

Lie Group Analysis of a Nonlinear Coupled System of Korteweg-de Vries Equations

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ABSTRACT. In this paper, we consider coupled Korteweg-de Vries equations that model the propagation of shallow water waves, ion-acoustic waves in plasmas, solitons, and nonlinear perturbations along internal surfaces between layers of different densities in stratified fluids, for example propagation of solitons of long internal waves in oceans. The method of Lie group analysis is used to on the system to obtain symmetry reductions. Soliton solutions are constructed by use of a linear combination of time and space translation symmetries. Furthermore, we compute conservation laws in two ways that is by multiplier method and by an application of new conservation theorem developed by Nail Ibragimov.

1. INTRODUCTION

The dynamics of shallow-water waves, ion-acoustic waves in plasmas, and long internal waves in oceans can be described by coupled KdV equations. The equations are derived from the classical kdV equation. This section extends the previous study of kdV equations to that of a coupled nonlinear system. From the Korteweg-de Vries equation

$$q_t + \alpha q q_x + \beta q_{xxx} = 0, \quad (1)$$

for α and β as constants, we let

$$q(t, x) = u(t, x) + iv(t, x), \quad (2)$$

where $i^2 = -1$. Then substituting (2) into (1) and separating the real and imaginary parts, we obtain

$$\Delta_1 \equiv u_t + \alpha u u_x - \alpha v v_x + \beta u_{xxx} = 0, \quad \Delta_2 \equiv v_t + \alpha u v_x + \alpha v u_x + \beta v_{xxx} = 0, \quad (3)$$

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which is a nonlinear system of coupled KdV equations. We perform Lie symmetry analysis on (3), that is, we obtain Lie point symmetries, invariant solutions and conservation laws of (3). This paper uses symmetry analysis method to construct exact solutions and conservation laws for a nonlinear coupled kdv system (3).

2. PRELIMINARIES

In this section, we outline preliminary concepts which are useful in the sequel. In Euclidean spaces \mathbb{R}^n of $x = x^i$ independent variables and \mathbb{R}^m of $u = u^\alpha$ dependent variables, we consider the transformations

$$T_\epsilon : \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad (4)$$

involving the continuous parameter ϵ which ranges from a neighbourhood $\mathcal{N}' \subset \mathcal{N} \subset \mathbb{R}$ of $\epsilon = 0$ where the functions φ^i and ψ^α differentiable and analytic in the parameter ϵ .

Definition 2.1. The set \mathcal{G} of transformations given by (4) is a local Lie group if it holds true that

- (1) (i). (Closure) Given $T_{\epsilon_1}, T_{\epsilon_2} \in \mathcal{G}$, for $\epsilon_1, \epsilon_2 \in \mathcal{N}' \subset \mathcal{N}$, then $T_{\epsilon_1} T_{\epsilon_2} = T_{\epsilon_3} \in \mathcal{G}$, $\epsilon_3 = \phi(\epsilon_1, \epsilon_2) \in \mathcal{N}$.
- (2) (ii). (Identity) There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$.
- (3) (iii). (Inverse) There exists a unique $T_{\epsilon^{-1}} \in \mathcal{G}$ for every transformation $T_\epsilon \in \mathcal{G}$, where $\epsilon \in \mathcal{N}' \subset \mathcal{N}$ and $\epsilon^{-1} \in \mathcal{N}$ such that $T_\epsilon T_{\epsilon^{-1}} = T_{\epsilon^{-1}} T_\epsilon = T_0$.

Remark 2.2. Associativity of the group \mathcal{G} in (4) follows from (1).

In the system,

$$\Delta_\alpha(x^i, u^\alpha, u_{(1)}, \dots, u_{(\pi)}) = \Delta_\alpha = 0, \quad (5)$$

the variables u^α are dependent. The partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \dots, u_{(\pi)} = \{u_{i_1 \dots i_\pi}^\alpha\}$, are of the first, second, \dots , up to the π th-orders.

Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad (6)$$

the total differentiation operator with respect to the variables x^i and δ_i^j , the Kronecker delta, we have

$$D_i(x^j) = \delta_i^j, \quad u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(D_i(u^\alpha)), \quad \dots, \quad (7)$$

where u_i^α defined in (7) are differential variables [7].

Consider the local Lie group \mathcal{G} given by the transformations

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i \Big|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha \Big|_{\epsilon=0} = u^\alpha, \quad (8)$$

where the symbol $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2.3. The construction of the group \mathcal{G} given by (8) is an equivalence of the computation of infinitesimal transformations

$$\bar{x}^i \approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i \Big|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha \approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha \Big|_{\epsilon=0} = u^\alpha, \quad (9)$$

obtained from (4) by a Taylor series expansion of $\varphi^i(x^i, u^\alpha, \epsilon)$ and $\psi^\alpha(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ , where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon} \Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon} \Big|_{\epsilon=0}. \quad (10)$$

Remark 2.4. The symbol of infinitesimal transformations, X , is used to write (9) as

$$\bar{x}^i \approx (1 + X)x^i, \quad \bar{u}^\alpha \approx (1 + X)u^\alpha, \quad (11)$$

where

$$X = \xi^i(x^i, u^\alpha) \frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha) \frac{\partial}{\partial u^\alpha}, \quad (12)$$

is the generator of the group \mathcal{G} given by (8).

