{aligned} \int_{\mathbb{E}_j} \left[\mathfrak{M}_\gamma^{\mathcal{D}} f(x)\right]^p dx &= \sum_{Q \in \Delta_j} \int_Q \left[\mathfrak{M}_\gamma^{\mathcal{D}} f(x)\right]^p dx \\ &\leq 2^p \sum_{Q \in \Delta_j} \left[|Q|^{\frac{1}{\alpha}-1} \int_Q |f(y)|dy\right]^p. \quad (****) \end{aligned}$$

(ii) By $(***)$, we have $\bigcup\limits_{j \geq 0} \mathbb{E}_j = \mathbb{R}^d$. Meanwhile, the definition of $\mathbb{E}_j (j \geq 0)$ shows that

$$\forall\, j', j'' \in \mathbb{N}, \text{ with } j' \neq j'', \mathbb{E}_{j'} \cap \mathbb{E}_{j''} = \emptyset.$$

Hence we have

$$\int_{\mathbb{R}^d} \left[\mathfrak{M}_\gamma^{\mathcal{D}} f(x)\right]^p dx = \sum_{j \geq 0} \int_{\mathbb{E}_j} \left[\mathfrak{M}_\gamma^{\mathcal{D}} f(x)\right]^p dx$$

and therefore, by $(****)$,

$$\|\mathfrak{M}_\gamma^{\mathcal{D}} f\|_p \leq 2 \left(\sum_{j \geq 0} \sum_{Q \in \Delta_j} \left[|Q|^{\frac{1}{\alpha}-1} \int_Q |f(y)|dy\right]^p\right)^{\frac{1}{p}}. \qquad (*****)$$

Note that, if j' and j'' are two integer such that $j'' > j' \geq 0$ then

$$\forall\,(Q', Q'') \in \Delta_{j'} \times \Delta_{j''}\,,\quad Q' \neq Q''$$

and so

$$\left\{Q_{k,m} : Q_{k,m} \in \Delta_{j'}\right\} \cap \left\{Q_{k,m} : Q_{k,m} \in \Delta_{j''}\right\} = \emptyset, \quad k \in \mathbb{Z}^d,\ m \in \mathbb{Z}.$$

Therefore

$$\sum_{j\geq 0}\sum_{Q\in\Delta_j}\left[|Q|^{\frac{1}{\alpha}-1}\int_Q |f(y)|dy\right]^p = \sum_{m\in\mathbb{Z}}\ \sum_{k\in\mathbb{Z}^d : Q_{k,m}\in \bigcup\limits_{j\geq 0}\Delta_j}\left[|Q_{k,m}|^{\frac{1}{\alpha}-1}\int_{Q_{k,m}}|f(y)|dy\right]^p$$
$$\leq \|f\|^p_{\mathcal{M}^\alpha_{1,p}}.$$

This inequality combined with $(*****)$ gives

$$\|\mathfrak{M}^{\mathcal{D}}_\gamma f\|_p \leq 2\,\|f\|_{\mathcal{M}^\alpha_{1,p}}.$$

2) By Lemma 5.2, there exists a sequence $(f_n)_{n\geq 1}$ of elements of $L^\infty_c \cap \mathcal{M}^\alpha_{q,p}$ such that $(|f_n|)_{n\geq 1} \uparrow |f|$ almost everywhere and $\lim\limits_{n\to\infty}\|f - f_n\|_{\mathcal{M}^\alpha_{q,p}} = 0$. Thus, the result obtained in Point 1) implies that

$$\begin{cases} \left\|\mathfrak{M}^{\mathcal{D}}_\gamma f_n\right\|_p \leq 2\,\|f_n\|_{\mathcal{M}^\alpha_{1,p}} \leq 2\,\|f\|_{\mathcal{M}^\alpha_{1,p}}\,, \quad n \geq 1 \\ \\ 0 \leq \left(\mathfrak{M}^{\mathcal{D}}_\gamma f_n\right)_{n\geq 1} \uparrow \mathfrak{M}^{\mathcal{D}}_\gamma f \text{ and so } \left(\left\|\mathfrak{M}^{\mathcal{D}}_\gamma f_n\right\|_p\right)_{n\geq 1} \uparrow \left\|\mathfrak{M}^{\mathcal{D}}_\gamma f\right\|_p \end{cases}$$

and therefore

$$\|\mathfrak{M}^{\mathcal{D}}_\gamma f\|_p \leq 2\,\|f\|_{\mathcal{M}^\alpha_{1,p}}.$$

The proof is complete. □

Now we prove Theorem 2.1 thanks to Lemma 5.1, Lemma 5.3 and Corollary 3.6.

