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On η-Local Functions in Ideal Topological Spaces

Junvon A. Almocera*, Lezel M. Tutanes

Department of Mathematics, College of Arts and Sciences, Bukidnon State University, Malaybalay City, Bukidnon, Philippines

1901103646@student.buksu.edu.ph, lezeltutanes@buksu.edu.ph

**Correspondence: 1901103646@student.buksu.edu.ph*

Abstract. This study introduces and investigates a new local function called η-local function in ideal topological space (X, τ, I) by using the notion of η-open sets in topological space (X, τ). The operator $(\cdot)^*_\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as $(\cdot)^*_\eta(A) = A^*_\eta = \{x \in X : A \cap U \notin I$ for every $U \in \eta\text{-}O(x)\}$ for each $A \subseteq X$, where $\eta\text{-}O(x)$ is the set of all η-open subset of X containing x. This study establishes some properties of A^*_η including its relationships to the local function and local function Γ^* in ideal topological space (X, τ, I). This study also introduces a new type of closure called the η-local closure in ideal topological space (X, τ, I) which is denoted by $Cl^*_\eta(A)$ for each $A \subseteq X$. Furthermore, this study establishes some properties of the η-local closure.

1. Introduction

The concept of ideal topological spaces was first studied by Kuratowski [4] and Vaidyanathaswamy [9]. The notion of topological spaces with ideals were investigated by Jankovic [3]. Thereafter, the study of ideal topological spaces attracts the attention of many topologists. Recently, Al-omari [1] have introduced and investigated the notion of local function Γ^* in an ideal topological space and showed that Γ^* is equivalent to the δ-local function due to Hatir et al. [2]. In this paper, the researcher defined a new type of local function called the η-local function in ideal topological spaces by using the η-open set of Subbulakshmi [8] and established some of its properties, including its relationship to the local function and local function Γ^* in ideal topological spaces. Subsequently, the η-local closure has been defined, and some of the properties are established.

2. Preliminaries

Throughout this paper (X, τ) and (X, τ, I) denote a topological space and an ideal topological space, respectively. The members of τ are called open sets and their complement are called closed sets. For any subset A of X, the closure and interior of A are denoted by $cl(A)$ and

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$int(A)$, respectively. A subset A of a space (X,τ) is said to be η-open (resp. η-closed) [8] if $A \subseteq int(cl(int(A))) \cup cl(int(A))$ (resp. $A \supseteq cl(int(cl(A))) \cap int(cl(A))$). The η-closure of A is defined by the intersection of all η-closed sets containing the set A and it is denoted by η-$cl(A)$ [8]. A subset A of a space (X,τ) is said to be semi-open [6] if $A \subseteq cl(int(A))$. A subset A of a space (X,τ) is said to be regular-open [7] if $A = int(cl(A))$. The familiy of all η-open (resp. semi-open, regular open) sets in X is denoted by η-$O(X)$ (resp. $SO(X)$, $RO(X)$).

An ideal I [5] on a topological spaces (X,τ) is a nonempty collection of subsets of X, which satisfies *(i)* $A \in I$ and $B \in I$ implies $A \cup B \in I$; and *(ii)* $A \in I$ and $B \subseteq A$ implies $B \in I$. Then the triplet (X,τ,I) is called an ideal topological space. If $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : \mathcal{P}(X) \to \mathcal{P}(X)$ called a local function [3,9] of A with respect to τ and I is defined as follows: for $A \subseteq X$, $A^*(I,\tau) = \{x \in X : U \cap A \notin I, \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. $A^*(I,\tau)$ can simply be written as A^*. For every ideal topological space, there exists a topology $\tau^*(I,\tau)$ or briefly τ^* [3], finer than τ, generated by the $\mathcal{B}(I,\tau) = \{U \setminus J : U \in \tau \text{ and } J \in I\}$, however, $\mathcal{B}(I,\tau)$ is not a topology in general. Additionally, $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator [5] for τ^*. A subset A of an ideal topological spaces is τ^*-closed set or $*$-closed set [3] if $A^* \subseteq A$. Let (X,τ,I) be an ideal topological spaces and A be a subset of X. Then $\Gamma^*(A)(I,\tau) = \{x \in X : A \cap U \notin I, \text{ for every } U \in RO(X)\}$ where $RO(X) = \{U \in RO(X) : x \in U\}$. $\Gamma^*(A)(I,\tau)$ can simply be denoted as $\Gamma^*(A)$ [1].

3. η-LOCAL FUNCTIONS

Definition 1. *Let (X,τ,I) be an ideal topological space. Then the operator $(\cdot)^*_\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as for $A \subseteq X$, $A^*_\eta\big(I,\eta\text{-}O(X)\big) = \{x \in X : A \cap U \notin I, \text{ for every } U \in \eta\text{-}O(x)\}$ where $\eta\text{-}O(x) = \{U \in \eta\text{-}O(X) : x \in U\}$ is called the η-$local\ function$ of A with respect to I and η-$O(X)$. $A^*_\eta\big(I,\eta\text{-}O(X)\big)$ can simply be denoted by A^*_η.*

Example 1. *Let (X,τ,I) be an ideal topological where $X = \{a,b,c,d\}$, $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$, and $I = \{\varnothing, \{c\}, \{d\}, \{c,d\}\}$. Then $\eta\text{-}O(X) = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,c,d\}, \{a,b,d\}, \{b,c,d\}\}$. Now, let $A = \{a,b,d\}$. Then by Definition 1, $A^*_\eta = \{a,b,c,d\} = X$.*

Theorem 1. *Let (X,τ,I) be an ideal topological space and A, B be subsets of X. Then for any η-local functions, the following properties hold:*

(i) *if $A \subseteq B$, then $A^*_\eta \subseteq B^*_\eta$;*
(ii) *$(A \cap B)^*_\eta \subseteq A^*_\eta \cap B^*_\eta$; and*
(iii) *$A^*_\eta \cup B^*_\eta \subseteq (A \cup B)^*_\eta$;*
(iv) *if $A = \varnothing$, then $A^*_\eta = \varnothing$;*
(v) *if $A^*_\eta \cap B \notin I$, then $A^*_\eta \cap B \neq \varnothing$;*
(vi) *$\big(A^*_\eta\big)^*_\eta \subseteq A^*_\eta$;*

(vii) *if $A \in I$, then $A^*_\eta = \varnothing$;*

(iix) *if $I = \{\varnothing\}$, then $A^*_\eta = \eta\text{-}cl(A)$;*

(ix) *if $I = \mathcal{P}(X)$, then $A^*_\eta = \varnothing$; and*

(x) *$A^*_\eta \subseteq \eta\text{-}cl(A)$;*

Proof.

(i) Let $A, B \subseteq X$ and $A \subseteq B$. Suppose $x \notin B^*_\eta$, then there exist $U \in \eta\text{-}O(x)$ such that $B \cap U \in I$. Since $A \subseteq B$, $A \cap U \subseteq B \cap U \in I$, by Definition of ideal, $A \cap U \in I$. Hence, $x \notin A^*_\eta$.