Remark 2.5. To obtain transformed derivatives from (4), we use a change of variable formulae

$$D_i = D_i(\varphi^j) \bar{D}_j, \quad (13)$$

where \bar{D}_j is the total differentiation in the variables \bar{x}^i . This means that

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \quad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha). \quad (14)$$

If we apply the change of variable formula given in (13) on \mathcal{G} given by (8), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j), \quad \bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j). \quad (15)$$

Expansion of (15) yields

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta} \right) \bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta}. \quad (16)$$

The variables \bar{u}_i^α can be written as functions of $x^i, u^\alpha, u_{(1)}$, that is

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha \Big|_{\epsilon=0} = u_i^\alpha. \quad (17)$$

Definition 2.6. The transformations in the space of the variables $x^i, u^\alpha, u_{(1)}$ given in (8) and (17) form the first prolongation group $\mathcal{G}^{[1]}$.

Definition 2.7. Infinitesimal transformation of the first derivatives is

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon). \quad (18)$$

Remark 2.8. In terms of infinitesimal transformations, the first prolongation group $\mathcal{G}^{[1]}$ is given by (9) and (18).

Definition 2.9. By using the relation given in (15) on the first prolongation group $\mathcal{G}^{[1]}$ given by Definition 2.6, we obtain [5]

$$D_i(x^j + \xi^j \epsilon)(u_j^\alpha + \zeta_j^\alpha \epsilon) = D_i(u^\alpha + \eta^\alpha \epsilon), \quad \text{which gives} \quad u_i^\alpha + \zeta_j^\alpha \epsilon + u_j^\alpha \epsilon D_i \xi^j = u_i^\alpha + D_i \eta^\alpha \epsilon, \quad (19)$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (20)$$

is the first prolongation formula.

Remark 2.10. Similarly, we get higher order prolongations [8],

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{i\kappa}^\alpha D_j(\xi^\kappa), \quad \dots, \quad \zeta_{i_1, \dots, i_\kappa}^\alpha = D_{i_\kappa}(\zeta_{i_1, \dots, i_{\kappa-1}}^\alpha) - u_{i_1, i_2, \dots, i_{\kappa-1}}^\alpha D_{i_\kappa}(\xi^j). \quad (21)$$

Remark 2.11. The prolonged generators of the prolongations $\mathcal{G}^{[1]}, \dots, \mathcal{G}^{[\kappa]}$ of the group \mathcal{G} are

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad \dots, \quad X^{[\kappa]} = X^{[\kappa-1]} + \zeta_{i_1, \dots, i_\kappa}^\alpha \frac{\partial}{\partial \zeta_{i_1, \dots, i_\kappa}^\alpha}, \quad \kappa \geq 1, \quad (22)$$

where X is the group generator given by (12).

Definition 2.12. A function $\Gamma(x^i, u^\alpha)$ is called an invariant of the group \mathcal{G} of transformations given by (4) if

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha). \quad (23)$$

Theorem 2.13. A function $\Gamma(x^i, u^\alpha)$ is an invariant of the group \mathcal{G} given by (4) if and only if it solves the following first-order linear PDE: [5]

$$X\Gamma = \xi^i(x^i, u^\alpha) \frac{\partial \Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha) \frac{\partial \Gamma}{\partial u^\alpha} = 0. \quad (24)$$

From Theorem (2.13), we have the following result.

Theorem 2.14. The local Lie group \mathcal{G} of transformations in \mathbb{R}^n given by (4) [7] has precisely $n - 1$ functionally independent invariants. One can take, as the basic invariants, the left-hand sides of the first integrals

$$\psi_1(x^i, u^\alpha) = c_1, \dots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \quad (25)$$

of the characteristic equations for (24):

$$\frac{dx^i}{\xi^i(x^i, u^\alpha)} = \frac{du^\alpha}{\eta^\alpha(x^i, u^\alpha)}. \quad (26)$$

Definition 2.15. The vector field X (12) is a Lie point symmetry of the PDE system (5) if the determining equations

$$X^{[\pi]} \Delta_\alpha \Big|_{\Delta_\alpha=0} = 0, \quad \alpha = 1, \dots, m, \quad \pi \geq 1, \quad (27)$$

are satisfied, where $\Big|_{\Delta_\alpha=0}$ means evaluated on $\Delta_\alpha = 0$ and $X^{[\pi]}$ is the π -th prolongation of X .

Definition 2.16. The Lie group \mathcal{G} is a symmetry group of the PDE system given in (5) if the PDE system (5) is form-invariant, that is

$$\Delta_\alpha (\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \dots, \bar{u}_{(\pi)}) = 0. \quad (28)$$

Theorem 2.17. Given the infinitesimal transformations in (8), the Lie group \mathcal{G} in (4) is found by integrating the Lie equations

$$\frac{d\bar{x}^i}{d\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i \Big|_{\epsilon=0} = x^i, \quad \frac{d\bar{u}^\alpha}{d\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha \Big|_{\epsilon=0} = u^\alpha. \quad (29)$$

Definition 2.18. A vector space \mathcal{V}_r of operators [5] X (12) is a Lie algebra if for any two operators, $X_i, X_j \in \mathcal{V}_r$, their commutator

$$[X_i, X_j] = X_i X_j - X_j X_i, \quad (30)$$

is in \mathcal{V}_r for all $i, j = 1, \dots, r$.