Proof of Theorem 2.1

Let f be in $\mathcal{M}^\alpha_{1,p}$. By Lemma 5.1 and Lemma 5.3, we have

$$\|\mathfrak{M}_\gamma f\|_p \leq 3^{d(1-\gamma)}\left\|\max_{t\in\{-1/3,0,1/3\}^d}\mathfrak{M}^{\mathcal{D}^t}_\gamma f\right\|_p \leq 3^{d(1-\gamma)}\sum_{t\in\{-1/3,0,1/3\}^d}\left\|\mathfrak{M}^{\mathcal{D}^t}_\gamma f\right\|_p$$
$$\leq 3^{d(1-\gamma)}\sum_{t\in\{-1/3,0,1/3\}^d} 2\,\|f\|_{\mathcal{M}^\alpha_{1,p}(\mathcal{D}^t)}\,.$$

Note that the hypotheses imply that $p < \infty$ and so Corollary 3.6 leads to

$$\|\mathfrak{M}_\gamma f\|_p \leq 3^{d(1-\gamma)}2^{d\left(2-\frac{1}{p}\right)}\,\|f\|_{\mathcal{M}^\alpha_{1,p}}\ \sharp\left(\{-1/3, 0, 1/3\}^d\right)$$
$$= 2^{d\left(2-\frac{1}{p}\right)}\,3^{d(2-\gamma)}\,\|f\|_{\mathcal{M}^\alpha_{1,p}}\,.$$

The proof is complete. □

As an immediate consequence of Theorem 2.1 we can now prove Theorem 2.2.

Proof of Theorem 2.2

Let f be in $\mathcal{M}^{\alpha}_{1,p}$. Note that the hypotheses imply that $p<\infty$. By [12, Theorem 1], we have

$$\|I_\gamma f\|_p \leq D \ \|\mathfrak{M}_\gamma f\|_p ,$$

where D is a real constant not depending on f. Therefore, Theorem 2.1 provides the desired inequality. □

6. Application

Theorem 2.2, Theorem 2.5 and the boundedness properties of Riesz tranforms in Lebesgue spaces lead to the following result, which contains Theorem 2.6.

Proposition 6.1. *Let us assume that $d\geq 3$, $1\leq q\leq\alpha<d$, $\frac{1}{p}=\frac{1}{\alpha}-\frac{1}{d}$ and f is an element of the Bourgain-Morrey space $\mathcal{M}^{\alpha}_{q,p}$. Then*
1) for $1\leq j\leq d$, the function $F_j=R_j\left(I_{\frac{1}{d}}f\right)$ belongs to L^p
2) there exists a real constant c_d depending only on d and such that $F=\left(c_d\ F_j\right)_{1\leq j\leq d}$ is a solution in $(L^p)^d$ of the equation (11).

Proof. Note that, the hypotheses imply that $1\leq q\leq\alpha<p<\infty$.
1) Since $\mathcal{M}^{\alpha}_{q,p}\subset\mathcal{M}^{\alpha}_{1,p}$ (see (1)), Theorem 2.2 implies that $I_{\frac{1}{d}}f\in L^p$. Furthermore, it is well known that the Riesz tranform R_j is bounded on L^p, for $1\leq j\leq d$. Therefore, we deduce that $F_j=R_j\left(I_{\frac{1}{d}}f\right)$ belongs to L^p.
2) a) Let φ be any element of $\mathcal{C}^{\infty}_{c}$. For $1\leq j\leq d$, the boundedness properties of R_j and $I_{\frac{1}{d}}$ show that $\psi_j=R_j\left(I_{\frac{1}{d}}\varphi\right)$ belongs to $\bigcap\limits_{r>\frac{d}{d-1}}L^r$. Since $\frac{d}{d-1}<2$, there exists a real number r such that $\frac{d}{d-1}<r<2$ and $\psi_j\in L^r$. Therefore, we can use the Fourier transform to obtain $c_d\sum\limits_{j=1}^{d}\partial_j\psi_j=\varphi$, where c_d is a real constant depending only on d (see [14, Formula (17), p.125]).
b) Fix an integer $n\geq 1$ and set $f_n=(f\omega_n)*\phi_n$. Since $f_n\in\mathcal{C}^{\infty}_{c}$, the result of Point a) implies that $\operatorname{div} F_n=f_n$, where $F_n=(F_{n_j})_{1\leq j\leq d}$ with

$$F_{n_j}=c_dR_j\left(I_{\frac{1}{d}}f_n\right)\in\bigcap_{r>\frac{d}{d-1}}L^r , \qquad 1\leq j\leq d.$$

- According to Theorem 2.5, $(f_n)_{n\geq 1}$ converges to f in $\mathcal{M}^{\alpha}_{1,p}$.
- For $1\leq j\leq d$, the boundedness properties of R_j and $I_{\frac{1}{d}}$ imply that $\left(F_{n_j}\right)_{n\geq 1}$ converges to $c_dF_j=c_dR_j\left(I_{\frac{1}{d}}f\right)$ in L^p.

Therefore, for any element φ of $\mathcal{C}_c^\infty$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \operatorname{div} F(x)\,\varphi(x)dx &= -\sum_{j=1}^{d}\int_{\mathbb{R}^d} c_d F_j(x)\,\partial_j\varphi(x)dx \\ &= \lim_{n\to\infty}\left[-\sum_{j=1}^{d}\int_{\mathbb{R}^d} F_{n_j}(x)\,\partial_j\varphi(x)dx\right] \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d}\left(\sum_{j=1}^{d}\partial_j F_{n_j}(x)\right)\varphi(x)dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d} \operatorname{div} F_n(x)\,\varphi(x)dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^d} f_n(x)\,\varphi(x)dx = \int_{\mathbb{R}^d} f(x)\,\varphi(x)dx.\end{aligned}$$

Hence, div $F = f$. Thus, we obtain the desired result. □

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