(ii) Let $A, B \subseteq X$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Theorem 1 *(i)*, $(A \cap B)^*_\eta \subseteq A^*_\eta$ and $(A \cap B)^*_\eta \subseteq B^*_\eta$, respectively. Hence, $(A \cap B)^*_\eta \subseteq A^*_\eta \cap B^*_\eta$.

(iii) Let $A, B \subseteq X$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by Theorem 1 *(i)*, $A^*_\eta \subseteq (A \cup B)^*_\eta$ and $B^*_\eta \subseteq (A \cup B)^*_\eta$, respectively. Hence, $A^*_\eta \cup B^*_\eta \subseteq (A \cup B)^*_\eta$.

(iv) Let $A = \varnothing$. Suppose $A^*_\eta \neq \varnothing$. Then there exists $x \in A^*_\eta$. It follows that $A \cap U = \varnothing \cap U = \varnothing \notin I$ for every $U \in \eta\text{-}O(x)$. Since I is an ideal, $\varnothing \in I$ which is a contradiction.

(v) Let $A^*_\eta \cap B \notin I$. Suppose $A^*_\eta \cap B = \varnothing$. Note that by definition of ideal, $\varnothing \in I$ for any ideal I. Now, since $A^*_\eta \cap B = \varnothing$ and I is an ideal, $A^*_\eta \cap B = \varnothing \in I$ implies $A^*_\eta \cap B \in I$, a contradiction.

(vi) Let $x \in \left(A^*_\eta\right)^*_\eta$. Then, for every $U \in \eta\text{-}O(x)$, $U \cap A^*_\eta \notin I$ and hence, by *(ii)*, $U \cap A^*_\eta \neq \varnothing$. Now, let $y \in U \cap A^*_\eta$. Then, $U \in \eta\text{-}O(y)$ and $y \in A^*_\eta$. Hence, we have $U \cap A \notin I$. Note that $U \in \eta\text{-}O(x)$ and $U \cap A \notin I$. It follows that $x \in A^*_\eta$. Therefore, $\left(A^*_\eta\right)^*_\eta \subseteq A^*_\eta$.

(vii) Let $A \in I$. Suppose $A^*_\eta \neq \varnothing$. Then there exists an element $x \in A^*_\eta$. Then $A \cap U \notin I$ for every $U \in \eta\text{-}O(x)$. Now, Since, $A \cap U \subseteq A \in I$ and I is an ideal, $A \cap U \in I$ which is a contradiction.

(iix) Let $I = \{\varnothing\}$. Suppose that $A^*_\eta \neq \eta\text{-}cl(A)$. Let $\eta\text{-}cl(A) \subset A^*_\eta$, then there exists an element $x \in A^*_\eta$ and $x \notin \eta\text{-}cl(A)$. It follows that for every $A \subseteq X$, since $x \in A^*_\eta$, $A \cap U \notin I$ for every $U \in \eta\text{-}O(x)$. Since $I = \{\varnothing\}$, $A \cap U \neq \varnothing$ for every $U \in \eta\text{-}O(x)$. Note that $U \in \eta\text{-}O(x)$ means $x \in U$ where U is η-open set. Since, $x \notin \eta\text{-}cl(A)$, $x \notin \bigcap\{K : K \text{ is } \eta\text{-closed and } A \subseteq K\}$. It follows that $x \notin K$ for some η-closed set K such that $A \subseteq K$. Hence, $x \in K^c$ for some η-open set K^c such that $A \cap K^c = \varnothing$. It implies that there exists an η-open set K^c where $x \in K^c$ and $A \cap K^c = \varnothing$, a contradiction.

(ix) Let $I = \mathcal{P}(X)$. Note that $A \subseteq X$, then $A \in \mathcal{P}(X)$. Since $I = \mathcal{P}(X)$, $A \in I$. Hence, by Theorem 1 *(vii)*, $A^*_\eta = \varnothing$.

(x) Let $x \notin \eta\text{-}cl(A)$. Then, $x \notin \bigcap\{K : K \text{ is } \eta\text{-closed and } A \subseteq K\}$. It follows that $x \notin K$ for some η-closed set K such that $A \subseteq K$. Hence, $x \in K^c$ for some η-open set K^c such that $A \cap K^c = \varnothing$. It implies that there exists $K^c \in \eta\text{-}O(x)$ such that $A \cap K^c = \varnothing$, and by definition of ideal, $\varnothing \in I$ for any ideal I. Hence, $A \cap K^c \in I$ for some $K^c \in \eta\text{-}O(x)$. This shows that $x \notin A^*_\eta$. □

Remark 1. *Let (X, τ, I) be an ideal topological space and A be any subset of X. Then for any η-local functions, the following properties hold:*

(i) *The reverse inclusion of Theorem 1 (iii) need not be true in general.*
(ii) *Neither $A \subseteq A^*_\eta$ nor $A^*_\eta \subseteq A$ in general.*
(iii) *A^*_η is an η-closed set iff $A^*_\eta = \eta\text{-}cl(A^*_\eta)$.*

In order to verify Remark 1 (i) and (ii), the following examples are shown.

Example 2.

(i) *Consider the ideal topological space (X, τ, I), where $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}\}$, and $I = \{\varnothing, \{a\}\}$. Then the η-open sets are $\varnothing$, X, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$, and $\{b, c, d\}$. Now, let $A = \{b\}$ and $B = \{c\}$, then $A \cup B = \{b, c\}$. Then by applying Definition 1, $A^*_\eta = \{b\}$, $B^*_\eta = \{c\}$, and $(A \cup B)^*_\eta = X$. Observe that $(A \cup B)^*_\eta = X$ and $A^*_\eta \cup B^*_\eta = \{b, c\}$. These shows that $(A \cup B)^*_\eta \nsubseteq A^*_\eta \cup B^*_\eta$.*
(ii) *Consider the ideal topological space (X, τ, I), where $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{b\}, \{c\}, \{b, c, d\}\}$, and $I = \{\varnothing, \{c\}\}$. Then the η-open sets are $\varnothing$, X, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, and $\{b, c, d\}$. Let $A, B \subset X$ where, $A = \{a, c, d\}$ and $B = \{a, b\}$. Then by Definition 1, $A^*_\eta = \{a, d\}$ and $B^*_\eta = \{a, b, d\}$. Obeserve that $A \nsubseteq A^*_\eta$ and $B^*_\eta \nsubseteq B$.*

Theorem 2. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then for any η-local functions, the following properties hold:*

(i) $(A \setminus B)^*_\eta \setminus B^*_\eta \subseteq A^*_\eta \setminus B^*_\eta$;
(ii) *if* $B \in I$, *then* $(A \cup B)^*_\eta = A^*_\eta = (A \setminus B)^*_\eta$;
(iii) $(A \setminus B)^*_\eta \cup (B \setminus A)^*_\eta \subseteq (A \cup B)^*_\eta$;
(iv) *if* $U \subseteq X$, *then* $U \cap (U \cap A)^*_\eta \subseteq U \cap A^*_\eta$;
(v) *if* $U \in I$, *then* $(A \cap U)^*_\eta = \varnothing$;
(vi) *if A is an η-closed set, then* $A^*_\eta \subseteq A$;
(vii) $\left(A \cap A^*_\eta\right)^*_\eta \subseteq A^*_\eta$;
(iix) *if* $A \cup B \in I$, *then* $(A \cup B)^*_\eta = A^*_\eta \cup B^*_\eta = \varnothing$.
(ix) $A^*_\eta = \eta\text{-}cl(A^*_\eta) \subseteq \eta\text{-}cl(A)$ *and A^*_η is an η-closed set; and*
(x) *if* $A \subseteq A^*_\eta$, *then* $A^*_\eta = \eta\text{-}cl(A^*_\eta) = \eta\text{-}cl(A)$.