Remark 2.19. The commutator satisfies the properties of bilinearity, skew symmetry and the Jacobi identity [5].

Theorem 2.20. The set of solutions of the determining equation given by (27) forms a Lie algebra [5].

The methods of (G'/G)-expansion method [20], Extended Jacobi elliptic function expansion [21] and Kudryashov [22] are usually applied after symmetry reductions. Let a system of π th-order PDEs be given by (5).

Definition 2.21. The Euler-Lagrange operator $\delta/\delta u^\alpha$ is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} (-1)^\kappa D_{i_1} \dots D_{i_\kappa} \frac{\partial}{\partial u_{i_1 i_2 \dots i_\kappa}^\alpha}, \quad (31)$$

and the Lie- Bäcklund operator in abbreviated form [5] is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \dots \quad (32)$$

Remark 2.22. The Lie- Bäcklund operator (32) in its prolonged form is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} \zeta_{i_1 \dots i_\kappa} \frac{\partial}{\partial u_{i_1 i_2 \dots i_\kappa}^\alpha}, \quad (33)$$

where

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \quad \dots, \zeta_{i_1 \dots i_k}^\alpha = D_{i_1 \dots i_k}(W^\alpha) + \xi^j u_{j i_1 \dots i_k}^\alpha, \quad j = 1, \dots, n. \tag{34}$$

and the Lie characteristic function is

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{35}$$

Remark 2.23. The characteristic form of Lie- Bäcklund operator (33) is

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1 \dots i_k}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\alpha}. \tag{36}$$

Remark 2.24. Noether’s Theorem is applicable to systems from variational problems

Definition 2.25. A function $\Lambda^\alpha(x^i, u^\alpha, u_{(1)}, \dots) = \Lambda^\alpha$, is a multiplier of the PDE system given by (5) if it satisfies the condition that [16]

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{37}$$

where $D_i T^i$ is a divergence expression.

Definition 2.26. To find the multipliers Λ^α , one solves the determining equations (38) [3],

$$\frac{\delta}{\delta u^\alpha} (\Lambda^\alpha \Delta_\alpha) = 0. \tag{38}$$

The technique [9] enables one to construct conserved vectors associated with each Lie point symmetry of the PDE system given by (5).

Definition 2.27. The adjoint equations of the system given by (5) are

$$\Delta_\alpha^* (x^i, u^\alpha, v^\alpha, \dots, u_{(\pi)}, v_{(\pi)}) \equiv \frac{\delta}{\delta u^\alpha} (v^\beta \Delta_\beta) = 0, \tag{39}$$

where v^α is the new dependent variable.

Definition 2.28. Formal Lagrangian \mathcal{L} of the system (5) and its adjoint equations (39) is [9]

$$\mathcal{L} = v^\alpha \Delta_\alpha(x^i, u^\alpha, u_{(1)}, \dots, u_{(\pi)}). \tag{40}$$

Theorem 2.29. Every infinitesimal symmetry X of the system given by (5) leads to conservation laws [9]

$$D_i T^i \Big|_{\Delta_\alpha=0} = 0, \tag{41}$$

where the conserved vector

$$T^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \dots \right] + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] + D_j D_k(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots \right]. \tag{42}$$

3. MAIN RESULTS

We now present our results in this section. An illustrative example with a simple kdV equation can be found in [6]. The infinitesimal transformations of the Lie group with parameter ϵ are

$$\bar{t} = t + \xi^t(t, x, u, v)\epsilon, \quad \bar{x} = x + \xi^x(t, x, u, v)\epsilon, \quad \bar{u} = u + \eta^u(t, x, u, v)\epsilon, \quad \bar{v} = v + \eta^v(t, x, u, v)\epsilon. \tag{43}$$

The vector field

$$X = \xi^t(t, x, u, v)\frac{\partial}{\partial t} + \xi^x(t, x, u, v)\frac{\partial}{\partial x} + \eta^u(t, x, u, v)\frac{\partial}{\partial u} + \eta^v(t, x, u, v)\frac{\partial}{\partial v}, \tag{44}$$

is a Lie point symmetry of (3) if

$$X^{[3]}\Delta_1 \Big|_{\Delta_1=0, \Delta_2=0} = 0, \quad X^{[3]}\Delta_2 \Big|_{\Delta_1=0, \Delta_2=0} = 0. \tag{45}$$

Expanding (45) and splitting on derivatives of v and u , we have an overdetermined system of ten PDEs, namely,

$$\begin{aligned} \xi_u^t = 0, \quad \xi_v^t = 0, \quad \xi_x^t = 0, \quad \xi_u^x = 0, \quad \xi_v^x = 0, \quad \xi_{tt}^t = 0, \quad \xi_{tt}^x = 0, \quad 3\xi_x^x - \xi_t^t = 0, \\ 3\eta^v + 2\xi_t^t v = 0, \quad 3\alpha\eta^u + 2\alpha\xi_t^t u - 3\xi_t^x = 0. \end{aligned} \tag{46}$$

