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Schwarz Algorithms for Stokes-Stokes Coupling

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Abstract. In this article, we exhibit the behavior of the Schwarz algorithms for the Steady Stokes equation in the case of two unbounded subdomains at the continuous level. The Schwarz methods have received a lot of attention during the last decades with the vast development of parallel computing devices. Hermann Amandus Schwarz, a German analyst, is considered to be the pioneer of the Domain Decomposition methods. We will closely observe how the overlapping and non overlapping Schwarz methods work for the steady Stokes problem. This problem has immediate practical application, modeling the flow of an incompressible fluid. For the analysis, we rely on Fourier analysis techniques and we provide comparison of the exhibited methods.

1. Introduction

Many people have been fascinated by the motion of fluids, and wonder how we are able to simulate the motion of fluids with such an accuracy. Of course, the answer is simple but at the same time complicated. Firstly, in order to model various phenomena, we use partial differential equations(PDEs). PDEs are equations that involve partial derivatives and most of the times we are not able to obtain solutions in closed form. As a result, we use numerical algorithms in order to obtain the approximate solution of a PDE. This field is called Numerical Analysis of PDEs and it is gaining increasing interest from mathematical and engineering communities worldwide. Especially, the last two decades Domain Decomposition methods [7], [8], [9], [10] are gaining ground due to the increased use of parallel computing. The pioneer of these methods was the German analyst Hermann Schwarz [4], [5], [6] who devised an algorithm to solve the Poisson equation in an irregular domain (union of rectangle and a circle), in order to fix a glitch in Riemann's mapping theorem. The algorithm is

$$\begin{cases} \Delta u_1^{(k)} = -f, \ in \ \Omega_1 \\ u_1^{(k)} = u_2^{(k-1)}, \ at \ \Gamma_1 \\ u_1^{(k)} = g_1, \ on \ \partial\Omega_1 \setminus \Gamma_1 \end{cases} , then \ \begin{cases} \Delta u_2^{(k)} = -f, \ in \ \Omega_2 \\ u_2^{(k)} = u_1^{(k)}, \ at \ \Gamma_2 \\ u_2^{(k)} = g_2, \ on \ \partial\Omega_2 \setminus \Gamma_2. \end{cases} \tag{1}$$

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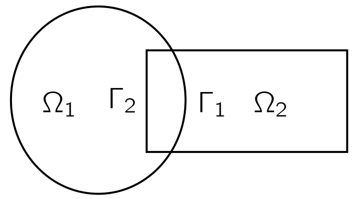

Figure 1. Domain decomposition of the global domain into a union of a circle and a rectangle.

Having a close look at the above figure we notice the following: Firstly, the Poisson problem is solved in the circle and then in the rectangle, going back and forth, passing the values at the interfaces Γ_1 and Γ_2. This iteration process is repeated until the convergence is reached. The index (k) denotes the iterations, and f is the source function. This is the so called Alternating Schwarz algorithm proposed by Schwarz back in 1870. After a significant amount of time, the Fields medalist Pierre Luis Lions [3], [13] proposed a modification in the Alternating Schwarz method (1). After imposing this modification, the algorithm (1) takes the form

$$\begin{cases} \Delta u_1^{(k)} = -f, \ in \ \Omega_1 \\ u_1^{(k)} = u_2^{(k-1)}, \ at \ \Gamma_1 \\ u_1^{(k)} = g_1, \ on \ \partial\Omega_1 \setminus \Gamma_1 \end{cases}, and \ \begin{cases} \Delta u_2^{(k)} = -f, \ in \ \Omega_2 \\ u_2^{(k)} = u_1^{(k-1)}, \ at \ \Gamma_2 \\ u_2^{(k)} = g_2, \ on \ \partial\Omega_2 \setminus \Gamma_2. \end{cases} \tag{2}$$

In this iterative algorithm, the two local subproblems are solved in parallel passing the Dirichlet values at the two interfaces. This algorithm (2) is known as the Parallel Schwarz algorithm. This iterative scheme provides two great benefits. The first is balancing the computational cost by breaking the global problem into smaller subproblems. The second benefit is that with the increasing amount of computational resources, the Schwarz method (2) is ideal for parallel computations. There has been an avalanche of new research results and there is a great avenue of research on Domain Decomposition methods. In this article, we will observe the behaviour of the Schwarz methods for the steady Stokes equation, for two unbounded subdomains using Fourier analysis techniques which is a standard approach in the literature [1], [2], [11], [12], [14], [15], [16], [17]. The steady Stokes equation is derived from the Navier-Stokes equation, which is a PDE for modeling the flow of incompressible fluids. It is a generalization of the equations proposed by the Swiss mathematician Leonhard Euler in the 18th century. In 1821, Claude-Luis Navier introduced the element of the viscosity. Later in the mid 19th century, Sir Gabriel Stokes worked extensively on the equation. The steady Stokes equation in strong form reads

$$\begin{cases} -\nu\Delta\vec{u} + \nabla p = \vec{f} \ in \ \Omega = (-\infty, +\infty) \times (-\infty, +\infty), \\ div\vec{u} = 0 \ in \ \Omega, \\ \vec{u}: \ bounded \ at \ \pm\infty, \\ p: \ bounded \ at \ \pm\infty \end{cases} \tag{3}$$

where $\nu > 0$ is the kinematic viscosity of the fluid, $\vec{u}$ is the velocity of the fluid and p is the pressure field. The function $\vec{f}$ is called the sink term. The function spaces for the velocity field, pressure field and sink term are $\left(H^1(\Omega)\right)^2$, $L_2(\Omega)$, $(L_2(\Omega))^2$ respectively. The space $H^1(\Omega)$ is classical Sobolev space, and $L_2(\Omega)$ is the space of square integrable functions. The $(3)_2$ denotes the incompressibility condition, with the divergence free velocity field. Furthermore as $(3)_3$, $(3)_4$ suggest, the velocity and the pressure field stay bounded at infinity.

2. Parallel Schwarz Method-Dirichlet IC

We decompose the domain $\Omega = \mathbb{R}^2$ into two subdomains $\Omega_1 = (-\infty, H) \times (-\infty, +\infty)$ and $\Omega_2 = (0, +\infty) \times (-\infty, +\infty)$. The Parallel Schwarz method in strong form reads

$$\begin{cases} -\nu\Delta\overrightarrow{u_1}^{(k)} + \nabla p_1^{(k)} = \vec{f} \; in \; \Omega_1, \\ div\overrightarrow{u_1}^{(k)} = 0 \; in \;\; \Omega_1, \\ \overrightarrow{u_1}^{(k)} = \overrightarrow{u_2}^{(k-1)} at \; x = H, \\ \overrightarrow{u_1}^{(k)} : bounded \; at \; -\infty, \\ p_1^{(k)} : bounded \; at \; -\infty, \end{cases} , and \begin{cases} -\nu\Delta\overrightarrow{u_2}^{(k)} + \nabla p_2^{(k)} = \vec{f} \; in \; \Omega_2, \\ div\overrightarrow{u_2}^{(k)} = 0 \; in \; \Omega_2, \\ \overrightarrow{u_2}^{(k)} = \overrightarrow{u_1}^{(k-1)} at \; x = 0, \\ \overrightarrow{u_2}^{(k)} : bounded \; at \; +\infty, \\ p_2^{(k)} : bounded \; at \; +\infty, \end{cases} \tag{4}$$

where two initial guesses $\overrightarrow{u_1}^{(0)}$, $\overrightarrow{u_2}^{(0)}$ are required to start the iterative process.