Proof.

(i) Let $A, B \subseteq X$. Since $A \setminus B \subseteq A$, by Theorem 1 *(i)*, $(A \setminus B)^*_\eta \subseteq A^*_\eta$ implies that $(A \setminus B)^*_\eta \setminus B^*_\eta \subseteq A^*_\eta \setminus B^*_\eta$.
(ii) Let $B \in I$. Suppose $x \in (A \cup B)^*_\eta$. Then for every $U \in \eta\text{-}O(x)$, $(A \cup B) \cap U \notin I$. Note that $(A \cap U) \cup (B \cap U) = (A \cup B) \cap U \notin I$ implies that $(A \cap U) \cup (B \cap U) \notin I$. It shows that $A \cap U \notin I$

or $B \cap U \notin I$, or both, and as a result, $x \in A^*_\eta$ or $x \in B^*_\eta$, or both. Hence, $x \in A^*_\eta \cup B^*_\eta$. Now, note that $B \in I$, then by Theorem 1 *(vii)*, $B^*_\eta = \varnothing$. Thus, $x \in A^*_\eta \cup B^*_\eta = A^*_\eta \cup \varnothing = A^*_\eta$. This implies that $x \in A^*_\eta$. Hence, it shows that $(A \cup B)^*_\eta \subseteq A^*_\eta$. In contrast, since $A \subseteq A \cup B$, by Theorem 1 *(i)*, it implies that $A^*_\eta \subseteq (A \cup B)^*_\eta$. Consequently, as a result, $A^*_\eta = (A \cup B)^*_\eta$. Now, suppose that $(A \setminus B)^*_\eta \neq A^*_\eta$. Let $(A \setminus B)^*_\eta \subset A^*_\eta$. Then there exists an element $x \in A^*_\eta$ such that $x \notin (A \setminus B)^*_\eta$. Note that $x \in A^*_\eta$ implies that for every $A \subseteq X$, $A \cap U \notin I$ for every $U \in \eta\text{-}O(x)$. Now, since $x \notin (A \setminus B)^*_\eta$, there exists $U \in \eta\text{-}O(x)$ such that $(A \setminus B) \cap U \in I$. Note that $(A \setminus B) \cap U = (A \cap U) \setminus (B \cap U) \in I$ and $B \cap U \subseteq B \in I$. Now, since I is an ideal, $B \cap U \in I$ and $\big[(A \cap U) \setminus (B \cap U)\big] \cup (B \cap U) \in I$, respectively. Again, note that $\big[(A\cap U)\setminus(B\cap U)\big]\cup(B\cap U) = (A\cap U)\cup(B\cap U) \in I$. Since $(A\cap U) \subseteq (A\cap U)\cup(B\cap U) \in I$, it implies that $A \cap U \in I$. So, there exists $U \in \eta\text{-}O(x)$ such that $A \cap U \in I$ which is a contradiction.

(iii) Let $A, B \subseteq X$. Note that $A \setminus B \subseteq A$ and $B \setminus A \subseteq B$. Then by Theorem 1 *(i)*, $(A \setminus B)^*_\eta \subseteq A^*_\eta$ and $(B \setminus A)^*_\eta \subseteq B^*_\eta$, and so, $(A \setminus B)^*_\eta \cup (B \setminus A)^*_\eta \subseteq A^*_\eta \cup B^*_\eta$. Now, by Theorem 1 *(iii)*, $A^*_\eta \cup B^*_\eta \subseteq (A \cup B)^*_\eta$. Hence, $(A \setminus B)^*_\eta \cup (B \setminus A)^*_\eta \subseteq (A \cup B)^*_\eta$.

(iv) Let $U \subseteq X$. Since $U \cap A \subseteq A$, by Theorem 1 *(i)*, $(U \cap A)^*_\eta \subseteq A^*_\eta$, and hence, $U \cap (U \cap A)^*_\eta \subseteq U \cap A^*_\eta$.

(v) Let $U \in I$. Since $A \cap U \subseteq U \in I$ and I is an ideal, $A \cap U \in I$. Hence, by Theorem 1 *(vii)*, $(A \cap U)^*_\eta = \varnothing$.

(vi) Let A be an η-closed set. Then $A = \eta\text{-}cl(A)$. Now, note that by Theorem 1 *(x)*, $A^*_\eta \subseteq \eta\text{-}cl(A)$. Hence, $A^*_\eta \subseteq A$.

(vii) Let $A \subseteq X$. Since $A \cap A^*_\eta \subseteq A^*_\eta$, by Theorem 1 *(i)*, $\big(A \cap A^*_\eta\big)^*_\eta \subseteq \big(A^*_\eta\big)^*_\eta$. Note that by Theorem 1 *(vi)*, $\big(A^*_\eta\big)^*_\eta \subseteq A^*_\eta$. Therefore, $\big(A \cap A^*_\eta\big)^*_\eta \subseteq A^*_\eta$.

(iix) Let $A \cup B \in I$. Since $A \cup B \in I$ and I is an ideal, $A \in I$ and $B \in I$. These imply by Theorem 1 *(vii)*, $(A \cup B)^*_\eta = \varnothing$, $A^*_\eta = \varnothing$, and $B^*_\eta = \varnothing$. Therefore, $(A \cup B)^*_\eta = A^*_\eta \cup B^*_\eta = \varnothing$.