Solving the system (46) yields

$$\xi^t = A_1 + 3A_2t, \quad \xi^x = A_2x + \alpha A_3t + A_4, \quad \eta^u = -2A_2u + A_3, \quad \eta^v = -2A_2v, \tag{47}$$

for arbitrary constants A_1, A_2, A_3, A_4 . Hence from (47), the infinitesimal symmetries of the coupled KdV Equations (3) is a Lie algebra generated by the vector fields

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_4 = 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v}. \tag{48}$$

The set of all infinitesimal symmetries of coupled KdV equations forms a Lie algebra and yield the following commutation relations in Table 1.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	αX_2	$3X_1$
X_2	0	0	0	X_2
X_3	$-\alpha X_2$	0	0	$-2X_3$
X_4	$-3X_1$	$-X_2$	$2X_3$	0

Table 1: A commutator table for the Lie algebra generated by the symmetries of coupled KdV equation.

The following Lie groups, for $i = 1, 2, 3, 4$, are obtained

$$T_{\epsilon_1} : \bar{t} = t + \epsilon_1, \bar{x} = x, \bar{u} = u, \bar{v} = v, \quad (49)$$

$$T_{\epsilon_2} : \bar{t} = t, \bar{x} = x + \epsilon_2, \bar{u} = u, \bar{v} = v, \quad (50)$$

$$T_{\epsilon_3} : \bar{t} = t, \bar{x} = x + \alpha\epsilon_3 t, \bar{u} = u + \epsilon_3, \bar{v} = v, \quad (51)$$

$$T_{\epsilon_4} : \bar{t} = te^{3\epsilon_4}, \bar{x} = xe^{\epsilon_4}, \bar{u} = ue^{-2\epsilon_4}, \bar{v} = ve^{-2\epsilon_4}. \quad (52)$$

The symmetries obtained yield the following symmetry reductions.

$$X_1 = \frac{\partial}{\partial t}. \quad (53)$$

Solving the characteristic equations

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0} = \frac{dv}{0}, \quad (54)$$

associated to the operator X_1 gives the invariants

$$J_1 = x, \quad J_2 = u, \quad J_3 = v. \quad (55)$$

Hence, we have

$$u = \varphi(x), \quad v = \psi(x), \quad (56)$$

for arbitrary functions φ and ψ . Substituting the expressions for u and v given by (56) into the system (3), we get a system of third order ordinary DEs namely,

$$\alpha [\varphi(x)\varphi'(x) - \psi(x)\psi'(x)] + \beta\varphi'''(x) = 0, \quad \alpha (\varphi(x)\psi(x))' + \beta\psi'''(x) = 0. \quad (57)$$

Integration of the system (57) yields;

$$\frac{\alpha}{2} [\varphi(x)^2 - \psi(x)^2] + \beta\varphi''(x) = C_1, \quad (58)$$

$$\alpha [\varphi(x)\psi(x)] + \beta\psi''(x) = C_2, \quad (59)$$

for arbitrary constants C_1 and C_2 . If we take

$$C_1 = C_2 = 0, \quad (60)$$

the system (58)-(59) becomes

$$\frac{\alpha}{2} [\varphi(x)^2 - \psi(x)^2] + \beta\varphi''(x) = 0, \quad (61)$$

$$\alpha [\varphi(x)\psi(x)] + \beta\psi''(x) = 0. \quad (62)$$

To find more solutions of the system (61)-(62), we determine its Lie point symmetries. Using the Lie's algorithm for computing point symmetries, we see that the Lie point symmetries of (61)-(62) are

$$X_1^* = \frac{\partial}{\partial x}, \quad X_2^* = x \frac{\partial}{\partial x} - 2\varphi \frac{\partial}{\partial \varphi} - 2\psi \frac{\partial}{\partial \psi}. \quad (63)$$

Proceeding as above, we see that the symmetry X_1^* yields the trivial solution

$$u = 0, \quad v = 0. \quad (64)$$

The second symmetry X_2^* has the characteristic equations

$$\frac{dx}{x} = \frac{d\varphi}{-2\varphi} = \frac{d\psi}{-2\psi}, \quad (65)$$

which provides the invariants

$$J_1 = x^2\varphi, \quad J_2 = x^2\psi. \quad (66)$$

Letting

$$\varphi = \frac{\lambda}{x^2}, \quad \psi = \frac{\mu}{x^2}, \quad (67)$$

substituting the values of φ and ψ into (61)-(62) and solving the resulting equations yield:

Case one. Taking

$$\mu = 0 \quad (68)$$

gives

$$\lambda = 0 \quad (69)$$

or

$$\lambda = -\frac{12\beta}{\alpha}. \quad (70)$$

When

$$\lambda = 0, \quad \text{and} \quad \mu = 0, \quad (71)$$

we also get the trivial solution (64). One can easily see that if

$$\lambda = -\frac{12\beta}{\alpha}, \quad \text{and} \quad \mu = 0, \quad (72)$$

then

$$\varphi = -\frac{12\beta}{\alpha x^2}, \quad \psi = 0, \quad (73)$$

which is a solution of the system (61)-(62). Hence

$$u_1(t, x) = -\frac{12\beta}{\alpha x^2}, \quad v_1(t, x) = 0, \quad (74)$$

is a solution of the coupled KdV system (3).