Theorem 1. *The convergence factor of the Parallel Schwarz algorithm using Dirichlet transmission conditions is given by the formula below*

$$r_{PSM,D}(\xi, H) = \left(1 + 2H^2|\xi|^2 + 2\,|\xi|\sqrt{H^2\left(1 + H^2|\xi|^2\right)}\right) e^{-2|\xi|H} \tag{5}$$

where ξ is the Fourier frequency and $H > 0$ is the size of the overlap.

Proof. In order to study the convergence behavior of the method, we go back to the local subproblems in (4) and we consider the homogeneous counterparts taking $\vec{f} = \vec{0}$. In addition, the velocity fields in the two subdomains are $\overrightarrow{u_1}^{(k)} = \left(u_{1,1}^{(k)}, u_{1,2}^{(k)}\right)$ and $\overrightarrow{u_2}^{(k)} = \left(u_{2,1}^{(k)}, u_{2,2}^{(k)}\right)$, where the first indices denote the subdomain and the second indices denote the component. Consequently, the Parallel Schwarz method prescribed by (4) can be written in the following form

$$\begin{cases} \frac{\partial^2 u_{1,1}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{1,1}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_1^{(k)}}{\partial x} \ in\ \Omega_1, \\ \frac{\partial^2 u_{1,2}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{1,2}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_1^{(k)}}{\partial y} \ in\ \Omega_1, \\ \frac{\partial u_{1,1}^{(k)}}{\partial x} + \frac{\partial u_{1,2}^{(k)}}{\partial y} = 0 \ in\ \Omega_1, \\ u_{1,1}^{(k)} = u_{2,1}^{(k-1)} \ at\ x = H, \\ u_{1,2}^{(k)} = u_{2,2}^{(k-1)} \ at\ x = H, \\ u_{1,1}^{(k)} : bounded\ at\ -\infty, \\ u_{1,2}^{(k)} : bounded\ at\ -\infty, \\ p_1^{(k)} : bounded\ at\ -\infty, \end{cases} , and \begin{cases} \frac{\partial^2 u_{2,1}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{2,1}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_2^{(k)}}{\partial x} \ in\ \Omega_2, \\ \frac{\partial^2 u_{2,2}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{2,2}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_2^{(k)}}{\partial y} \ in\ \Omega_2, \\ \frac{\partial u_{2,1}^{(k)}}{\partial x} + \frac{\partial u_{2,2}^{(k)}}{\partial y} = 0 \ in\ \Omega_2, \\ u_{2,1}^{(k)} = u_{1,1}^{(k-1)} \ at\ x = 0, \\ u_{2,2}^{(k)} = u_{1,2}^{(k-1)} \ at\ x = 0, \\ u_{2,1}^{(k)} : bounded\ at\ +\infty, \\ u_{2,2}^{(k)} : bounded\ at\ +\infty, \\ p_2^{(k)} : bounded\ at\ +\infty. \end{cases} \tag{6}$$

Going back to $(4)_1$, for $\vec{f} = \vec{0}$, taking the divergence on both sides for the first subproblem, we obtain

$$\begin{aligned} div\left(\Delta \overrightarrow{u_1}^{(k)}\right) &= \frac{1}{\nu}\ \Delta p_1^{(k)} \\ &= \left(\frac{\partial^3 u_{1,1}^{(k)}}{\partial x^3} + \frac{\partial^3 u_{1,2}^{(k)}}{\partial y \partial x^2}\right) + \left(\frac{\partial^3 u_{1,2}^{(k)}}{\partial y^3} + \frac{\partial^3 u_{1,1}^{(k)}}{\partial x \partial y^2}\right) \\ &= 0 \ in\ \Omega_1 \end{aligned}$$

exploiting the equation $(4)_2$ (divergence free velocity in subdomain Ω_1). In the same fashion we obtain that $div\left(\Delta \overrightarrow{u_2}^{(k)}\right) = \frac{1}{\nu}\ \Delta p_2^{(k)} = 0\ in\ \Omega_2$. As a consequence, we have to solve two Laplace problems in each subdomain where the unknown is the pressure field. We deal with $\Delta p_1^{(k)} = 0\ in\ \Omega_1$ and by taking the Fourier transform in the y direction we obtain the homogeneous equation $\frac{\partial^2 \hat{p}_1^{(k)}}{\partial x^2} - |\xi|^2 \hat{p}_1^{(k)} = 0$. The general solution of this equation is $\hat{p}_1^{(k)} = \mathcal{C}_1^{(k)} e^{-|\xi|x} + \mathcal{D}_1^{(k)} e^{|\xi|x}$. By the boundedness assumption of the pressure field in Ω_1 as $x \to -\infty$, we obtain that $\hat{p}_1^{(k)} = \mathcal{D}_1^{(k)} e^{|\xi|x}$. We proceed to solve the equation $\Delta p_2^{(k)} = 0\ in\ \Omega_2$, and the first step is to take the Fourier transform in the y direction. As a result, the equation $\frac{\partial^2 \hat{p}_2^{(k)}}{\partial x^2} - |\xi|^2 \hat{p}_2^{(k)} = 0$ has a general solution of the form $\hat{p}_2^{(k)} = \mathcal{C}_2^{(k)} e^{-|\xi|x} + \mathcal{D}_2^{(k)} e^{|\xi|x}$.By exploiting the property that the pressure field $\hat{p}_2^{(k)}$ remains bounded as $x \to +\infty$, we obtain that $\hat{p}_2^{(k)} = \mathcal{C}_2^{(k)} e^{-|\xi|x}$. The next move is to go to the two local Schwarz subproblems in $(6)_1$ and to take the Fourier transform in the y direction. This will give

$$\frac{\partial^2 \hat{u}_{1,1}^{(k)}}{\partial x^2} - |\xi|^2 \hat{u}_{1,1}^{(k)} = \frac{|\xi| \mathcal{D}_1^{(k)} e^{|\xi|x}}{\nu}, \tag{7}$$

$$\frac{\partial^2 \hat{u}_{2,1}^{(k)}}{\partial x^2} - |\xi|^2 \hat{u}_{2,1}^{(k)} = \frac{-|\xi| \mathcal{C}_2^{(k)} e^{-|\xi|x}}{\nu}. \tag{8}$$

We solve (7), (8) to obtain the two solutions in closed form

$$\hat{u}_{1,1}^{(k)} = \left(\mathcal{B}_1^{(k)} + \frac{x}{2\nu}\mathcal{D}_1^{(k)}\right) e^{|\xi|x}, \tag{9}$$

$$\hat{u}_{2,1}^{(k)} = \left(\mathcal{B}_2^{(k)} + \frac{x}{2\nu}\mathcal{C}_2^{(k)}\right) e^{-|\xi|x}. \tag{10}$$

We go back to (6)$_3$ and by taking the Fourier trasform in the y direction and exploiting the solutions (9), (10) we obtain

$$\hat{u}_{1,2}^{(k)} = \frac{i}{\xi}\left(|\xi|\mathcal{B}_1^{(k)} + \left(\frac{1+x|\xi|}{2\nu}\right)\mathcal{D}_1^{(k)}\right)e^{|\xi|x}, \tag{11}$$

$$\hat{u}_{2,2}^{(k)} = \frac{i}{\xi}\left(-|\xi|\mathcal{B}_2^{(k)} + \left(\frac{1-x|\xi|}{2\nu}\right)\mathcal{C}_2^{(k)}\right)e^{-|\xi|x}. \tag{12}$$