(ix) Suppose that $A^*_\eta \neq \eta\text{-}cl(A^*_\eta)$. Let $\eta\text{-}cl(A^*_\eta) \subset A^*_\eta$. Then there exists an element $x \in A^*_\eta$ such that $x \notin \eta\text{-}cl(A^*_\eta)$. Note that since $x \in A^*_\eta$, for every $U \in \eta\text{-}O(x)$, $A \cap U \notin I$. Now, note that $x \notin \eta\text{-}cl(A^*_\eta)$. Then $x \notin \bigcap\{K : K \text{ is } \eta\text{-closed and } A^*_\eta \subseteq K\}$. This shows that $x \notin K$ for some η-closed set K such that $A^*_\eta \subseteq K$. This implies that $x \in K^c$ for some η-open set K^c such that $K^c \cap A^*_\eta = \varnothing$. Note that $x \in K^c$ and $K^c \cap A^*_\eta = \varnothing$, then it follows that $x \notin A^*_\eta$, and so, for some $K^c \in \eta\text{-}O(x)$, $A \cap K^c \in I$, which is a contradiction. Consequently, $A^*_\eta = \eta\text{-}cl(A^*_\eta)$, then by Remark 1 (iii), A^*_η is an η-closed set. Now, note that $A^*_\eta = \eta\text{-}cl(A^*_\eta)$ and by Theorem 1 *(x)*, hence, $A^*_\eta = \eta\text{-}cl(A^*_\eta) \subseteq \eta\text{-}cl(A)$.

(x) Let $A \subseteq A^*_\eta$. Suppose that $x \in \eta\text{-}cl(A)$. Then $x \in \bigcap\{K : K \text{ is } \eta\text{-closed and } A \subseteq K\}$. This shows that $x \in K$ for every η-closed set K such that $A \subseteq K$. Note that by Theorem 2 *(ix)*, A^*_η is an η-closed set. Now, note that since $A \subseteq A^*_\eta$ and A^*_η is an η-closed set,

$A^*_\eta \in \{K : K \text{ is } \eta\text{-closed and } A \subseteq K\}$. This implies that $x \in A^*_\eta$. Hence, $\eta\text{-}cl(A) \subseteq A^*_\eta$. As a result, by Theorem 1 *(x)* and 2 *(ix)*, $A^*_\eta = \eta\text{-}cl(A^*_\eta) = \eta\text{-}cl(A)$. □

Theorem 3. *Let (X, τ, I) be an ideal topological space where $\eta\text{-}O(X)$ is closed under any two intersections. Then for any A, B subsets of X, the following properties hold:*

(i) $(A \cup B)^*_\eta = A^*_\eta \cup B^*_\eta$;

(ii) *for* $U \in \eta\text{-}O(x)$, $U \cap A^*_\eta = U \cap (U \cap A)^*_\eta \subseteq (U \cap A)^*_\eta$; *and*

(iii) $A^*_\eta \setminus B^*_\eta = (A \setminus B)^*_\eta \setminus B^*_\eta \subseteq (A \setminus B)^*_\eta$.

Proof.

(i) Let $\eta\text{-}O(X)$ be closed under any two intersections. Suppose that $x \notin A^*_\eta \cup B^*_\eta$, then $x \notin A^*_\eta$ and $x \notin B^*_\eta$ implying that there exist $U, V \in \eta\text{-}O(X)$ such that $A \cap U \in I$ and $B \cap V \in I$. Note that $A \cap U \in I$, $B \cap V \in I$, and I is an ideal. Then $(A \cap U) \cup (B \cap V) \in I$. Since $U \cap V \subseteq U$ and $U \cap V \subseteq V$,

$$\begin{aligned}(A \cap U) \cup (B \cap V) &\supseteq \big[A \cap (U \cap V)\big] \cup \big[B \cap (U \cap V)\big] \\ &= (A \cup B) \cap (U \cap V).\end{aligned}$$

It implies that $(A \cup B) \cap (U \cap V) \subseteq (A \cap U) \cup (B \cap V) \in I$. Again, since I is an ideal, $(A \cup B) \cap (U \cap V) \in I$. Now, note that by assumption, $\eta\text{-}O(X)$ is closed under any two intersections, and so, there exists $U \cap V \in \eta\text{-}O(x)$ such that $(A \cup B) \cap (U \cap V) \in I$. This shows that $x \notin (A \cup B)^*_\eta$. Hence, $(A \cup B)^*_\eta \subseteq A^*_\eta \cup B^*_\eta$. Now, by Theorem 1 *(iii)*, therefore, $(A \cup B)^*_\eta = A^*_\eta \cup B^*_\eta$.

(ii) Let $\eta\text{-}O(X)$ be closed under any two intersections. For $U \in \eta\text{-}O(X)$, suppose that $x \in U \cap A^*_\eta$. Then $x \in U$ and $x \in A^*_\eta$. To show that $x \in (U \cap A)^*_\eta$, let $V \in \eta\text{-}O(x)$. Since $x \in U$ and $U \in \eta\text{-}O(X)$, we can write it as $U \in \eta\text{-}O(x)$. Hence, by assumption, $U \cap V \in \eta\text{-}O(x)$. Note that since $x \in A^*_\eta$ and $U \cap V \in \eta\text{-}O(x)$, then $A \cap (U \cap V) \notin I$ for every $U \cap V \in \eta\text{-}O(x)$. Now, by associativity and commutativity,

$$\begin{aligned}A \cap (U \cap V) &= (A \cap U) \cap V \notin I \\ &= (U \cap A) \cap V \notin I.\end{aligned}$$

This shows that for every $V \in \eta\text{-}O(x)$, $(U \cap A) \cap V \notin I$. It implies that $x \in (U \cap A)^*_\eta$. Hence, $U \cap A^*_\eta \subseteq (U \cap A)^*_\eta$. Now, note that $U \cap A^*_\eta \subseteq (U \cap A)^*_\eta$, then $U \cap (U \cap A^*_\eta) \subseteq U \cap (U \cap A)^*_\eta$. Since $U \cap (U \cap A^*_\eta)$, by associativity again,

$$\begin{aligned}U \cap (U \cap A^*_\eta) &= (U \cap U) \cap A^*_\eta \\ &= U \cap A^*_\eta.\end{aligned}$$

This implies that $U \cap A^*_\eta \subseteq U \cap (U \cap A)^*_\eta$. In contrast, note that $U \cap A \subseteq A$, then by Theorem 1 *(i)*, $(U \cap A)^*_\eta \subseteq A^*_\eta$. Thus, $U \cap (U \cap A)^*_\eta \subseteq U \cap A^*_\eta$. Consequently, as a result, $U \cap A^*_\eta = U \cap (U \cap A)^*_\eta$. Note that $U \cap (U \cap A)^*_\eta \subseteq (U \cap A)^*_\eta$. This shows that $U \cap A^*_\eta = U \cap (U \cap A)^*_\eta \subseteq (U \cap A)^*_\eta$.