Case two. Taking

$$\lambda = -\frac{6\beta}{\alpha} \quad (75)$$

gives

$$\mu = \pm \frac{6\beta i}{\alpha}, \quad (76)$$

with $i^2 = -1$. Consequently,

$$u_2(t, x) = -\frac{6\beta}{\alpha x^2}, \quad v_2(t, x) = \frac{6i\beta}{\alpha x^2}, \quad (77)$$

and

$$u_3(t, x) = -\frac{6\beta}{\alpha x^2}, \quad v_3(t, x) = -\frac{6i\beta}{\alpha x^2}, \quad (78)$$

are solutions of the coupled KdV system. Hence Lie group analysis has given us three steady-state solutions for the coupled KdV system under the time translation symmetry $X_1 = \frac{\partial}{\partial t}$.

$$X_2 = \frac{\partial}{\partial x}. \quad (79)$$

Solving the characteristic equations

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0} = \frac{dv}{0}, \quad (80)$$

associated to X_2 gives the invariants

$$J_1 = t, \quad J_2 = u, \quad J_3 = v. \quad (81)$$

Therefore, the group-invariant solution is

$$u = \phi(t), \quad v = h(t), \quad (82)$$

for arbitrary functions h and ϕ . Substitution of the solutions from (82) into (3), we get a system of first order ordinary DEs, namely,

$$\phi'(t) = 0, \quad h'(t) = 0, \quad (83)$$

which is integrated once with respect to t to yield,

$$\phi(t) = C_1, \quad h(t) = C_2, \quad (84)$$

for arbitrary constants C_1 and C_2 . Consequently, the space translation group-invariant solution of the system (3) is

$$u(t, x) = C_1, \quad v(t, x) = C_2. \quad (85)$$

$$X_3 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \quad (86)$$

Solving the characteristic equations

$$\frac{dt}{0} = \frac{dx}{\alpha t} = \frac{du}{1} = \frac{dv}{0}, \quad (87)$$

associated to Galilean boost gives the invariants

$$J_1 = t, \quad J_2 = v, \quad J_3 = -u + \frac{x}{\alpha t}, \quad t \neq 0. \quad (88)$$

Thus the invariant solution of (3) is

$$u = \frac{x}{\alpha t} - g(t), \quad v = f(t), \quad t \neq 0, \quad (89)$$

for arbitrary functions f and g . Substitution of the values of u and v from (89) into the System (3), we get a nonlinear system of coupled first order ordinary DEs, namely,

$$tg'(t) + g(t) = 0, \quad tf'(t) + f(t) = 0, \quad (90)$$

whose solutions are

$$g(t) = \frac{C_1}{t}, \quad f(t) = \frac{C_2}{t}, \quad (91)$$

for arbitrary constants C_1 and C_2 . Hence the Galilean boost group-invariant solution of the system (3) is

$$u(t, x) = \frac{x + A}{\alpha t}, \quad v(t, x) = \frac{C_2}{t} \quad (92)$$

where $A = -\alpha C_1$ and $t \neq 0$.

The scaling

$$X_4 = 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v} \quad (93)$$

. By solving of the characteristic equations

$$\frac{dt}{3t} = \frac{dx}{x} = -\frac{du}{2u} = -\frac{dv}{2v}, \quad (94)$$

associated to this symmetry, we obtain the invariants

$$J_1 = \frac{x^3}{t}, \quad J_2 = ux^2, \quad J_3 = vx^2. \quad (95)$$

Generally, the group-invariant solution pair is

$$u(t, x) = \frac{f(\lambda)}{x^2}, \quad v(t, x) = \frac{g(\lambda)}{x^2}, \quad \text{where } \lambda = \frac{x^3}{t}, \quad (96)$$

and the functions f and g satisfy the system of third order nonlinear coupled ordinary DEs

$$2\alpha(g^2 - f^2) - \lambda^2 f' + 3\alpha\lambda(ff' - gg') + \beta(-24f + 24\lambda f' + 27\lambda^3 f''') = 0, \quad (97)$$

$$-4\alpha fg - \lambda^2 g' + 3\alpha\lambda(fg)' + \beta(-24g + 24\lambda g' + 27\lambda^3 g''') = 0. \quad (98)$$

$$X = X_1 + cX_2. \quad (99)$$

We consider a symmetry X , which is a linear combination of the time and space translations symmetries, that is,

$$X = \frac{\partial}{\partial t} + c \frac{\partial}{\partial x}, \tag{100}$$

for a constant c . The invariants associated to this symmetry X are

$$J_1 = x - ct, \quad J_2 = u, \quad J_3 = v. \tag{101}$$

Hence, the invariant solution for the symmetry X is

$$u = f(x - ct), \quad v = g(x - ct), \tag{102}$$

for arbitrary functions f and g . Substitution of u and v from (102) into the system (3) yields a system of nonlinear third order ordinary DEs, namely

$$-cf'(\xi) + \alpha \{f(\xi)f'(\xi) - g(\xi)g'(\xi)\} + \beta f'''(\xi) = 0, \quad -cg'(\xi) + \alpha(f(\xi)g(\xi))' + \beta g'''(\xi) = 0, \tag{103}$$

which on integrating once with respect to ξ yields

$$-cf + \frac{1}{2}\alpha(f^2 - g^2) + \beta f'' + C_1 = 0, \quad -cg + \alpha fg + \beta g'' + C_2 = 0, \tag{104}$$

for arbitrary constants C_1 and C_2 .