We further proceed, substituting the solutions (9), (10), (11), (12) back to the interface conditions (6)$_4$, (6)$_5$ to obtain the following equations

$$\mathcal{B}_1^{(k)} + \frac{H}{2\nu}\mathcal{D}_1^{(k)} = \left(\mathcal{B}_2^{(k-1)} + \frac{H}{2\nu}\mathcal{C}_2^{(k-1)}\right)e^{-2|\xi|H}, \tag{13}$$

$$|\xi|\mathcal{B}_1^{(k)} + \left(\frac{1+H|\xi|}{2\nu}\right)\mathcal{D}_1^{(k)} = \left(-|\xi|\mathcal{B}_2^{(k-1)} + \left(\frac{1-H|\xi|}{2\nu}\right)\mathcal{C}_2^{(k-1)}\right)e^{-2|\xi|H}, \tag{14}$$

$$\mathcal{B}_2^{(k)} = \mathcal{B}_1^{(k-1)}, \tag{15}$$

$$-|\xi|\mathcal{B}_2^{(k)} + \frac{\mathcal{C}_2^{(k)}}{2\nu} = |\xi|\mathcal{B}_1^{(k-1)} + \frac{\mathcal{D}_1^{(k-1)}}{2\nu}. \tag{16}$$

We combine the equations (15), (16) to obtain $\mathcal{D}_1^{(k)} = \mathcal{C}_2^{(k+1)} - 4|\xi|\nu\mathcal{B}_2^{(k+1)}$. We substitute the coefficients $\mathcal{D}_1^{(k)}$ back to equation (13) to obtain

$$\mathcal{B}_2^{(k+1)}(2\nu - 4\nu H|\xi|) + H\mathcal{C}_2^{(k+1)} = \mathcal{B}_2^{(k-1)}2\nu e^{-2|\xi|H} + H\mathcal{C}_2^{(k-1)}e^{-2|\xi|H}. \tag{17}$$

In the same spirit, we replace the iteration coefficients $\mathcal{D}_1^{(k)}$ back to (14) to get

$$\mathcal{B}_2^{(k+1)}\left(2\nu|\xi| + 4\nu H|\xi|^2\right) - \mathcal{C}_2^{(k+1)}(1+H|\xi|) = 2\nu|\xi|\mathcal{B}_2^{(k-1)}e^{-2|\xi|H} + (H|\xi|-1)\mathcal{C}_2^{(k-1)}e^{-2|\xi|H}. \tag{18}$$

We take the two equations (17), (18) and write them in matrix form

$$\begin{bmatrix}(2\nu - 4\nu H|\xi|) & H \\ (2\nu|\xi| + 4\nu H|\xi|^2) & -(1+H|\xi|)\end{bmatrix}\begin{bmatrix}\mathcal{B}_2^{(k+1)} \\ \mathcal{C}_2^{(k+1)}\end{bmatrix} = \begin{bmatrix}2\nu e^{-2|\xi|H} & He^{-2|\xi|H} \\ 2\nu|\xi|e^{-2|\xi|H} & (H|\xi|-1)e^{-2|\xi|H}\end{bmatrix}\begin{bmatrix}\mathcal{B}_2^{(k-1)} \\ \mathcal{C}_2^{(k-1)}\end{bmatrix}. \tag{19}$$

We recast the equation (19) in the form

$$\begin{bmatrix}\mathcal{B}_2^{(k+1)} \\ \mathcal{C}_2^{(k+1)}\end{bmatrix} = \underbrace{\begin{bmatrix}(1+2H|\xi|)e^{-2|\xi|H} & \frac{H^2|\xi|e^{-2|\xi|H}}{\nu} \\ 8\nu H|\xi|^2 e^{-2|\xi|H} & \left(4H^2|\xi|^2 - 2H|\xi| + 1\right)e^{-2|\xi|H}\end{bmatrix}}_{\Psi_{PSM,D}}\begin{bmatrix}\mathcal{B}_2^{(k-1)} \\ \mathcal{C}_2^{(k-1)}\end{bmatrix} \tag{20}$$

where $\Psi_{PSM,D}$ is the Schwarz iteration matrix. The Spectrum of $\Psi_{PSM,D}$ is $\sigma(\Psi_{PSM,D}) = \{\lambda_+,\ \lambda_-\}$, where λ_+ and λ_- are the corresponding eigenvalues given by the formulas

$$\lambda_+ = \left(1 + 2H^2|\xi|^2 + 2\,|\xi|\sqrt{H^2\left(1+H^2|\xi|^2\right)}\right)e^{-2|\xi|H},$$

$$\lambda_- = \left(1 + 2H^2|\xi|^2 - 2\,|\xi|\sqrt{H^2\left(1+H^2|\xi|^2\right)}\right)e^{-2|\xi|H}.$$

Consequently, the convergence factor of the Parallel Schwarz algorithm is

$$r_{PSM,D} = \rho(\Psi_{PSM,D}) = \max\{|\lambda_+|, |\lambda_-|\} = \left(1 + 2H^2|\xi|^2 + 2\,|\xi|\sqrt{H^2\left(1 + H^2|\xi|^2\right)}\right) e^{-2|\xi|H}$$

, where $\rho(\Psi_{PSM,D})$ is the spectral radius of the Schwarz iteration matrix, H is the size of the overlap between the subdomains, ξ is the Fourier frequency. □

3. Alternating Schwarz Method-Neumann IC

The interface conditions play critical role on the convergence of the Schwarz method. In this section, we introduce the Alternating Schwarz algorithm employing Neumann interface conditions. We go back to the iterative scheme prescribed by (4) and we modify the transmission conditions in $(4)_3$. As a consequence, the Schwarz method in strong form reads

$$\begin{cases} -\nu\Delta\overrightarrow{u_1}^{(k)} + \nabla p_1^{(k)} = \vec{f} \ in \ \Omega_1, \\ div\overrightarrow{u_1}^{(k)} = 0 \ in \ \Omega_1, \\ \nu\nabla\overrightarrow{u_1}^{(k)}\vec{n} - p_1^{(k)}\vec{n} = \nu\nabla\overrightarrow{u_2}^{(k-1)}\vec{n} - p_2^{(k-1)}\vec{n} \ at \ x = H, \\ \overrightarrow{u_1}^{(k)} : bounded \ at \ -\infty, \\ p_1^{(k)} : bounded \ at \ -\infty, \end{cases} \quad \begin{cases} -\nu\Delta\overrightarrow{u_2}^{(k)} + \nabla p_2^{(k)} = \vec{f} \ in \ \Omega_2, \\ div\overrightarrow{u_2}^{(k)} = 0 \ in \ \Omega_2, \\ \nu\nabla\overrightarrow{u_2}^{(k)}\vec{n} - p_2^{(k)}\vec{n} = \nu\nabla\overrightarrow{u_1}^{(k)}\vec{n} - p_1^{(k)}\vec{n} \ at \ x = 0, \\ \overrightarrow{u_2}^{(k)} : bounded \ at \ +\infty, \\ p_2^{(k)} : bounded \ at \ +\infty, \end{cases} \tag{21}$$

where $\vec{n}$ is the outward normal vector. The initial guess $\nu\nabla\overrightarrow{u_2}^{(0)}\vec{n} - p_2^{(0)}\vec{n}$ is required to start the iterative procedure.