(iii) Let $A, B \subseteq X$. Note that $A = (A \setminus B) \cup (B \cap A)$. Thus $A^*_\eta = \left[(A \setminus B) \cup (B \cap A)\right]^*_\eta$. Note that by assumption, η-$O(X)$ is closed under any two intersections, then by Theorem 3 *(i)*,

$$\begin{aligned} A^*_\eta &= \left[(A \setminus B) \cup (B \cap A)\right]^*_\eta \\ &= (A \setminus B)^*_\eta \cup (B \cap A)^*_\eta. \end{aligned}$$

So, $A^*_\eta = (A \setminus B)^*_\eta \cup (B \cap A)^*_\eta$. Now, note that $A^*_\eta \setminus B^*_\eta = A^*_\eta \cap \left(B^*_\eta\right)^c$, and since $A^*_\eta = (A \setminus B)^*_\eta \cup (B \cap A)^*_\eta$,

$$\begin{aligned} A^*_\eta \setminus B^*_\eta &= A^*_\eta \cap \left(B^*_\eta\right)^c \\ &= \left[(A \setminus B)^*_\eta \cup (B \cap A)^*_\eta\right] \cap \left(B^*_\eta\right)^c \\ &= \left[(A \setminus B)^*_\eta \cap \left(B^*_\eta\right)^c\right] \cup \left[(B \cap A)^*_\eta \cap \left(B^*_\eta\right)^c\right] \\ &= \left[(A \setminus B)^*_\eta \setminus B^*_\eta\right] \cup \left[(B \cap A)^*_\eta \setminus B^*_\eta\right]. \end{aligned}$$

Hence, $A^*_\eta \setminus B^*_\eta = \left[(A \setminus B)^*_\eta \setminus B^*_\eta\right] \cup \left[(B \cap A)^*_\eta \setminus B^*_\eta\right]$. Note that $B \cap A \subseteq B$, then by Theorem 1 *(i)*, $(B \cap A)^*_\eta \subseteq B^*_\eta$ implies that $(B \cap A)^*_\eta \setminus B^*_\eta = \varnothing$. Now, since $A^*_\eta \setminus B^*_\eta = \left[(A \setminus B)^*_\eta \setminus B^*_\eta\right] \cup \left[(B \cap A)^*_\eta \setminus B^*_\eta\right]$ and $(B \cap A)^*_\eta \setminus B^*_\eta = \varnothing$, it follows that

$$\begin{aligned} A^*_\eta \setminus B^*_\eta &= \left[(A \setminus B)^*_\eta \setminus B^*_\eta\right] \cup \left[(B \cap A)^*_\eta \setminus B^*_\eta\right] \\ &= \left[(A \setminus B)^*_\eta \setminus B^*_\eta\right] \cup \varnothing \\ &= (A \setminus B)^*_\eta \setminus B^*_\eta \\ &\subseteq (A \setminus B)^*_\eta. \end{aligned}$$

As a result, it shows that $A^*_\eta \setminus B^*_\eta = (A \setminus B)^*_\eta \setminus B^*_\eta \subseteq (A \setminus B)^*_\eta$. □

Theorem 4. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then for any η-local function, the following properties hold:*

(i) $A^*_\eta \subseteq A^*$;

(ii) $A^*_\eta \subseteq \Gamma^*(A)$; *and*

(iii) $A^* \subseteq \Gamma^*(A)$.

Proof.

(i) Let $x \in A^*_\eta$ and $U \in \tau(x)$. Since every open set is η-open set, $U \in \eta$-$O(x)$. Also, since $x \in A^*_\eta$ and $U \in \eta$-$O(x)$, $A \cap U \notin I$. Note that $A \cap U \notin I$ and $U \in \tau(x)$. Hence, $A \cap U \notin I$ for every $U \in \tau(x)$, and so, $x \in A^*$. Therefore, $A^*_\eta \subseteq A^*$.

(ii) Let $x \in A^*_\eta$ and $U \in RO(x)$. Since every regular-open set is η-open set, $U \in \eta$-$O(x)$. Also, since $x \in A^*_\eta$ and $U \in \eta$-$O(x)$, $A \cap U \notin I$. Hence, $A \cap U \notin I$ for every $U \in RO(x)$, and so, $x \in \Gamma^*(A)$. Therefore, $A^*_\eta \subseteq \Gamma^*(A)$.

(iii) Let $x \in A^*$ and $U \in RO(x)$. Since every regular-open set is open set, $U \in \tau$. Also, since $x \in A^*$ and $U \in \tau$, $A \cap U \notin I$. Hence, $A \cap U \notin I$ for every $U \in RO(x)$, and so, $x \in \Gamma^*(A)$. Therefore, $A^* \subseteq \Gamma^*(A)$. □

Remark 2. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then for any η-local functions, the following properties hold:*

(ii) $A^*_\eta \subseteq A^* \subseteq \Gamma^*(A)$;

(ii) *if* η-$O(X) = \tau$, *then* $A^*_\eta = A^*$; *and*

(iii) *if* η-$O(X) = RO(X)$, *then* $A^*_\eta = \Gamma^*(A)$

Theorem 5. *Let (X, τ) be a topological space with ideals I_1 and I_2 on X and $A \subseteq X$. Then, for any η-local functions, the following properties hold:*

(i) if $I_1 \subseteq I_2$, *then* $A^*_\eta\big(I_2, \eta\text{-}O(X)\big) \subseteq A^*_\eta\big(I_1, \eta\text{-}O(X)\big)$; *and*

(ii) $A^*_\eta\big((I_1 \cap I_2), \eta\text{-}O(X)\big) = A^*_\eta\big(I_1, \eta\text{-}O(X)\big) \cup A^*_\eta\big(I_2, \eta\text{-}O(X)\big)$.

Proof.

(i) Let $I_1 \subseteq I_2$ and $x \in A^*_\eta\big(I_2, \eta\text{-}O(X)\big)$. Then for every $U \in \eta\text{-}O(x)$, $A \cap U \notin I_2$. Since $I_1 \subseteq I_2$, $A \cap U \notin I_1$ for every $U \in \eta\text{-}O(x)$. Hence, $A^*_\eta\big(I_2, \eta\text{-}O(X)\big) \subseteq A^*_\eta\big(I_1, \eta\text{-}O(X)\big)$.

(ii) Let I_1 and I_2 be ideals on X. Note that $I_1 \cap I_2 \subseteq I_1$ and $I_1 \cap I_2 \subseteq I_2$. Then by Theorem 5 *(i)*,

$$A^*_\eta\big(I_1, \eta\text{-}O(X)\big) \subseteq A^*_\eta\big((I_1 \cap I_2), \eta\text{-}O(X)\big)$$

and

$$A^*_\eta\big(I_2, \eta\text{-}O(X)\big) \subseteq A^*_\eta\big((I_1 \cap I_2), \eta\text{-}O(X)\big),$$

and hence,

$$A^*_\eta\big(I_1, \eta\text{-}O(X)\big) \cup A^*_\eta\big(I_2, \eta\text{-}O(X)\big) \subseteq A^*_\eta\big((I_1 \cap I_2), \eta\text{-}O(X)\big).$$