Remark 3.1. If we take the constants $C_1 = C_2 = 0$, then when the wave velocity $c = 0$, we can recover the stationary solutions given in (3).

Remark 3.2. Traveling wave solutions of the system (3) must satisfy the system (104).

Computation of conservation laws for the coupled KdV Equations (3) is done using two methods; the method of multipliers and a theorem due to Ibragimov. We seek local conservation law multipliers for the system (3), whose determining equations are

$$\frac{\delta}{\delta u} [\Lambda^1 \Delta_1 + \Lambda^2 \Delta_2] = 0, \quad \frac{\delta}{\delta v} [\Lambda^1 \Delta_1 + \Lambda^2 \Delta_2] = 0, \tag{105}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \dots, \tag{106}$$

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_x^2 \frac{\partial}{\partial v_{xx}} - D_x^3 \frac{\partial}{\partial v_{xxx}} + \dots, \tag{107}$$

are the Euler-Lagrange operators and

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_x} + u_{tt} \frac{\partial}{\partial u_t} + v_{tt} \frac{\partial}{\partial v_t} + \dots, \tag{108}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + u_{tx} \frac{\partial}{\partial u_t} + v_{tx} \frac{\partial}{\partial v_t} + \dots, \tag{109}$$

are total derivatives operators. We look for second order multipliers, that is,

$$\Lambda^n = \Lambda^n(t, x, u, u_x, u_{xx}, v, v_x, v_{xx}), \quad n = 1, 2. \tag{110}$$

The determining Equations (105) become

$$\frac{\delta}{\delta u} [\Lambda^1 \{u_t + \alpha uu_x - \alpha v v_x + \beta u_{xxx}\} + \Lambda^2 \{v_t + \alpha u v_x + \alpha v u_x + \beta v_{xxx}\}] = 0, \tag{111}$$

$$\frac{\delta}{\delta v} [\Lambda^1 \{u_t + \alpha uu_x - \alpha v v_x + \beta u_{xxx}\} + \Lambda^2 \{v_t + \alpha u v_x + \alpha v u_x + \beta v_{xxx}\}] = 0. \tag{112}$$

Expanding (111)-(112) and splitting on derivatives of u and v yields an overdetermined system of 22 PDEs, namely

$$\begin{aligned} \Lambda_{xx}^1 = 0, \Lambda_{xx}^2 = 0, \Lambda_{vx}^1 = 0, \Lambda_{vx}^2 = 0, \Lambda_{xvxx}^1 = 0, \Lambda_{xvxx}^2 = 0, \beta \Lambda_{vv}^1 - \alpha \Lambda_{vxx}^2 = 0, \\ \beta \Lambda_{vv}^2 + \alpha \Lambda_{vxx}^1 = 0, \Lambda_{vxx}^1 = 0, \Lambda_{vxx}^2 = 0, \Lambda_{vxxvxx}^1 = 0, \Lambda_{vxxvxx}^2 = 0, \Lambda_u^1 + \Lambda_v^2 = 0, \\ \Lambda_t^1 + \alpha (\Lambda_x^2 v + \Lambda_x^1 u) = 0, \Lambda_t^2 + \alpha (\Lambda_x^2 u - \Lambda_x^1 v) = 0, \Lambda_u^2 - \Lambda_v^1 = 0, \Lambda_{u_x}^1 = 0, \Lambda_{u_x}^2 = 0, \\ \Lambda_{u_{xx}}^1 + \Lambda_{v_{xx}}^2 = 0, \Lambda_{u_{xx}}^2 - \Lambda_{v_{xx}}^1 = 0, \Lambda_{v_x}^2 = 0, \Lambda_{v_x}^1 = 0. \end{aligned} \tag{113}$$

Calculations reveal the solution of the system (113) as

$$\begin{aligned} \Lambda^1 &= \frac{\alpha}{2\beta} (c_3 \{u^2 - v^2\} + 2c_4 uv) + (c_2 t + c_5)u + (c_1 t + c_6)v + c_3 u_{xx} + c_4 v_{xx} + c_7 - \frac{1}{\alpha} c_2 x, \\ \Lambda^2 &= \frac{\alpha}{2\beta} (c_4 \{u^2 - v^2\} - 2c_3 uv) + (c_1 t + c_6)u - (c_2 t + c_5)v + c_4 u_{xx} - c_3 v_{xx} + c_8 - \frac{1}{\alpha} c_1 x, \end{aligned} \tag{114}$$

for arbitrary constants c_1, \dots, c_8 .

Remark 3.3. Essentially, the nonlinear coupled system of KdV Equations (3) has eight sets of local conservation law multipliers.

