Eur. J. Math. Anal. 4 (2024) 18
doi: 10.28924/ada/ma.4.18

The Jacobi Mate of an Oval

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Abstract. We introduce and study the Jacobi mate C_j of an Euclidean oval C. We focus here on the curvature of C_j and on some examples.

1. Introduction

The enormous influence of convexity in practically every area of mathematics is widely known. We highlight the idea of *convex curve* by limiting the discussion to geometry, namely Euclidean plane geometry. The recent book [2] dedicates an entire chapter, specifically chapter 6, to this topic.

This brief note aims to relate, via the first two Jacobi elliptic functions, a second curve, C_j, to a given specific convex curve C, called *oval*. Given that these elliptic functions are 1-parametric extensions of the standard cosinus and sinus functions, which determine C, this link makes sense. The support function defining C serves as the foundation for the full analysis of this pair of curves. More specifically, we concentrate on the curvature, which is the only differential invariant for a plane curve. As possible area of applications for our results we mention the very recent (computer based) Shape Analysis or Topology Optimization.

The following is a list of the contents. The differential (and integral) geometry of the ovals is reviewed in the second section. Our new idea of *Jacobi mate* of the given oval C is presented in the next section. It is important to note that, apart from the pair (C, C_j), there exists another curve P that is naturally connected to the support function p of C and hence we will call *the support curve*. In fact, we study three curves. After the computation of P and C_j curvatures, we focus on a few cases. We point out that certain complicated calculations require software and we make use of WolframAlpha.

2. The differential geometry of Euclidean ovals

A brief overview of the differential geometry of ovals is given in this first part. Hence, our framework is the Euclidean linear space $\mathbb{E}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle)$ with to the canonical inner product:

Received: 28 May 2024.
Key words and phrases. Jacobi elliptic functions; oval; support function; curvature.

$$\langle u, v\rangle = x^1y^1 + x^2y^2, \quad u = (x^1, x^2) \in \mathbb{R}^2, \quad v = (y^1, y^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2.1}$$

Fix an open interval $I \subseteq \mathbb{R}$ and consider $C \subset \mathbb{E}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)), \quad r \in C^\infty, \quad \|r'(t)\| > 0, \quad t \in I. \tag{2.2}$$

Suppose that C is closed, simple and strictly convex; then will be called *oval*. All its geometry is provided by a smooth *support function* $p : I = [0, L > 0] \to (0, +\infty)$ with:

$$p(0) = p(L), \quad p(t) + p''(t) > 0, \quad t \in I \tag{2.3}$$

through the relations:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := R(t) \cdot \begin{pmatrix} p(t) \\ p'(t) \end{pmatrix}, R(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2) = S^1, \|r(t)\|^2 = (p(t))^2 + (p'(t))^2. \tag{2.4}$$

We point out that the function p was firstly considered by Minkowski and the function $t \to \|r(t)\| > 0$ is exactly the first Legendre transformation of the convex function p. Let $\mathcal{F}(C) = \{T, N\}$ be the Frenet frame of C and $k : I = [0, L] \to \mathbb{R}^*_+ = (0, +\infty)$ its curvature function. Then, it is well known that these main functions are given by:

$$p(t) := -\langle r(t), N(t)\rangle > 0, \quad k(t) := \frac{1}{p(t) + p''(t)} = \frac{1}{\|r'(t)\|} > 0 \tag{2.5}$$

since:

$$T(t) = (-\sin t, \cos t) = ie^{it}, \quad N(t) = iT(t) = -e^{it} = (-\cos t, -\sin t) \tag{2.6}$$

which means that the Frenet frame is universal for the set of ovals defined on the same interval I.

The geometry of the ovals has two well-known integral relations:
i) the Cauchy formula:

$$L = \int_0^{2\pi} p(t)dt. \tag{2.7}$$

ii) the Blaschke formula for the area $\mathcal{A}(C)$ enclosed by C:

$$\mathcal{A}(C) = \frac{1}{2}\int_0^{2\pi} [(p(t))^2 - (p'(t))^2]dt \leq \frac{1}{2}\int_0^{2\pi} \|r(t)\|^2 dt, \quad 4\pi\mathcal{A}(C) \leq L^2 \tag{2.8}$$

with equality in the isoperimetric inequality (2.8) provided by the circle; we will treat the circle as oval in the example 3.4.