Next, let $x \in A^*_\eta\big((I_1 \cap I_2), \eta\text{-}O(X)\big)$, then for every $U \in \eta\text{-}O(x)$, $A \cap U \notin I_1 \cap I_2$. This implies that $A \cap U \notin I_1$ or $A \cap U \notin I_2$. This shows that $x \in A^*_\eta\big(I_1, \eta\text{-}O(X)\big)$ or $x \in A^*_\eta\big(I_2, \eta\text{-}O(X)\big)$. Hence, $x \in A^*_\eta\big(I_1, \eta\text{-}O(X)\big) \cup A^*_\eta\big(I_2, \eta\text{-}O(X)\big)$, and so,

$$A^*_\eta\big((I_1 \cap I_2), \eta\text{-}O(X)\big) \subseteq A^*_\eta\big(I_1, \eta\text{-}O(X)\big) \cup A^*_\eta\big(I_2, \eta\text{-}O(X)\big).$$

As a result, thus,

$$A^*_\eta\big((I_1 \cap I_2), \eta\text{-}O(X)\big) = A^*_\eta\big(I_1, \eta\text{-}O(X)\big) \cup A^*_\eta\big(I_2, \eta\text{-}O(X)\big).$$ □

4. η-Local Closure

Definition 2. *Let (X,τ,I) be an ideal topological space. The η-local closure of A denoted by $Cl^*_\eta(A)$ is defined by the union of A and the η-local function of A, i.e, $Cl^*_\eta(A) = A \cup A^*_\eta$ for any $A \subseteq X$.*

Example 3. *Let (X,τ,I) be an ideal topological space where $X = \{a,b,c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$, and $I = \{\varnothing, \{b\}\}$. Then the η-open sets of X are $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a,b\}$, $\{a,c\}$, and $\{b,c\}$. Let $A = \{a,b\}$, Then by Definition 1 and 2, $A^*_\eta = \{a\}$ and $Cl^*_\eta(A) = \{a,b\} \cup \{a\} = \{a,b\}$, respectively.*

Theorem 6. *Let (X,τ,I) be an ideal topological space and $A, B \subseteq X$. Then the following properties hold:*

(i) *if $A \subseteq B$, then $Cl^*_\eta(A) \subseteq Cl^*_\eta(B)$;*
(ii) *$Cl^*_\eta(A \cap B) \subseteq Cl^*_\eta(A) \cap Cl^*_\eta(B)$;*
(iii) *if A is an η-closed set, then $Cl^*_\eta(A) = \eta\text{-}cl(A)$;*
(iv) *if $A \in I$, then $Cl^*_\eta(A) = A$;*
(v) *$Cl^*_\eta(A^*_\eta) = A^*_\eta$;*
(vi) *$Cl^*_\eta(A) = \eta\text{-}cl(A)$; and*
(vii) *$\left(Cl^*_\eta(A)\right)^*_\eta = A^*_\eta$.*

Proof.

(i) Let $A, B \subseteq X$ and $A \subseteq B$. By Definition 2, $Cl^*_\eta(A) = A \cup A^*_\eta$ and $Cl^*_\eta(B) = B \cup B^*_\eta$. Since $A \subseteq B$, by Theorem 1 *(i)*, $A^*_\eta \subseteq B^*_\eta$. This shows that $A \cup A^*_\eta \subseteq B \cup B^*_\eta$, and hence, $Cl^*_\eta(A) \subseteq Cl^*_\eta(B)$.

(ii) Let $A, B \subseteq X$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by Theorem 6 *(i)*, $Cl^*_\eta(A \cap B) \subseteq Cl^*_\eta(A)$ and $Cl^*_\eta(A \cap B) \subseteq Cl^*_\eta(B)$. Hence, it implies $Cl^*_\eta(A \cap B) \subseteq Cl^*_\eta(A) \cap Cl^*_\eta(B)$.

(iii) Let A be an η-closed set, then $A = \eta\text{-}cl(A)$. Now, suppose that $x \notin A$. It implies that $x \notin \eta\text{-}cl(A)$, then $x \notin \bigcap\{K : K \text{ is } \eta\text{-closed and } A \subseteq K\}$. It follows that $x \notin K$ for some η-closed set K such that $A \subseteq K$. Hence, $x \in K^c$ for some η-open set K^c such that $A \cap K^c = \varnothing$. It implies that there exists $K^c \in \eta\text{-}O(x)$ such that $A \cap K^c = \varnothing$, and by Definition of ideal, $\varnothing \in I$ for any ideal I. Hence, $A \cap K^c \in I$ for some $K^c \in \eta\text{-}O(x)$. This shows that $x \notin A^*_\eta$, and hence, $A^*_\eta \subseteq A$. It follows that $Cl^*_\eta(A) = A \cup A^*_\eta = A$. Note that $A = \eta\text{-}cl(A)$. Therefore, $Cl^*_\eta(A) = \eta\text{-}cl(A)$.

(iv) Let $A \in I$. Then by Definition 2 and Theorem 1 *(vii)*, $Cl^*_\eta(A) = A \cup A^*_\eta = A \cup \varnothing = A$. Consequently, $Cl^*_\eta(A) = A$.

(v) Let $A \subseteq X$. Then by Definition 2 and Theorem 1 *(vi)*, $Cl^*_\eta(A^*_\eta) = A^*_\eta \cup \left(A^*_\eta\right)^*_\eta = A^*_\eta$. It follows that, $Cl^*_\eta(A^*_\eta) = A^*_\eta$.

(vi) Let $A \subseteq X$. Suppose that $Cl^*_\eta(A) \neq \eta\text{-}cl(A)$. Let $\eta\text{-}cl(A) \subset Cl^*_\eta(A)$. Then there exists an element $x \in Cl^*_\eta(A)$ such that $x \notin \eta\text{-}cl(A)$. Note that since $x \in Cl^*_\eta(A)$, by Definition 2, $x \in A \cup A^*_\eta$ implies that $x \in A$ or $x \in A^*_\eta$, or both. Suppose $x \in A^*_\eta$. Then for every $U \in \eta\text{-}O(x)$, $A \cap U \notin I$. Now, Since $x \notin \eta\text{-}cl(A)$, $x \notin \bigcap\{K : K \text{ is } \eta\text{-closed and } A \subseteq K\}$. It follows that $x \notin K$ for some η-closed set K such that $A \subseteq K$. Hence, $x \in K^c$ for some η-open set K^c such that $A \cap K^c = \varnothing$, and by Definition of an ideal, $\varnothing \in I$ for any ideal I. It implies that there exists $K^c \in \eta\text{-}O(x)$ such that $A \cap K^c \in I$, and hence, $x \notin A^*_\eta$. Also, note that since $x \in K^c$ and $A \cap K^c = \varnothing$, $x \notin A$. This shows that $x \notin A$ and $x \notin A^*_\eta$, a contradiction.

