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Slicing of Negative Plurisubharmonic Currents Arising From Analytic Subsets

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Abstract. In this paper, first we focus on the slicing of negative plurisubharmonic currents which are finite sums involving currents arising from analytic subsets with geometric complete intersection. Next, we provide significant results on the integrability across analytic subsets, of the coefficients of such currents and of their slices.

1. Introduction

In this paper, we let Ω be a domain in $\mathbb{C}^n$ such that the unit polydisc Δ^n satisfies $\Delta^n \Subset \Omega$ and φ be a plurisubharmonic function (psh for short), locally bounded on Ω. We let N, k, p and n be nonzero fixed arbitrary natural numbers such that $k \leq p \leq n$ and we consider the n-complex space $\mathbb{C}^n$ with variables z such that

$$\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}, \quad z = (z', z''), \quad z' \in \mathbb{C}^k, \quad z'' \in \mathbb{C}^{n-k}.$$

Assuming that φ depends only on the variable $z' \in \mathbb{C}^k$ and the support S_φ of the associated Mong-Ampère measure

$$\mu_\varphi = (dd^c\varphi)^k$$

is Δ^k, then, the slice (or the φ–slice) denoted $\langle T, \pi, a\rangle_\varphi$ of a current $T \in \mathscr{D}'_{p,p}(\Omega)$, at point $a \in S_\varphi$, is studied and well defined in [10], such that

$$< T, \pi, a >_\varphi (\Psi) := \lim_{\varepsilon \to 0} \frac{1}{\mu_\varphi(B_k(a, \varepsilon))} \int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} T \wedge (dd^c\tilde{\varphi})^k \wedge \Psi. \qquad (1.1)$$

The definition (1.1) makes sense as well as the limit exists in $\mathbb{C}$ for any test form $\Psi \in \mathscr{D}_{(p-k,p-k)}(\Omega)$, where, $\tilde{\varphi} = \varphi \circ \pi$ stands for the composite of φ with the projection map

$$\pi : \begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^k \\ (z', z'') & \longmapsto & z' \end{array}$$

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and $B_k(a,\varepsilon)$ is the open ball in $\mathbb{C}^k$ centered at point a and of radius $\varepsilon > 0$.

In the present work we are concerned by topics studied in [9–12]. Particularly, we will study topics based on the definition (1.1) recently investigated in [11], for negative psh currents of small support. A fundamental result was provided showing that for any psh function v non identically equals $-\infty$, there exists a pluripolar subset E in $\mathbb{C}^k$, such that the slice $< v, \pi, a >_\varphi$ of v at point a is well defined and is expressed explicitly by $v(a,.)$ as well as the point a lies outside E.

As a consequence, an existence slicing result was deduced for negative psh currents having their supports contained in a strip. In particular, for positive or negative closed currents with support contained in a strip, it was established that the slices are well defined and are vanishing everywhere as soon as they are vanishing outside a pluripolar subset.

In this paper, we shall develop the work in [11] by providing others slicing results for negative psh currents which are arising from analytic subsets having suitable intersections.

Among several currents, we are interested on those written as finite sums of the form

$$\sum_{1\le j\le N}(\log|f_j|)\big[X_j\big],\quad \sum_{1\le j\le N} v_j[X_j],\quad \sum_{1\le j\le N} U_{[X_j]}\wedge[Y_j],\quad \text{or}\quad \sum_{1\le j\le N}[X_j]\wedge U_{[Y_j]},$$

where $(f_j)_{1\le j\le N}$ is a finite family of holomorphic functions non identically vanishing on Δ^n, $(X_j)_{1\le j\le N}$ is a finite family of analytic subsets all of pure dimension p, $(Y_j)_{1\le j\le N}$ is another finite family of analytic subsets such that each Y_j forms a complete intersection with X_j in Δ^n, $(v_j)_{1\le j\le N}$ is a finite family of psh functions non identically $-\infty$ on Δ^n and $U_{[X_j]}$ is the Lelong-Skoda potential associated to the current $[X_j], (1\le j\le N)$.

The first result describes properly the slices of the current $\sum_{1\le j\le N}(\log|f_j|)\big[X_j\big]$.