Theorem 2. *The convergence factor of the Schwarz algorithm using Neumann transmission conditions is given by the formula below*

$$r_{ASM,N}(\xi, H) = \left|\left(\frac{2|\xi|^2H^2}{9} - \frac{2|\xi|H}{9} + 1 + \frac{2\sqrt{\,|\xi|^4H^4 + 2|\xi|^3H^3 + \ 8\ |\xi|^2H^2}}{9}\right)\right| e^{-2|\xi|H} \tag{22}$$

where ξ is the Fourier frequency and $H > 0$ is the size of the overlap.

Proof. As a first step, we go back to the local subproblems in (21) and we consider the homogeneous counterparts taking $\vec{f} = \vec{0}$. We recast the method prescribed by (21) in the following form

$$\begin{cases} \frac{\partial^2 u_{1,1}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{1,1}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_1^{(k)}}{\partial x} \ in \ \Omega_1, \\ \frac{\partial^2 u_{1,2}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{1,2}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_1^{(k)}}{\partial y} \ in \ \Omega_1, \\ \frac{\partial u_{1,1}^{(k)}}{\partial x} + \frac{\partial u_{1,2}^{(k)}}{\partial y} = 0 \ in \ \Omega_1, \\ \nu\frac{\partial}{\partial x}u_{1,1}^{(k)} - p_1^{(k)} = \nu\frac{\partial}{\partial x}u_{2,1}^{(k-1)} - p_2^{(k-1)} \ at \ x = H, \\ \frac{\partial}{\partial x}u_{1,2}^{(k)} = \frac{\partial}{\partial x}u_{2,2}^{(k-1)} \ at \ x = H, \\ u_{1,1}^{(k)} : bounded \ at \ -\infty, \\ u_{1,2}^{(k)} : bounded \ at \ -\infty, \\ p_1^{(k)} : bounded \ at \ -\infty, \end{cases} \quad \begin{cases} \frac{\partial^2 u_{2,1}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{2,1}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_2^{(k)}}{\partial x} \ in \ \Omega_2, \\ \frac{\partial^2 u_{2,2}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{2,2}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_2^{(k)}}{\partial y} \ in \ \Omega_2, \\ \frac{\partial u_{2,1}^{(k)}}{\partial x} + \frac{\partial u_{2,2}^{(k)}}{\partial y} = 0 \ in \ \Omega_2, \\ \nu\frac{\partial}{\partial x}u_{2,1}^{(k)} - p_2^{(k)} = \nu\frac{\partial}{\partial x}u_{1,1}^{(k)} - p_1^{(k)} \ at \ x = 0, \\ \frac{\partial}{\partial x}u_{2,2}^{(k)} = \frac{\partial}{\partial x}u_{1,2}^{(k)} \ at \ x = 0, \\ u_{2,1}^{(k)} : bounded \ at \ +\infty, \\ u_{2,2}^{(k)} : bounded \ at \ +\infty, \\ p_2^{(k)} : bounded \ at \ +\infty. \end{cases} \tag{23}$$

We apply the Fourier transform in the y direction to the Schwarz subproblems prescribed by (23). The Fourier transformed velocity components are given by the formulas (9), (10), (11), (12). The Fourier transformed pressure fields are given by the relations: $\hat{p}_1^{(k)} = \mathcal{D}_1^{(k)} e^{|\xi|x}$, $\hat{p}_2^{(k)} = \mathcal{C}_2^{(k)} e^{-|\xi|x}$. We plug in the Fourier transformed velocities and pressure fields back to the interface conditions $(23)_4$, $(23)_5$ and by doing a little algebra we obtain

$$2\nu|\xi|e^{|\xi|H}\mathcal{B}_1^{(k)} + \mathcal{D}_1^{(k)}(H|\xi|-1)e^{|\xi|H} = -2\nu|\xi|\mathcal{B}_2^{(k-1)}e^{-|\xi|H} - \mathcal{C}_2^{(k-1)}e^{-|\xi|H}(1+H|\xi|), \quad (24)$$

$$2\nu\mathcal{B}_1^{(k)}|\xi|^2e^{|\xi|H} + \mathcal{D}_1^{(k)}e^{|\xi|H}\left(2|\xi| + H\,|\xi|^2\right) = 2\nu|\xi|^2\mathcal{B}_2^{(k-1)}e^{-|\xi|H} + \mathcal{C}_2^{(k-1)}e^{-|\xi|H}\left(H|\xi|^2 - 2|\xi|\right), \quad (25)$$

$$2\nu|\xi|\mathcal{B}_2^{(k)} + C_2^{(k)} = -2\nu|\xi|\mathcal{B}_1^{(k)} + \mathcal{D}_1^{(k)}, \quad (26)$$

$$\nu|\xi|^2\mathcal{B}_2^{(k)} - |\xi|C_2^{(k)} = \nu\mathcal{B}_1^{(k)}|\xi|^2 + |\xi|\mathcal{D}_1^{(k)}. \quad (27)$$

We multiply (26) by $-|\xi|$ then add (27), and solve with respect to the coefficient $\mathcal{B}_1^{(k)}$ obtaining

$$\mathcal{B}_1^{(k)} = -\frac{1}{3}\mathcal{B}_2^{(k)} - \frac{2}{3}\frac{1}{\nu|\xi|}\mathcal{C}_2^{(k)}. \quad (28)$$

The next step is to obtain a formula for the coefficient $\mathcal{D}_1^{(k)}$. In order to achieve that, we multiply (26) by $|\xi|$ then add (27) to obtain

$$\mathcal{D}_1^{(k)} = \frac{4}{3}\nu|\xi|\mathcal{B}_2^{(k)} - \frac{1}{3}\mathcal{C}_2^{(k)}. \quad (29)$$

We substitute the expressions (28), (29) back to (24) and (25) and this yields

$$\mathcal{B}_2^{(k)}e^{|\xi|H}\left(6\nu|\xi| - 4\nu H|\xi|^2\right) + \mathcal{C}_2^{(k)}e^{|\xi|H}(3+H|\xi|) = 6\nu|\xi|\mathcal{B}_2^{(k-1)}e^{-|\xi|H} + 3\mathcal{C}_2^{(k-1)}e^{-|\xi|H}(1+H|\xi|),$$

$$\mathcal{B}_2^{(k)}e^{|\xi|H}\left(6\nu|\xi|^2 + 4\nu H|\xi|^3\right) - \mathcal{C}_2^{(k)}e^{|\xi|H}\left(6|\xi| + H|\xi|^2\right) = 6\nu|\xi|^2\mathcal{B}_2^{(k-1)}e^{-|\xi|H} + 3\mathcal{C}_2^{(k-1)}e^{-|\xi|H}\left(H|\xi|^2 - 2|\xi|\right).$$

We write the above equations in matrix form and by doing some algebraic manipulations we derive the stationary iteration

$$\begin{bmatrix}\mathcal{B}_2^{(k)}\\ C_2^{(k)}\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{-12\nu H|\xi|^3-54\nu|\xi|^2}{54e^{2|\xi|H}|\xi|^2\nu} & \frac{H(H|\xi|+4)}{9e^{2|\xi|H}\nu}\\ \frac{8|\xi|^2\nu H}{9e^{2|\xi|H}} & \frac{4|\xi|^2H^2-2|\xi|H+9}{9e^{2|\xi|H}}\end{bmatrix}}_{\Psi_{ASM,N}}\begin{bmatrix}\mathcal{B}_2^{(k-1)}\\ C_2^{(k-1)}\end{bmatrix} \quad (30)$$

where $\Psi_{ASM,N}$ is the Schwarz iteration matrix. The spectrum of $\Psi_{ASM,N}$ is $\sigma(\Psi_{ASM,N}) = \{\mu_+, \mu_-\}$, where μ_+ and μ_- are the eigenvalues of the Schwarz iteration matrix provided by the formulas

$$\mu_+ = \left(\frac{2|\xi|^2H^2}{9} - \frac{2|\xi|H}{9} + 1 + \frac{2\sqrt{|\xi|^4H^4 + 2|\xi|^3H^3 + 8\,|\xi|^2H^2}}{9}\right)e^{-2|\xi|H},$$

$$\mu_- = \left(\frac{2|\xi|^2H^2}{9} - \frac{2|\xi|H}{9} + 1 - \frac{2\sqrt{|\xi|^4H^4 + 2|\xi|^3H^3 + 8\,|\xi|^2H^2}}{9}\right)e^{-2|\xi|H}.$$