(vii) Let $A \subseteq X$. Then by Definition 2 and Theorem 1 *(iii)*, $\big(Cl^*_\eta(A)\big)^*_\eta = \big(A \cup A^*_\eta\big)^*_\eta \supseteq A^*_\eta \cup \big(A^*_\eta\big)^*_\eta$. Note that by Theorem 1 *(vi)*, $\big(A^*_\eta\big)^*_\eta \subseteq A^*_\eta$, then $A^*_\eta \cup \big(A^*_\eta\big)^*_\eta = A^*_\eta$. It implies that $A^*_\eta \subseteq \big(Cl^*_\eta(A)\big)^*_\eta$. Now, let $x \in \big(Cl^*_\eta(A)\big)^*_\eta$. Then for every $U \in \eta\text{-}O(x)$, $Cl^*_\eta(A) \cap U \notin I$. Now, by Definition 2, $Cl^*_\eta(A) \cap U = (A \cup A^*_\eta) \cap U \notin I = (A \cap U) \cup (A^*_\eta \cap U) \notin I$. It implies that $A \cap U \notin I$ or $A^*_\eta \cap U \notin I$, or both, and so, $x \in A^*_\eta$ or $x \in \big(A^*_\eta\big)^*_\eta$, or both. It follows that $x \in A^*_\eta \cup \big(A^*_\eta\big)^*_\eta$. Note that $A^*_\eta \cup \big(A^*_\eta\big)^*_\eta = A^*_\eta$, and so, $x \in A^*_\eta$. Consequently, $\big(Cl^*_\eta(A)\big)^*_\eta \subseteq A^*_\eta$. Thus, $\big(Cl^*_\eta(A)\big)^*_\eta = A^*_\eta$. □

Theorem 7. *Let (X, τ, I) be an ideal topological space and $A, B \subseteq X$. Then the following properties hold:*

(i) $A \subseteq Cl^*_\eta(A)$ *and* $A^*_\eta \subseteq Cl^*_\eta(A)$;
(ii) $Cl^*_\eta(\varnothing) = \varnothing$ *and* $Cl^*_\eta(X) = X$;
(iii) $Cl^*_\eta(A) \cup Cl^*_\eta(B) \subseteq Cl^*_\eta(A \cup B)$; *and*
(iv) $\big(Cl^*_\eta(A)\big)^*_\eta \subseteq Cl^*_\eta(A) = Cl^*_\eta\big(Cl^*_\eta(A)\big)$.

Proof.

(i) Let $A \subseteq X$. Note that $A \subseteq A \cup A^*_\eta$. Then by Definition 2, $A \subseteq Cl^*_\eta(A)$. Next, note that $A^*_\eta \subseteq A \cup A^*_\eta$, by Definition 2 again, it implies that $A^*_\eta \subseteq Cl^*_\eta(A)$.

(ii) By Definition 2 and Theorem 1 *(iv)*, $Cl^*_\eta(\varnothing) = \varnothing \cup (\varnothing)^*_\eta = \varnothing \cup \varnothing = \varnothing$. Next, note that X is a universal set, then $(X)^*_\eta \subseteq X$. Hence, by Definition 2, $Cl^*_\eta(X) = X \cup (X)^*_\eta = X$.

(iii) Let $A, B \subseteq X$. By Definition 2 and Thoerem 1 *(iii)*,

$$\begin{aligned} Cl^*_\eta(A \cup B) &= (A \cup B) \cup (A \cup B)^*_\eta \\ &\supseteq (A \cup B) \cup (A^*_\eta \cup B^*_\eta) \\ &= (A \cup A^*_\eta) \cup (B \cup B^*_\eta) \\ &= Cl^*_\eta(A) \cup Cl^*_\eta(B). \end{aligned}$$

This shows that $Cl^*_\eta(A) \cup Cl^*_\eta(B) \subseteq Cl^*_\eta(A \cup B)$.

(iv) Let $A \subseteq X$. Note that by Theorem 6 *(vii)*, $\big(Cl^*_\eta(A)\big)^*_\eta = A^*_\eta$, and by Theorem 7 *(i)*, $A^*_\eta \subseteq Cl^*_\eta(A)$. Hence, it shows that $\big(Cl^*_\eta(A)\big)^*_\eta \subseteq Cl^*_\eta(A)$. Next, by Definition 2, $Cl^*_\eta\big(Cl^*_\eta(A)\big) = Cl^*_\eta(A) \cup \big(Cl^*_\eta(A)\big)^*_\eta$. Note that since $\big(Cl^*_\eta(A)\big)^*_\eta \subseteq Cl^*_\eta(A)$,

$$\begin{aligned} Cl^*_\eta\big(Cl^*_\eta(A)\big) &= Cl^*_\eta(A) \cup \big(Cl^*_\eta(A)\big)^*_\eta \\ &= Cl^*_\eta(A). \end{aligned}$$

It follows that, $Cl^*_\eta(A) = Cl^*_\eta\big(Cl^*_\eta(A)\big)$. □

Remark 3. *The reverse inclusion of Theorem 7 (iii) need not be true in general as shown from the following example.*

Example 4. *Let (X, τ, I) be an ideal topological space where $X = \{a, b, c, d\}$, $\tau = \big\{\varnothing, X, \{c\}, \{d\}, \{c, d\}\big\}$, and $I = \big\{\varnothing, \{a\}, \{b\}, \{a, b\}\big\}$. Then the η-open sets of X are $\varnothing$, X, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$, and $\{b, c, d\}$. Let $A = \{c\}$ and $B = \{d\}$ such that $A \cup B = \{c, d\}$, Then by Definition 1, $A^*_\eta = \{c\}$, $B^*_\eta = \{d\}$, and $(A \cup B)^*_\eta = X$. Now, By Definition 2, $Cl^*_\eta(A) = \{c\}$, $Cl^*_\eta(B) = \{d\}$, and $Cl^*_\eta(A \cup B) = X$. Observe that $Cl^*_\eta(A \cup B) = X$ and $Cl^*_\eta(A) \cup Cl^*_\eta(B) = \{c, d\}$. These shows that $Cl^*_\eta(A \cup B) \nsubseteq Cl^*_\eta(A) \cup Cl^*_\eta(B)$. Hence, the above assertion has been verified.*

Theorem 8. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then $Cl^*_\eta(A) \subseteq Cl^*(A)$.*

Proof. Let $A \subseteq X$. By Definition 2 and Kuratowski closure operator, $Cl^*_\eta(A) = A \cup A^*_\eta$ and $Cl^*(A) = A \cup A^*$, respectively. Since by Theorem 4 *(i)*, $A^*_\eta \subseteq A^*$, $A \cup A^*_\eta \subseteq A \cup A^*$. Hence, $Cl^*_\eta(A) \subseteq Cl^*(A)$. □

Theorem 9. *Let (X, τ, I) be an ideal topological space and A be any subset of X. Then A is an η-closed set iff $A = Cl^*_\eta(A)$.*

Proof. Let A be an η-closed set. Then $A = \eta\text{-}cl(A)$ and by Theroem 6 *(vi)*, $Cl^*_\eta(A) = \eta\text{-}cl(A)$, respectively. Note that $A = \eta\text{-}cl(A)$ and $\eta\text{-}cl(A) = Cl^*_\eta(A)$, then by transitive property, it implies that $A = Cl^*_\eta(A)$. Now, on the other hand, let $A = Cl^*_\eta(A)$. Note that by Theorem 6 *(vi)*, $Cl^*_\eta(A) = \eta\text{-}cl(A)$. Now that $A = Cl^*_\eta(A)$ and $Cl^*_\eta(A) = \eta\text{-}cl(A)$, by transitive property again, $A = \eta\text{-}cl(A)$. Therefore, A is an η-closed set. □

Note that Theorem 7 *(i)*, *(ii)*, and *(iv)* satisfy three of the Kuratowski closure axioms. However, Theorem 7 *(iii)* did not satisfy one of the Kuratowski closure axioms because it is an inclusion property. As a result, the following remark is obtained.