Remarks 2.1 i) The decomposition of the position vector field r in the Frenet basis is:

$$r(t) = p'(t)T(t) - p(t)N(t). \tag{2.9}$$

A plane curve satisfying $k(t) = \frac{1}{\|r'(t)\|}$ for all t is called *flat-flow curve* in [6]. Hence, any oval is such a curve, a fact that explains the equality with 2π of its total curvature.
ii) An important tool in one-dimensional dynamics is the Fermi-Walker derivative. Let $\mathfrak{X}(C)$ be the

set of vector fields along the curve C. Then the Fermi-Walker derivative is the map ([6, p. 420]) $\nabla^{FW}: \mathfrak{X}(C) \to \mathfrak{X}(C)$:

$$\nabla^{FW}(X) := \frac{d}{dt}X + \|r'(\cdot)\|k[\langle X, N\rangle T - \langle X, T\rangle N]. \tag{2.10}$$

The Frenet frame is Fermi-Walker conserved: $\nabla^{FW}(T) = \nabla^{FW}(N) = 0$. For our oval C we derive:

$$\nabla^{FW}(r)(t) = r'(t) - \|r'(t)\|k(t)[p(t)T(t) + p'(t)N(t)] = p''(t)T(t) - p'(t)N(t). \tag{2.11}$$

Hence if we denote $r = Rotation(p)$ then the curve $t \to \nabla^{FW}(r)(t)$ is exactly the curve $Rotation(p')$.

iii) Associated to the support function p there exists *the width function* $W : [0, L/2] \to (0, +\infty)$, $W(t) := p(t) + p\left(t + \frac{L}{2}\right)$. Hence, its period is $\frac{L}{2}$.

iv) Concerning the possible relationship between the periodicity and the curvature of a plane curve a very interesting problem is solved in the paper [1]: when is a periodic function the curvature of a closed plane curve? □

3. The Jacobi mate of an oval

Fix the real number $\rho \in (-1, 1)$ as *the modulus* for the differential system ([7, p. 130]):

$$\begin{cases} \frac{du}{dt} = -wv, & u(0) = 1, \\ \frac{dv}{dt} = wu, & v(0) = 0, \\ \frac{dw}{dt} = -\rho^2 uv, & w(0) = 1. \end{cases} \tag{3.1}$$

Recall that its solutions are called *Jacobi elliptic functions* and there are usually denoted $cn(\cdot, \rho)$, $sn(\cdot, \rho)$ respectively $dn(\cdot, \rho)$; we prefer the simple notation used above. As solutions of the ODE system (3.1) these functions satisfy two remarkable identities:

$$u^2 + v^2 = 1, \quad \rho^2 v^2 + w^2 = 1. \tag{3.2}$$

Also, both functions $u(\cdot)$ and $v(\cdot)$ are periodic with $L = 4\tilde{L}$ for ([7, p. 131]):

$$\tilde{L} = \tilde{L}(\rho) := \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\rho^2 s^2)}} \tag{3.3}$$

while w is periodic of period $2\tilde{L}$. In particular, $\tilde{L}(0) = \arcsin s|_0^1 = \frac{\pi}{2}$ for the usual trigonometrical functions $cn(\cdot, 0) = \cos(\cdot)$ and $sn(\cdot, 0) = \sin(\cdot)$. The *complementary modulus* is $\rho' := \sqrt{1-\rho^2} \in (0, 1]$ and the third Jacobi function is bounded by:

$$0 < \rho' \leq w(t) \leq 1. \tag{3.4}$$

The *self-complementary case* $\rho' = \rho$ is provided by $\rho = \frac{1}{\sqrt{2}}$ and being in the interval $(0, 1)$ is the eccentricity of an ellipse, called *self-complementary* and studied in [5].