As a result, the reduction factor of the Schwarz method (Neumann Interface conditions) is given by

$$
\begin{aligned}
r_{ASM,N} &= \rho(\Psi_{ASM,N}) = \max\{|\mu_+|, |\mu_-|\} \\
&= \left| \left(\frac{2|\xi|^2 H^2}{9} - \frac{2|\xi| H}{9} + 1 + \frac{2\sqrt{\;|\xi|^4 H^4 + 2|\xi|^3 H^3 + \;8\;|\xi|^2 H^2}}{9} \right) \right| e^{-2|\xi| H}.
\end{aligned}
$$

□

4. Non-overlapping Optimized Schwarz Algorithm-Robin IC

The domain $\Omega = \mathbb{R}^2$ is decomposed into two non-overlapping subdomains $\Omega_1 = (-\infty, 0) \times (-\infty, +\infty)$ and $\Omega_2 = (0, +\infty) \times (-\infty, +\infty)$.The Optimized Schwarz methods employ mixed interface boundary conditions, and more Precisely Robin. In this way, they facilitate both Neumann and Dirichlet conditions and there is a tuning parameter to tune the method accordingly. The Optimized Schwarz iterative scheme is given in strong form

$$
\begin{cases}
-\nu \Delta \overrightarrow{u_1}^{(k)} + \nabla p_1^{(k)} = \vec{f} \ in \ \Omega_1, \\
div \overrightarrow{u_1}^{(k)} = 0 \ in \ \ \Omega_1, \\
\nu \nabla \overrightarrow{u_1}^{(k)} \vec{n} - p_1^{(k)} \vec{n} + \gamma \overrightarrow{u_1}^{(k)} = \nu \nabla \overrightarrow{u_2}^{(k-1)} \vec{n} - p_2^{(k-1)} \vec{n} + \gamma \overrightarrow{u_2}^{(k-1)} \ at \ x = 0, \\
\overrightarrow{u_1}^{(k)} : bounded \ at \ -\infty, \\
p_1^{(k)} : bounded \ at \ -\infty,
\end{cases}
\tag{31}
$$

$$
\begin{cases}
-\nu \Delta \overrightarrow{u_2}^{(k)} + \nabla p_2^{(k)} = \vec{f} \ in \ \Omega_2, \\
div \overrightarrow{u_2}^{(k)} = 0 \ in \ \Omega_2, \\
\nu \nabla \overrightarrow{u_2}^{(k)} \vec{n} - p_2^{(k)} \vec{n} + \gamma \overrightarrow{u_2}^{(k)} = \nu \nabla \overrightarrow{u_1}^{(k-1)} \vec{n} - p_1^{(k-1)} \vec{n} \ + \gamma \overrightarrow{u_1}^{(k-1)} \ at \ x = 0, \\
\overrightarrow{u_2}^{(k)} : bounded \ at \ +\infty, \\
p_2^{(k)} : bounded \ at \ +\infty,
\end{cases}
\tag{32}
$$

where γ is the tuning parameter of the method. Two initial guesses are needed for the iterative method.

Theorem 3. *The contraction factor of the non-overlapping Schwarz algorithm is given by the mathematical expression*

$$
r^2_{OSM}(\xi, \nu, \gamma) = \frac{|3\nu^2|\xi|^2 - 4\nu|\xi|\gamma + \gamma^2|^2}{|3\nu^2|\xi|^2 + 4\nu|\xi|\gamma + \gamma^2|^2}. \tag{33}
$$

Proof. We recast the local Schwarz subproblems (31), (32) in the form

$$\begin{cases} \frac{\partial^2 u_{1,1}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{1,1}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_1^{(k)}}{\partial x} \ \ in\ \Omega_1, \\ \frac{\partial^2 u_{1,2}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{1,2}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_1^{(k)}}{\partial y} \ \ in\ \Omega_1, \\ \frac{\partial u_{1,1}^{(k)}}{\partial x} + \frac{\partial u_{1,2}^{(k)}}{\partial y} = 0 \ in\ \Omega_1, \\ \nu\frac{\partial}{\partial x}u_{1,1}^{(k)} - p_1^{(k)} + \gamma u_{1,1}^{(k)} = \nu\frac{\partial}{\partial x}u_{2,1}^{(k-1)} - p_2^{(k-1)} + \gamma u_{2,1}^{(k-1)} \ at\ x=0, \\ \nu\frac{\partial}{\partial x}u_{1,2}^{(k)} + \gamma u_{1,2}^{(k)} = \nu\frac{\partial}{\partial x}u_{2,2}^{(k-1)} + \gamma u_{2,2}^{(k-1)} \ at\ x=0, \\ u_{1,1}^{(k)} : bounded\ at\ -\infty, \\ u_{1,2}^{(k)} : bounded\ at\ -\infty, \\ p_1^{(k)} : bounded\ at\ -\infty, \end{cases} \tag{34}$$

$$\begin{cases} \frac{\partial^2 u_{2,1}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{2,1}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_2^{(k)}}{\partial x} \ \ in\ \Omega_2, \\ \frac{\partial^2 u_{2,2}^{(k)}}{\partial x^2} + \frac{\partial^2 u_{2,2}^{(k)}}{\partial y^2} = \frac{1}{\nu}\frac{\partial p_2^{(k)}}{\partial y} \ \ in\ \Omega_2, \\ \frac{\partial u_{2,1}^{(k)}}{\partial x} + \frac{\partial u_{2,2}^{(k)}}{\partial y} = 0 \ in\ \Omega_2, \\ \nu\frac{\partial}{\partial x}u_{2,1}^{(k)} - p_2^{(k)} - \gamma u_{2,1}^{(k)} = \nu\frac{\partial}{\partial x}u_{1,1}^{(k-1)} - p_1^{(k-1)} - \gamma u_{1,1}^{(k-1)} \ at\ x=0, \\ \nu\frac{\partial}{\partial x}u_{2,2}^{(k)} - \gamma u_{2,2}^{(k)} = \nu\frac{\partial}{\partial x}u_{1,2}^{(k-1)} - \gamma u_{1,2}^{(k-1)} \ at\ x=0, \\ u_{2,1}^{(k)} : bounded\ at\ +\infty, \\ u_{2,2}^{(k)} : bounded\ at\ +\infty, \\ p_2^{(k)} : bounded\ at\ +\infty. \end{cases} \tag{35}$$