Solving (105), we obtain conserved vectors corresponding to each set of multipliers as shown below.

(i) The multiplier

$$(\Lambda_1^1, \Lambda_1^2) = \left(tv, tu - \frac{x}{\alpha} \right), \tag{115}$$

has the conserved vectors

$$T_1^t = tuv - \frac{xv}{\alpha}, \quad T_1^x = \beta \left[t \{vu_{xx} + uv_{xx} - v_x u_x\} + \frac{1}{\alpha} \{v_x - xv_{xx}\} \right] + \alpha \left[t \left(u^2 v - \frac{v^3}{3} \right) \right] \tag{116}$$

$$-xuv. \tag{117}$$

(ii) The multiplier

$$(\Lambda_2^1, \Lambda_2^2) = \left(tu - \frac{x}{\alpha}, -tv \right), \tag{118}$$

has the conserved vectors

$$T_2^t = \frac{t}{2}\{u^2 - v^2\} - \frac{xu}{\alpha}, \quad T_2^x = \beta \left[t \left(uu_{xx} - vv_{xx} + \frac{1}{2}\{v_x^2 - u_x^2\} \right) + \frac{1}{\alpha}\{u_x - xv_{xx}\} \right] + \alpha t \left[\frac{u^3}{3} - uv^2 \right] + \frac{x}{2}\{v^2 - u^2\}. \quad (119)$$

(iii) The multiplier

$$(\Lambda_3^1, \Lambda_3^2) = \left(\frac{\alpha}{2\beta}\{u^2 - v^2\} + u_{xx}, -\left\{ \frac{\alpha uv}{\beta} + v_{xx} \right\} \right), \quad (120)$$

has the conserved vectors

$$T_3^t = \frac{\alpha}{2\beta} \left(\frac{u^3}{3} - uv^2 \right), \quad T_3^x = \frac{\alpha}{2} [(u^2 - v^2)u_{xx} - v^2v_{xx}] - \alpha uvv_{xx} + \quad (121)$$

$$\frac{\beta}{2} [u_{xx}^2 - v_{xx}^2] + u_t u_x - v_t v_x + \frac{\alpha^2}{4\beta} \left[\frac{1}{2}\{u^4 + v^4\} - 3u^2v^2 \right]. \quad (122)$$

(iv) The multiplier

$$(\Lambda_4^1, \Lambda_4^2) = \left(\left\{ \frac{\alpha uv}{\beta} + v_{xx} \right\}, \frac{\alpha[u^2 - v^2]}{2\beta} + u_{xx} \right), \quad (123)$$

has the conserved vectors

$$T_4^t = \frac{\alpha}{2\beta} \left(u^2v - \frac{v^3}{3} \right), \quad (124)$$

$$T_4^x = \frac{\alpha^2}{2\beta} [(u^3v - uv^3)] + v_t u_x + u_t v_x + \frac{\alpha}{2}(u^2 - v^2)v_{xx} + \{\alpha uv + \beta v_{xx}\}u_{xx}. \quad (125)$$

(v) The multiplier

$$(\Lambda_5^1, \Lambda_5^2) = (u, -v), \quad (126)$$

has the conserved vectors

$$T_5^t = \frac{1}{2}\{u^2 - v^2\}, \quad T_5^x = \beta \left(uu_{xx} - vv_{xx} + \frac{v_x^2 - u_x^2}{2} \right) + \alpha \left(\frac{u^3}{3} - uv^2 \right). \quad (127)$$

(vi) The multiplier

$$(\Lambda_6^1, \Lambda_6^2) = (v, u), \quad (128)$$

has the conserved vectors

$$T_6^t = uv, \quad T_6^x = \beta (vu_{xx} + uv_{xx} - u_x v_x) + \alpha \left(u^2v - \frac{v^3}{3} \right). \quad (129)$$

(vii) The multiplier

$$(\Lambda_7^1, \Lambda_7^2) = (1, 0), \quad (130)$$

has the conserved vectors

$$T_7^t = u, \quad T_7^x = \frac{\alpha}{2}\{u^2 - v^2\} + \beta u_{xx}. \quad (131)$$

(viii) The multiplier has

$$(\Lambda_8^1, \Lambda_8^2) = (0, 1), \quad (132)$$

the conserved vectors

$$T_8^t = v, \quad T_8^x = \alpha uv + \beta v_{xx}. \quad (133)$$

Remark 3.4. It can be verified that

$$D_t T_i^t + D_x T_i^x \Big|_{\Delta_1=0, \Delta_2=0} = 0, \quad (134)$$

for $i = 1, \dots, 8$.

Remark 3.5. The expressions in (134) are eight conservation laws for the coupled KdV system (3).

Remark 3.6. The presence of multipliers

$$(\Lambda_7^1, \Lambda_7^2) = (1, 0), \quad (\Lambda_8^1, \Lambda_8^2) = (0, 1) \quad (135)$$

manifest that the coupled KdV equations are themselves conservation laws.