Due to the increasing interest in the geometry of ovals this short note defines the Jacobi mate for the given oval C. As basic tool we use the new rotation matrix:

$$Jacobi(t,\rho) := \begin{pmatrix} u(t) & -v(t) \\ v(t) & u(t) \end{pmatrix} \in SO(2) = S^1. \tag{3.5}$$

Definition 3.1 The curve C_j is the ρ*-Jacobi mate* of C if its parametrization is:

$$r_j(t) = \begin{pmatrix} x_j \\ y_j \end{pmatrix}(t) := Jacobi(t,\rho)\begin{pmatrix} p \\ p' \end{pmatrix}(t) = \begin{pmatrix} p(t)u(t) - p'(t)v(t) \\ p'(t)u(t) + p(t)v(t) \end{pmatrix}, \quad t \in I = [0,L]. \tag{3.6}$$

Since the derivative of r_j is:

$$r_j'(t) = (p'(t)u(t)(1-w(t)) - v(t)(p(t)w(t)+p''(t)), p'(t)v(t)(1-w(t)) + u(t)(p(t)w(t)+p''(t))) \tag{3.7}$$

it results:

$$\|r_j'(t)\|^2 = (p'(t))^2[1-w(t)]^2 + [p(t)w(t)+p''(t)]^2 \in ((\rho' p(t)+p''(t))^2, (p'(t))^2+[p(t)+p''(t)]^2) \tag{3.8}$$

and then C_j is a regular curve. It results also immediately:

$$\begin{cases} x_j'' = p''u(1-2w) + p'v(\rho^2u^2 + w^2 - 2w) + v(p\rho^2uv - p''') - pw^2u \\ y_j'' = p''v(1-2w) + p'u(\rho^2v^2 - w^2 + 2w) - u(p\rho^2uv - p''') - pw^2v \end{cases} \tag{3.9}$$

and then, considering the map $(\cdot,\rho) \to r_j(\cdot)$ as a flow of curves, we compute its first derivative with a possible application to a parabolic flow (for example, of curve shortening type, see the chapter 2 in [3]):

$$\begin{cases} \frac{\partial}{\partial\rho} r_j''(t) = 2\rho u(t)v(t)[p'(t)(u(t),v(t)) + p(t)(v(t),-u(t))] = 2\rho u(t)v(t)[-ir_j(t)] = 2w'(t)[ir_j(t)], \\ \|\frac{\partial}{\partial\rho} r_j''(t)\| = 2|\rho||u(t)||v(t)|\|r(t)\|. \end{cases} \tag{3.10}$$

Therefore, $\frac{\partial}{\partial\rho} r_j''(t)$ is orthogonal to $r_j(t)$, for all $t \in [0,L]$.

Remark 3.2 We point out that following the approach of [8] we can think C and C_j as the Euclidean and Jacobi *deformations* of the *support curve* $t \to P(t) := (p(t), p'(t))$. We have $\|r(t)\| = \|P(t)\| = \|r_j(t)\|$, for all t. The expression of P recalls the well-known Weierstrass parametrization $(\wp(u), \wp'(u))$ of the elliptic curve $E(g_2,g_3) : y^2 = 4x^3 - g_2x - g_3$; see [9, p. 77]. □

Our main theoretical result computes the curvature of the mate C_j through a long but straightforward computation:

Theorem 3.3 i) *If p is not a constant then the support curve P is a regular one having the Euclidean curvature:*

$$k_P(t) = \frac{p'(t)p'''(t) - (p''(t))^2}{[(p'(t))^2 + (p''(t))^2]^{\frac{3}{2}}}. \tag{3.11}$$

Let $r(t_0)$ be a vertex of the oval C i.e. $p'''(t_0) = -p'(t_0)$. Then the curvature of P in t_0 is:

$$k_P(t_0) = \frac{-1}{[(p'(t_0))^2 + (p''(t_0))^2]^{\frac{1}{2}}} < 0.$$

ii) *The curvature of the ρ-Jacobi mate C_j of the oval C is a quadratic function in ρ:*

$$k_j = \frac{(p')^2(1-w)(2w-w^2) + p'[(1-w)p''' - \rho^2 uv(p+p'')] + [pw^2 + p''(2w-1)](pw+p'')}{[(p')^2(1-w)^2 + (pw+p'')^2]^{\frac{3}{2}}}. \tag{3.12}$$

If the modulus ρ is zero then $w \equiv 1$ and k_j reduces to the usual curvature k from (2.5). *Moreover, for the flow interpretation before the remark 3.2 we have*:

$$\frac{\partial^2 k_j(t)}{\partial \rho^2}|_{\rho=0} = -2u(t)v(t)p'(t)[k(t)]^2. \tag{3.13}$$

We focus now on some concrete examples.