We employ the Fourier transform for the local Schwarz subproblems (34), (35). The Fourier transformed velocity components are given by the mathematical expressions (9), (10), (11), (12). The Fourier transformed pressure fields are given by $\hat{p}_1^{(k)} = \mathcal{D}_1^{(k)} e^{|\xi|x}$, $\hat{p}_2^{(k)} = \mathcal{C}_2^{(k)} e^{-|\xi|x}$. We substitute the velocities and pressure fields back to the transmission conditions $(34)_4$, $(34)_5$, $(35)_4$, $(35)_5$, and by doing some algebraic manipulations we obtain

$$\mathcal{B}_1^{(k)}(2\gamma + 2\nu|\xi|) - \mathcal{D}_1^{(k)} = \mathcal{B}_2^{(k-1)}(2\gamma - 2\nu|\xi|\ ) - \mathcal{C}_2^{(k-1)}, \tag{36}$$

$$\mathcal{B}_1^{(k)}\left(2\nu^2|\xi|^2 + 2\nu\gamma|\xi|\right) + \mathcal{D}_1^{(k)}(2\nu|\xi|\ + \gamma) = \mathcal{B}_2^{(k-1)}\left(2\nu^2|\xi|^2 - 2\nu\gamma|\xi|\right) + \mathcal{C}_2^{(k-1)}(\gamma - 2\nu|\xi|), \tag{37}$$

$$\mathcal{B}_2^{(k)}(\ 2\gamma + 2\nu|\xi|) + \mathcal{C}_2^{(k)} = \mathcal{B}_1^{(k-1)}(\ 2\gamma - 2\nu|\xi|) + \mathcal{D}_1^{(k-1)}, \tag{38}$$

$$\mathcal{B}_2^{(k)}\left(2\nu\gamma|\xi| + 2\nu^2|\xi|^2\right) - \mathcal{C}_2^{(k)}(2\nu|\xi| + \gamma) = \mathcal{B}_1^{(k-1)}\left(2\nu^2|\xi|^2 - 2\nu\gamma|\xi|\right) + \mathcal{D}_1^{(k-1)}(2\nu|\xi| - \gamma). \tag{39}$$

We pick (38) and we obtain the coefficients

$$\mathcal{D}_1^{(k)} = \mathcal{B}_2^{(k+1)}(2\gamma + 2\nu|\xi|) + \mathcal{C}_2^{(k+1)} - \mathcal{B}_1^{(k)}(\ 2\gamma - 2\nu|\xi|). \tag{40}$$

We substitiute the coefficients (40) back to (36) and we obtain

$$4\bigcirc\mathcal{B}_1^{(k)} = \mathcal{B}_2^{(k+1)}(2\gamma + 2\nu|\xi|) + \mathcal{C}_2^{(k+1)} + \mathcal{B}_2^{(k-1)}(2\gamma - 2\nu|\xi|) - \mathcal{C}_2^{(k-1)}. \tag{41}$$

The equation (40) can take the following form

$$4\bigcirc\mathcal{D}_1^{(k)} = \mathcal{B}_2^{(k+1)}(2\gamma+2\nu|\xi|)^2+\mathcal{C}_2^{(k+1)}(2\gamma+2\nu|\xi|)-\mathcal{B}_2^{(k-1)}(2\gamma-2\nu|\xi|)^2+\mathcal{C}_2^{(k-1)}(2\gamma-2\nu|\xi|) \tag{42}$$

by employing (41). We take the relation (37), multiply with 4γ, and then plug in (41), (42) to obtain

$$k_1\mathcal{B}_2^{(k+1)} + k_2\mathcal{C}_2^{(k+1)} = k_3\mathcal{B}_2^{(k-1)} + k_4\mathcal{C}_2^{(k-1)} \tag{43}$$

where k_1, k_2, k_3, k_4 are given by the relations below

$$\begin{aligned}
k_1 &= 12\left(\nu|\xi| + \frac{\gamma}{3}\right)(\nu|\xi| + \gamma)^2, \\
k_2 &= 6\nu^2|\xi|^2 + 8\nu|\xi|\gamma + 2\gamma^2, \\
k_3 &= 12|\xi|^3\nu^3 - 4|\xi|^2\gamma\nu^2 - 12|\xi|\gamma^2\nu + 4\gamma^3, \\
k_4 &= 6\nu^2|\xi|^2 - 8\nu|\xi|\gamma + 2\gamma^2.
\end{aligned}$$

In the same fashion, we pick (39), multiply with 4γ, and then exploit the expressions (41), (42) to derive the equation

$$q_1\mathcal{B}_2^{(k+1)} + q_2\mathcal{C}_2^{(k+1)} = q_3\mathcal{B}_2^{(k-1)} + q_4\mathcal{C}_2^{(k-1)} \tag{44}$$

where q_1, q_2, q_3, q_4 are provided by the expressions

$$\begin{aligned}
q_1 &= 12|\xi|^3\nu^2 + 4|\xi|^2\gamma\nu^2 - 12|\xi|\gamma^2\nu - 4\gamma^3, \\
q_2 &= 6\nu^2|\xi|^2 + 8\nu\gamma|\xi| + 2\gamma^2, \\
q_3 &= 12\left(\nu|\xi| - \frac{\gamma}{3}\right)(\nu|\xi| - \gamma)^2, \\
q_4 &= 6\nu^2|\xi|^2 - 8\nu\gamma|\xi| + 2\gamma^2.
\end{aligned}$$

We take (43), (44) and after some algebraic manipulations we obtain a stationary iteration

$$\begin{bmatrix} \mathcal{B}_2^{(k+1)} \\ \mathcal{C}_2^{(k+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3\nu^2|\xi|^2-4\nu|\xi|\gamma+\gamma^2}{3\nu^2|\xi|^2+4\nu|\xi|\gamma+\gamma^2} & 0 \\ 0 & \frac{3\nu^2|\xi|^2-4\nu|\xi|\gamma+\gamma^2}{3\nu^2|\xi|^2+4\nu|\xi|\gamma+\gamma^2} \end{bmatrix}}_{\Psi_{OSM}} \begin{bmatrix} \mathcal{B}_2^{(k-1)} \\ \mathcal{C}_2^{(k-1)} \end{bmatrix}. \tag{45}$$

The eigenvalue of the Schwarz iteration matrix Ψ_{OSM} of multiplicity two is provided by the formula

$$\mu_d = \frac{3\nu^2|\xi|^2 - 4\nu|\xi|\gamma + \gamma^2}{3\nu^2|\xi|^2 + 4\nu|\xi|\gamma + \gamma^2}.$$

As a consequence, the contraction factor is

$$r_{OSM}^2 = |\mu_d|^2 = \frac{|3\nu^2|\xi|^2 - 4\nu|\xi|\gamma + \gamma^2|^2}{|3\nu^2|\xi|^2 + 4\nu|\xi|\gamma + \gamma^2|^2}.$$

□

5. Non-overlapping Optimized Schwarz Algorithm-Second Order IC

More Sophisticated Schwarz methods arise by the appropriate modification of the interface conditions. We can employ the Optimized Schwarz algorithms imposing second order transmission conditions. More precisely, we go back to the algorithm prescribed by (31), (32), go to the interface conditions and instead of γ we use the symbol $\mathcal{S}$, where $\mathcal{S} = q\left(1+\xi^2\right)$.

Theorem 4. *The contraction factor of the non-overlapping Schwarz algorithm (Second order IC) is given by the mathematical expression*

$$r^2_{OSM,SOIC}(\nu, q, \xi) = \frac{|3\nu^2|\xi|^2 - 4\nu|\xi| q\left(1+\xi^2\right) + q^2\left(1+\xi^2\right)^2|^2}{|3\nu^2|\xi|^2 + 4\nu|\xi| q\left(1+\xi^2\right) + q^2\left(1+\xi^2\right)^2|^2} \tag{46}$$

where $q > 0$.