Remark 4. *The η-local closure, i.e, Cl^*_η, need not be a Kuratowski closure operator with respect to η in general.*

Theorem 10. *Let (X, τ, I) be an ideal topological space where η-$O(X)$ is closed under any two intersections and $A, B \subseteq X$. Then $Cl^*_\eta(A \cup B) = Cl^*_\eta(A) \cup Cl^*_\eta(B)$.*

Proof. Let η-$O(X)$ be closed under any two intersections. Then by Definition 2 and Theorem 3 *(i)*, it follows that

$$\begin{aligned} Cl^*_\eta(A \cup B) &= (A \cup B) \cup (A \cup B)^*_\eta \\ &= (A \cup B) \cup (A^*_\eta \cup B^*_\eta) \\ &= (A \cup A^*_\eta) \cup (B \cup B^*_\eta) \\ &= Cl^*_\eta(A) \cup Cl^*_\eta(B). \end{aligned}$$

Hence, $Cl^*_\eta(A \cup B) = Cl^*_\eta(A) \cup Cl^*_\eta(B)$. □

Note that Theorem 7 *(i)*, *(ii)*, *(iv)* and Theorem 10 using the condition, for any ideal topological spaces (X, τ, I) where η-$O(X)$ is closed under any two intersections, satisfy the Kuratowski closure axioms. As a result, the following remark is obtained

Remark 5. *Let (X, τ, I) be an ideal topological space where η-$O(X)$ is closed under any two intersections, the η-local closure, i.e, Cl^*_η, is a Kuratowski closure operator (or Almocera closure operator) with respect to η.*

Note that by Remark 5, Cl^*_η is a Kuratowski closure operator (or Almocera closure operator) with respect to η for any ideal topological space (X, τ, I) where η-$O(X)$ is closed under any two intersections. Now, let A be a τ^*_η-closed set iff $A^*_\eta \subseteq A$ in any ideal topological space (X, τ, I) where η-$O(X)$ is closed under any two intersections. Then the following lemma is obtained.

Lemma 1. *Let A be a τ^*_η-closed set iff $A^*_\eta \subseteq A$ in any ideal topological space (X, τ, I) where η-$O(X)$ is closed under any two intersections. Then A is τ^*_η-closed set iff $Cl^*_\eta(A) = A$.*

Proof. Let A be a τ^*_η-closed in (X, τ, I) where η-$O(X)$ is closed under any two intersections. Now, since A is a τ^*_η-closed, by assumption, $A^*_\eta \subseteq A$. It follows that $A \cup A^*_\eta = A$. Now, by Definiton 2, $Cl^*_\eta(A) = A \cup A^*_\eta = A$. Therefore, $Cl^*_\eta = A$. On the other hand, let $Cl^*_\eta(A) = A$. Now, by Definiton 2, $Cl^*_\eta(A) = A \cup A^*_\eta = A$. So, $A \cup A^*_\eta = A$ implies $A^*_\eta \subseteq A$, and so, by assumption, A is τ^*_η-closed. □

Theorem 11. *Let (X, τ, I) be an ideal topological space where η-$O(X)$ is closed under any two intersections. Let $\tau^*_\eta = \{J \subseteq X : Cl^*_\eta(J^c) = J^c\}$. Then τ^*_η is a topology for X such that $\tau^* \subseteq \tau^*_\eta$ and η-$O(X) \subseteq \tau^*_\eta$.*

Proof. Let η-$O(X)$ be closed under any two intersections. Note that by Remark 5, Cl^*_η is a Kuratowski closure operator with respect to η. Therefore, τ^*_η is a topology generated by Cl^*_η. Now, to show that $\tau^* \subseteq \tau^*_\eta$, let A be a τ^*-open. Then A^c is a τ^*-closed. Then by Definition of τ^*-closed,

$\left(A^c\right)^* \subseteq A^c$. Hence, $Cl^*(A^c) = A^c \cup \left(A^c\right)^* = A^c$ implies that $Cl^*(A^c) = A^c$. Then by Theorem 8, $Cl^*_\eta(A^c) \subseteq A^c$. Now, since $Cl^*_\eta(A^c) \subseteq A^c$, by Theorem 7 *(i)*, $Cl^*_\eta(A^c) = A^c$. Hence, by Lemma 1, A^c is a τ^*_η-closed, and so, A is a τ^*_η-open. As a result, thus, $\tau^* \subseteq \tau^*_\eta$. Next, to show that η-$O(X) \subseteq \tau^*_\eta$, let A be an η-open. Then A^c is an η-closed and by Theorem 9, $A^c = Cl^*_\eta(A^c)$. It follows that by Lemma 1, A^c is a τ^*_η-closed implies that A is a τ^*_η-open. Hence, η-$O(X) \subseteq \tau^*_\eta$. $\square$

5. Conclusion

The concept of the η-local function and the closure Cl^*_η has been introduced and demonstrated through illustrative examples. Additionally, certain properties have been studied and explored. It can be concluded that the closure Cl^*_η can only be a Kuratowski closure operator (Almocera closure operator) if η-$O(X)$ is closed under two intersections. Under this condition, τ^*_η can form a topology, making τ^*_η a more generalized version of τ^* and η-$O(X)$.

References

[1] A. Al-Omari, T. Noiri, Local function Γ^* in ideal topological spaces, Sci. Stud. Res. Ser. Math. Inform. 26 (1) (2016) 5-16.

[2] E. Hatir, A. Al-Omari, S. Jafari, δ-local functions and its properties in ideal topological spaces, Fasciculi Math. 53 (2014) 53-64.

[3] D. Jankovic, T.R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly 97 (4) (1990) 295-310.

[4] K. Kuratowski, Topology I, Warszawa, 1933.

[5] K. Kuratowski, Topology, Academic Press, New York, 1966.

[6] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963) 36-41.

[7] P.L. Powar, K. Rajak, Some new concepts of continuity in generalized topological space, Int. J. Com. Appl. 38 (5) (2012) 12-17.

[8] D. Subbulakshmi, K. Sumathi, K. Indiran, η-open sets in topological, Int. J. Innov. Techno. Explor. Eng. 8 (10S) (2019), 276-282.

[9] R. Vaidyanathaswamy, The localization theory in set-topology, Proc. Indian Acad. Sci. Sect. 20 (1944) 51-61.