Example 3.4 The circle $\mathcal{C}(O, R>0)$ of the Euclidean plane geometry is the oval provided by the constant support function $p \equiv R$ and hence $W \equiv 2R$; the curve P consists in the unique point $(R, 0)$. Its ρ-Jacobi mate coincides cu $\mathcal{C}(O, R)$, hence $k_j = k \equiv \frac{1}{R}$, but now with the parametrization:

$$circle_j(t) = R(u(t), v(t)), circle_j'(t) = Rw(t)(-v(t), u(t)), \|circle_j'(t)\| = Rw(t) \in [R\rho', R]. \tag{3.14}$$

Hence, the Frenet frame is:

$$T(t) = (-v(t), u(t)), \quad N(t) = iT(t) = (-u(t), -v(t)) \tag{3.15}$$

as natural generalization of (2.6). By defining a new function:

$$W(t) = \int_0^t w(\lambda)d\lambda \tag{3.16}$$

we can write the parametrization by arc-length:

$$circle_j(s) = R\left(u \circ W^{-1}\left(\frac{s}{R}\right), v \circ W^{-1}\left(\frac{s}{R}\right)\right), \quad s \in [0, 2\pi R]. \tag{3.17}$$

The value of $\tilde{L}$ from (3.3) in the self-complementary case is:

$$\tilde{L}\left(\frac{1}{\sqrt{2}}\right) \simeq 1.85 > \frac{\pi}{2} \simeq 1.57 \tag{3.18}$$

while the second identity from (3.2) provides, in the case $\rho \neq 0$, a second Jacobi parametrization of the circle:

$$Scircle_j(t) = R(\rho v(t), w(t)), \quad Scircle_j'(t) = R\rho u(t)(w(t), -\rho v(t)). \tag{3.19}$$

Now, this second parametrization has singularities, namely the zeros $\tilde{L}$, $3\tilde{L}$ of the function u. □

Example 3.5 Fix the smooth real function $p(t) := R - \cos 3t$; hence $p(t) = p(t+2\pi)$ and again the width is constant $W \equiv 2R$. In [4, p. 23] it is proved that if $R > 8$ then p is the support function of an oval C. If $R > 8$ is a positive integer then the oval C contains the integral point $(-(R+1), 0)$ corresponding to $t = \pi$ while the curve P contains the 4 integral points $(R-1, 0)$,

$(R, 3)$, $(R+1, 0)$, $(R, -3)$ corresponding respectively to $t = 0$, $t = \frac{\pi}{2}$, $t = \pi$ and $t = \frac{3\pi}{2}$. With the derivatives:

$$p'(t) = 3\sin 3t, \quad p''(t) = 9\cos 3t, \quad p'''(t) = -27\sin 3t \tag{3.20}$$

it results the curvatures:

$$k_P(t) = \frac{-3}{[(\sin 3t)^2 + 9(\cos 3t)^2]^{\frac{3}{2}}} < 0, \quad k(t) = \frac{1}{R + 8\cos 3t} \in \left[\frac{1}{R+8}, \frac{1}{R-8}\right]. \tag{3.21}$$

We note that k_P does not depend on R while the curvature k solves the differential equation $\frac{\partial k}{\partial R} = -k^2$. The Cauchy and the Blaschke formulae give:

$$L(C) = 2\pi R, \quad \mathcal{A}(C) = \pi(R^2 - 4) > 60\pi. \tag{3.22}$$

The length of the curve P is:

$$L(P) = 3\int_0^{2\pi} \sqrt{(\sin 3t)^2 + 9(\cos 3t)^2} dt = 3\int_0^{2\pi} \sqrt{5 + 4\cos 6t} dt \simeq 40.09 \tag{3.23}$$

and we point out that the argument $3t$ involved in its components recalls the Cayley sextic, which is not an oval but a closed curve:

$$Cayley(t) := \cos^3 t(\cos 3t, \sin 3t), \quad t \in [0, 2\pi], \quad L(Cayley) = 3\pi. \tag{3.24}$$