Proof. The calculations follow through in the same spirit as the Optimized Schwarz methods with the Robin transmission conditions. Instead of γ, the symbol $\mathcal{S}$ is used and the convergence factor is obtained naturally. □

Corollary 1. *The reduction factor of The Parallel Schwarz method (Dirichlet IC) given by* (5) *satisfies the following*

$$r_{PSM,D}(\xi, H) = \begin{cases} 1, & H = 0 \\ 0, & |\xi| \to +\infty \\ 0, & H \to +\infty \\ <1, & \xi > 0. \end{cases} \tag{47}$$

Proof. The result $(47)_1$ occurs by replacing $H = 0$ back to the formula (5). As a consequence, it means that the Schwarz method stagnates without overlap, something which is very usual in the literature. The $(47)_2$ is obtained by taking the limit of (5) as the Fourier frequency tends to $+\infty$. The $(47)_3$ is coming from the fact that when the overlap is sufficiently large, the convergence factor turns to be zero. The ultimate result $(47)_4$ comes from the fact that for non zero Fourier frequency, the convergence factor is strictly less than 1. □

Corollary 2. *The reduction factor of the Alternating Schwarz method (Neumann IC) given by* (22), *satisfies the relations*

$$r_{ASM,N}(\xi, H) = \begin{cases} 1, & H = 0 \\ 0, & |\xi| \to +\infty \\ 0, & H \to +\infty \\ <1, & \xi > 0. \end{cases} \tag{48}$$

Proof. The reduction factor is a function that depends on the size of the overlap and the Fourier frequency. Consequently, for zero overlap, the function becomes one and this leads to stagnation of the algorithm ($(48)_1$). When the Fourier number grows large, the function goes to zero as prescribed by $(48)_2$. Moving to $(48)_3$, a big overlap leads to better convergence because the contraction factor rapidly tends to zero. Last but not least, for finite Fourier number, the convergence factor is strictly less than one ($(48)_4$). □

Corollary 3. *The contraction factor of the non-overlapping Optimized Schwarz method (Robin IC) given by* (33) *satisfies the properties*

$$r^2_{OSM}(\xi,\nu,\gamma) = \begin{cases} 1, \gamma = 0 \\ 1, \gamma \to +\infty \\ 1,\ |\xi| \to +\infty \\ < 1,\ \xi\ \in (0,+\infty) \\ 0,\ \gamma = \gamma_+ = 3\nu|\xi|, \gamma = \gamma_- = \nu|\xi|. \end{cases} \tag{49}$$

Proof. The contraction factor depends on the kinematic viscosity, the Fourier frequency and the parameter γ. The first three properties in (49) are straighforward to obtain. Taking the Robin parameter to be zero or tend to infinity gives a stagnant Schwarz algorithm. In addition, when the Fourier frequency tends to infinity, the contraction factor becomes 1. For finite Fourier frequency (not growing to infinity) the reduction factor is less than 1. Lastly, the values of the Robin parameter that make the contraction factor zero are $\gamma_+ = 3\nu|\xi|$ and $\gamma_- = \nu|\xi|$ and can obtained by solving a trinomial equation appearing in the numerator of the contraction factor. □

Corollary 4. *If $\gamma = m\nu|\xi|$, $m \in \mathbb{Z}^+ - \{1,3\}$, then the convergence factor* (33) *does not depend on viscosity and Fourier frequency.*

Proof. By substitution, we obtain

$$\begin{aligned} r^2_{OSM}(\xi,\nu,\gamma) &= \frac{|3\nu^2|\xi|^2 - 4\nu|\xi|\gamma + \gamma^2|^2}{|3\nu^2|\xi|^2 + 4\nu|\xi|\gamma + \gamma^2|^2} \\ &= \frac{|3\nu^2|\xi|^2 - 4\nu|\xi|m\nu|\xi| + m^2|\xi|^2\nu^2|^2}{|3\nu^2|\xi|^2 + 4\nu|\xi|m\nu|\xi| + m^2|\xi|^2\nu^2|^2} \\ &= \frac{|\nu^2|\xi|^2\left(m^2 - 4m + 3\right)|^2}{|\nu^2|\xi|^2\left(m^2 + 4m + 3\right)|^2} \\ &= \frac{|m^2 - 4m + 3|^2}{|m^2 + 4m + 3|^2}. \end{aligned}$$

□

Corollary 5. *The contraction factor of non-overlapping optimised Schwarz method (Second order IC) given by* (46) *satisfies the properties*

$$r^2_{OSM,SOIC}(\nu, q, \xi) = \begin{cases} 1, q = 0 \\ 1, q \to +\infty \\ 1, \ |\xi| \to +\infty \\ < 1, \ \xi \ \in (0, +\infty) \\ 0, \ q = q_+ = \frac{3\nu|\xi|}{1+\xi^2}, \ q = q_- = \frac{\nu|\xi|}{1+\xi^2}. \end{cases} \tag{50}$$

Proof. The first three relations in (50) can directly be derived by taking the appropriate limits for the parameter q and the Fourier frequency ξ. For finite Fourier frequency, the reduction factor is less than one. Ultimately, for the indicated parameters q_- and q_+ the contraction factor becomes zero. □

Corollary 6. *If $q = m\nu|\xi|(1+\xi^2)^{-1}$, $m \in \mathbb{Z}^+ - \{1, 3\}$, then the convergence factor (46) does not depend on viscosity and Fourier frequency.*

Proof. The proof follows by substitution of the q parameter back to (46). The expression obtained is identical to the one appearing in the corollary 4. □

6. Numerical Evidence-Convergence Curves

In this section, the convergence curves are presented for each one of the Schwarz algorithms. In the cases of Oprimised Schwarz methods with Robin and second order transmission conditions, we consider $\gamma = \nu$ and $q = \nu$. The convergence curves are presented below.

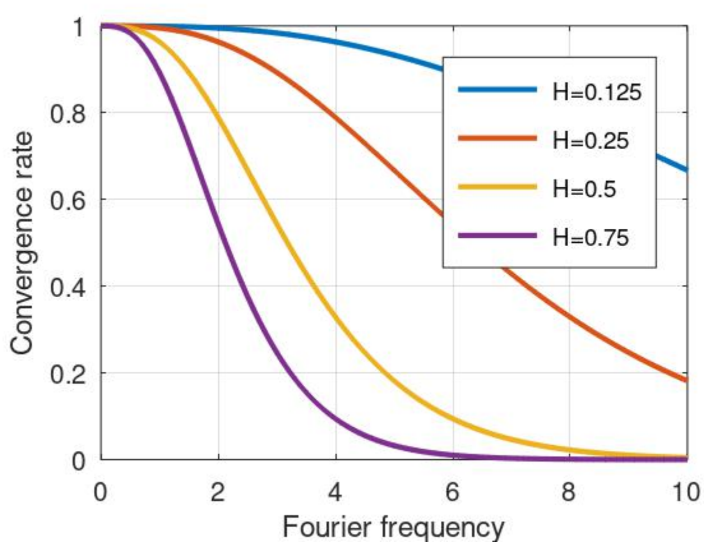

Figure 2. Convergence rate of Schwarz method using Dirichlet IC for varying overlap.