At this point, we derive conserved vectors for coupled KdV equations (3) by a new theorem due to Ibragimov. The adjoint equations for the nonlinear system coupled KdV Equations (3) are

$$\Delta_1^* \equiv f_t + \alpha u f_x + \alpha v g_x + \beta f_{xxx} = 0, \quad \Delta_2^* g_t - \alpha v f_x + \alpha u g_x + \beta g_{xxx} = 0. \quad (136)$$

The formal Lagrangian \mathcal{L} for the nonlinear coupled system of the KdV Equations (3) and its adjoint Equations (136) is given by

$$\mathcal{L} = f\{u_t + \alpha uu_x - \alpha vv_x + \beta u_{xxx}\} + g\{v_t + \alpha uv_x + \alpha vu_x + \beta v_{xxx}\}, \quad (137)$$

where f and g are new variables. We shall use the Lie point symmetries of the system (3), namely

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \alpha t \partial_x + \partial_u, \quad X_4 = 3t \partial_t + x \partial_x - 2u \partial_u - 2v \partial_v, \quad (138)$$

to derive conserved vectors corresponding to each symmetry below.

Case (i) The symmetry $X_1 = \frac{\partial}{\partial t}$, yields Lie characteristic functions given by

$$W_1^1 = -u_t, \quad W_1^2 = -v_t. \quad (139)$$

Hence by Ibragimov's theorem [9], the associated conserved vector is given by

$$\begin{aligned} T_1^t &= \alpha [f\{uu_x - vv_x\} + g\{vu_x + uv_x\}] + \beta\{f u_{xxx} + g v_{xxx}\}, \\ T_1^x &= \alpha [f\{-uu_t + vv_t\} - g\{vu_t + uv_t\}] \\ &\quad + \beta\{f_x u_{tx} + g_x v_{tx} - u_t f_{xx} - v_t g_{xx} - f u_{txx} - g v_{txx}\}. \end{aligned} \quad (140)$$

Case (ii) The symmetry $X_2 = \frac{\partial}{\partial x}$, yields Lie characteristic functions

$$W_2^1 = -u_x, \quad W_2^2 = -v_x. \quad (141)$$

Therefore by Ibragimov's theorem [9], the associated conserved vector is

$$T_2^t = -u_x f - v_x g, \quad T_2^x = f u_t + g v_t + \beta \{-u_x f_{xx} - v_x g_{xx} + f_x u_{xx} + g_x v_{xx}\}. \quad (142)$$

Case (iii) The symmetry

$$X_3 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \quad (143)$$

yields Lie characteristic functions given by

$$W_3^1 = 1 - \alpha t u_x, \quad W_3^2 = -\alpha t v_x. \quad (144)$$

Hence by Ibragimov's theorem [9], the associated conserved vector is given by

$$T_3^t = f - \alpha t \{u_x f + v_x g\}, \quad (145)$$

$$T_3^x = \alpha \left[f u + g v + t \{u_t f + v_t g\} + \beta t \left\{ \frac{f_{xx}}{\alpha t} - u_x f_{xx} - v_x g_{xx} + f_x u_{xx} + g_x v_{xx} \right\} \right].$$

Case (iv) The symmetry

$$X_4 = 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v} \quad (146)$$

yields the Lie characteristic functions

$$W_4^1 = -2u - 3t u_t - x u_x, \quad W_4^2 = -2v - 3t v_t - x v_x. \quad (147)$$

Consequently by Ibragimov's theorem [9], the corresponding conserved vector is given by

$$T_4^t = \alpha [3t \{f u u_x - f v v_x + g u v_x + g v u_x\}] + \beta [3t \{f u_{xxx} + g v_{xxx}\}]$$

$$- 2\{f u + g v\} - x \{f u_x + g v_x\},$$

$$T_4^x = x \{f u_t + g v_t\} + \beta \left[3 \left(f_x u_x + g_x v_x + t \{f_x u_{tx} + g_x v_{tx}\} \right) \right] \quad (148)$$

$$- \alpha \left[2 \left(f \{u^2 - v^2\} + 2g u v \right) + 3t \left(f \{u u_t - v v_t\} + g \{v u_t + u v_t\} \right) \right]$$

$$- \beta [x \{u_x f_{xx} + v_x g_{xx} - f_x u_{xx} - g_x v_{xx}\} + 2\{u f_{xx} + v g_{xx}\}]$$

$$- \beta [3t \{f_{xx} u_t + g_{xx} v_t + f u_{txx} + g v_{txx}\} + 4\{f u_{xx} + g v_{xx}\}].$$

Remark 3.7. The appearance of arbitrary functions $f(t, x)$ and $g(t, x)$ in the conserved vectors proves the existence of infinite conservation laws for coupled KdV system obtained by Ibragimov's method.

4. CONCLUSION

In this paper, Lie group analysis was employed in studying a nonlinear coupled kdV system. A four-dimensional Lie algebra of symmetries was found for the nonlinear coupled system KdV equations. This was spanned by space and time translations, Galilean boost and scaling symmetries where the scaling symmetry acts on four variables. Associated to each symmetry, we obtained symmetry reductions that gave six nontrivial solutions for the coupled system. All the group-invariant solutions describe the various states of the system. The obtained solutions can be used as a benchmark against numerical simulations. Lastly, we constructed infinite conservation laws of a nonlinear coupled KdV system by using multipliers and a theorem proposed by Nail Ibragimov.

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