For the ρ-Jacobi mate we compute only the velocity since its curvature has a complicated expression:

$$\begin{cases} \|r_j'(t)\|^2 = 9(\sin 3t)^2[1 - w(t)]^2 + [9\cos 3t + w(t)(R - \cos 3t)]^2, \\ 0 < (\rho' R + (9 - \rho')\cos 3t)^2 < \|r_j'(t)\|^2 < 81(\sin 3t)^2 + (R + 8\cos 3t)^2. \end{cases} \tag{3.25}$$

□

Example 3.6 For $\alpha \in [1, +\infty)$ the 2π-periodic function $p_\alpha : [0, 2\pi] \to \mathbb{R}^*_+$, $p_\alpha(t) := \frac{1}{\alpha}\sqrt{\alpha^4 \cos^2 t + \sin^2 t}$ is the support function of an ellipse since:

$$p_\alpha(t) + p''_\alpha(t) = \frac{\alpha^3}{(\alpha^4 \cos^2 t + \sin^2 t)^{\frac{3}{2}}} > 0. \tag{3.26}$$

Now, the width function is non-constant being $2p_\alpha$. As example, with WolframAlpha we obtain the length $L\left(P_{\alpha=2}\right) \simeq 45.51$. The RHS of the inequality (3.8) reads:

$$\|r_j'(t)\|^2 < \frac{(\alpha^4 - 1)^2}{2\alpha^2(\alpha^4 \cos^2 t + \sin^2 t)} + \frac{\alpha^6}{(\alpha^4 \cos^2 t + \sin^2 t)^3}. \tag{3.27}$$

With the same possible flow interpretation in mind we compute the first derivative of the support function:

$$\frac{\partial p_\alpha}{\partial \alpha}(t) = \frac{\alpha^4 \cos^2 t - \sin^2 t}{\alpha^2 \sqrt{\alpha^4 \cos^2 t + \sin^2 t}}. \tag{3.28}$$

We note also that for $\alpha > 1$ the given support function does not has an *indicatrix* i.e. the plane curve defined implicitly by $\{(\alpha, t) \in \mathbb{R}^2; p_\alpha(t) = 1\}$ is empty. The same fact holds for the support function of the previous example when $R > 8$. □

References

[1] J. Arroyo, O. J. Garay, J. J. Mencia, When is a periodic function the curvature of a closed plane curve, Am. Math. Mon. 115 (2005), 405-414. https://doi.org/10.1080/00029890.2008.11920543

[2] H. Alencar, W. Santos, G. Silva Neto, Differential geometry of plane curves, American Mathematical Society, Providence, Rhode Island, 2022. https://doi.org/10.1090/stml/096.

[3] B. Andrews, B. Chow, C. Guenther, M. Langford, Extrinsic geometric flows, American Mathematical Society, Providence, Rhode Island, 2020. https://doi.org/10.1090/gsm/206.

[4] W. Cieślak, W. Mozgawa, P. Wlaź, On the closest distance between a point and a convex body, Bull. Soc. Sci. Lettr. Łódź, Sér.: Rech. Déform. 67 (2017), 21-30. https://doi.org/10.26485/0459-6854/2017/67.2/2.

[5] M. Crasmareanu, Magic conics, their integer points and complementary ellipses, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. 67 (2021), 129-148.

[6] M. Crasmareanu, The flow-curvature of plane parametrized curves, Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat. 72 (2023), 417-428. https://doi.org/10.31801/cfsuasmas.1165123.

[7] R.H. Cushman, L.M. Bates, Global aspects of classical integrable systems, Springer Basel, Basel, 2015. https://doi.org/10.1007/978-3-0348-0918-4.

[8] B. Mazur, Perturbations, deformations, and variations (and "near-misses") in geometry, physics, and number theory, Bull. Amer. Math. Soc. 41 (2004), 307–336. https://doi.org/10.1090/S0273-0979-04-01024-9.

[9] R. Takloo-Bighash, A pythagorean introduction to number theory: right triangles, sums of squares, and arithmetic, Springer, Cham, 2018. https://doi.org/10.1007/978-3-030-02604-2.