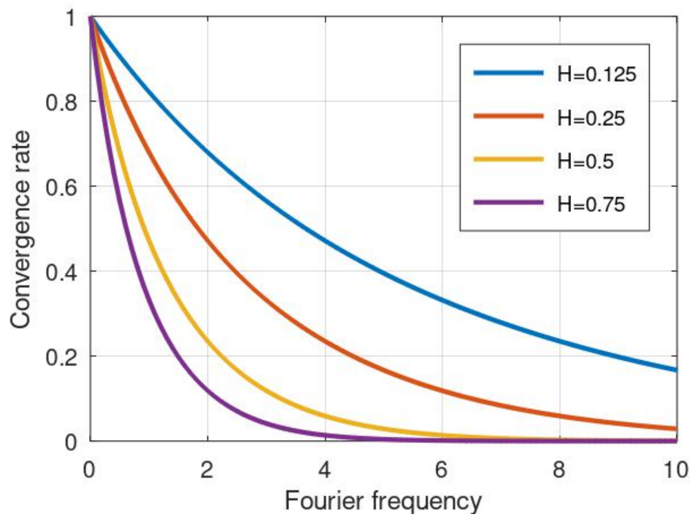

FIGURE 3. Convergence rate of Schwarz method using Neumann IC for varying overlap.

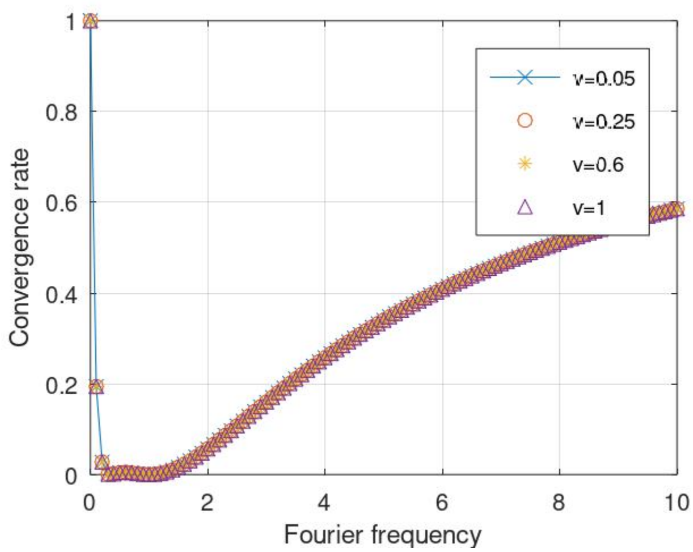

FIGURE 4. Convergence rate of Schwarz method using Robin IC for varying viscosity.

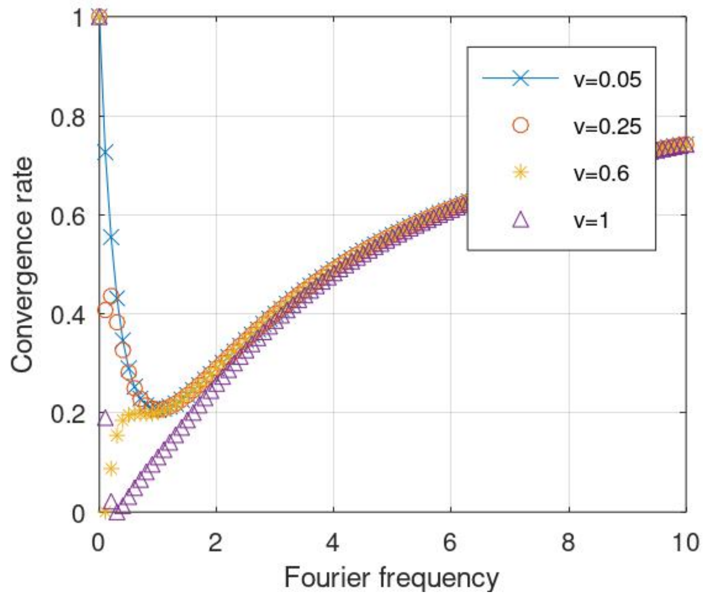

FIGURE 5. Convergence rate of Schwarz method using second order IC for varying viscosity.

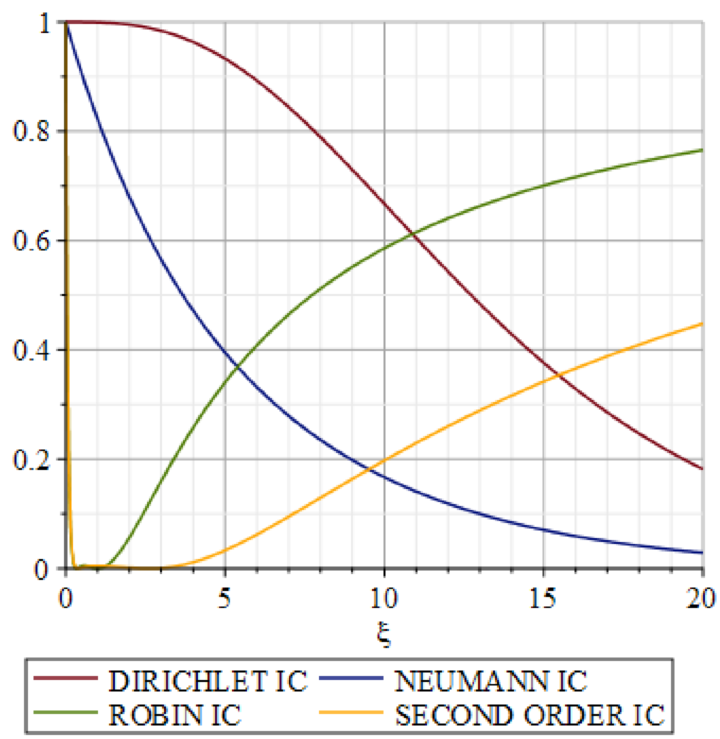

FIGURE 6. Comparison of convergence rates for all Schwarz methods.

Employing the graphs of the convergence rates in FIGURE 2 and FIGURE 3, we can compare the Schwarz methods using Dirichlet and Neumann Interface conditions. We notice that when

Neumann transmission conditions are imposed, the convergence rate decays rapidly for increasing Fourier modes, whereas using Dirichlet conditions makes the convergence slower. In addition, it is evident that when the overlap between the subdomains is larger then this enhances the overall convergence which is the expected result when using the Classical Schwarz methods. In figures 4 and 5 we have the convergence curves of non-overlapping Optimized Schwarz methods (FIGURE 4-Robin IC, FIGURE 5-Second Order IC) for varying values of the viscosity. We notice that these algorithms have better convergence for small Fourier frequencies but as the Fourier frequency grows to infinity, the reduction factor tends to 1. We also notice that when we tune the parameters of these Optimised methods, we can choose values to make the convergence rate equal to zero. Last but not least, we compare all the convergence curves and obtain FIGURE 6, which indicates that the Schwarz methods with Neummann and Dirichlet transmission conditions are slower for low frequencies, and the Optimised methods perform better in this regime. However, the problems occur when the frequencies are large which means that the reduction factor tends to 1, which is not desirable when dealing with Schwarz algorithms.

7. Conclusions

In this work we focused on the convergence analysis of the Schwarz algorithms for Stokes-Stokes configuration for varying interface conditions. We carried out the analysis using partial Fourier transform and we obtained the contraction factors for each one of the methods introduced. After conducting the convergence analysis, we notice that the Neumann conditions result in faster decay of reduction factor when ξ grows sufficiently large compared to the Dirichlet IC. The Optimised Schwarz methods have advantage in the low frequency regime, but as the Fourier number grows the contraction rate tends to one which is not desirable behavior. The convergence analysis for Stokes-Stokes configuration is useful for studying the behavior of Schwarz algorithms and getting a general insight. So far there is such analysis for Stokes-Darcy coupling [18], as a result this work could enrich the existing mathematical literature.

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