Theorem 1.1. *Assume we have in Δ^n, a finite family $(X_j)_{1\le j\le N}$ of analytic subsets all of the same pure dimension $q>k$ and a finite family $(f_j)_{1\le j\le N}$ of holomorphic functions non identically vanishing such that for all $1\le j\le N$ the hypersurface $Y_j=\{f_j=0\}$ yields a complete intersection near the origin with X_j. If $\sigma_j : X_j\cap\pi^{-1}(a)\longrightarrow\Delta^n$ denotes the canonical injection, then, there exist, a closed pluripolar subset E of Δ^k, an integer $m\in\mathbb{N}$ and homogenous polynomials $Q_{m,j}$ on X_j, $1\le j\le N$, such that for all $a\in\Delta^k\smallsetminus E$ the slice $\langle\sum_{1\le j\le N}(log|f_j|)[X_j],\pi,a\rangle_\varphi$ is defined such that*

(1) *If $\sigma_j{}^*f_j\not\equiv 0$ on $X_j\cap\pi^{-1}(a)$, $\forall\, 1\le j\le N$, then*

$$\langle\sum_{1\le j\le N}(log|f_j|)[X_j],\pi,a\rangle_\varphi=\sum_{1\le j\le N}\big(\log|\sigma_j{}^*f_j|\big)\left[X_j\cap\pi^{-1}(a)\right],$$

(2) *If $\sigma_j{}^*f_j\equiv 0$ and $Q_{m,j}{}_{|X_j\cap\pi^{-1}(a)}\not\equiv 0$, $\forall\, 1\le j\le N$, then*

$$\langle\sum_{1\le j\le N}(log|f_j|)[X_j],\pi,a\rangle_\varphi=\sum_{1\le j\le N}\big(\log|\sigma_j{}^*Q_{m,j}|\big)\left[X_j\cap\pi^{-1}(a)\right].$$

The second result generalizes Theorem 1.1 to psh functions. It provides the existence and the expression of the slices of a negative psh current of the form $\sum\limits_{1\leq j\leq N} v_j[X_j]$.

Theorem 1.2. *Assume we have in Δ^n, a finite family $(X_j)_{1\leq j\leq N}$ of analytic subsets all of pure dimension p and a finite family $(v_j)_{1\leq j\leq N}$ of negative psh functions such that for all $1\leq j\leq N$ the set of singular points of v_j is contained in a hypersurface Y_j that yields a complete intersection with X_j. Then, there is a pluripolar subset E of Δ^k, such that for all $a\in\Delta^k\smallsetminus E$, the current $\sum_{1\leq j\leq N} v_j[X_j]$ admits a slice expressed by*

$$\langle \sum_{1\leq j\leq N} v_j.[X_j],\pi,a\rangle_\varphi = \sum_{1\leq j\leq N} v_{j|X_j\cap\pi^{-1}(a)}.\big[X_j\cap\pi^{-1}(a)\big].$$

Next, we let $N(x)=-\frac{1}{(n-1)(4\pi)^n}\frac{1}{|x|^{2n-2}}$ be the Newton kernel in $\mathbb{C}^n$ and we let η be a positive smooth function with compact support in Ω such that $0\leq\eta\leq 1$, $\eta\equiv 1$ on a neighborhood of $\overline{\Delta^n}$. For a given analytic subset X in Δ^n, we take the Lelong-Skoda potential $U_{[X]}$ associated to the current $[X]$, as the $(n-p-1,n-p-1)$-negative current defined by the following integral

$$U_{[X]}(z)=\int_{\xi\in\mathbb{C}^n}\eta(\xi)N(z-\xi)[X](\xi)\wedge\left(dd^c|z-\xi|^2\right)^{n-1}.$$

We prove the third result of this paper on the existence and the expression of the slices of the wedge product current $\sum\limits_{1\leq j\leq N} U_{[X_j]}\wedge[Y_j]$ introduced and studied in [9].

Theorem 1.3. *Assume we have in Δ^n, finite families $(X_j)_{1\leq j\leq N}$ and $(Y_l)_{1\leq l\leq N}$ of analytic subsets such that for all $1\leq j\leq N$,, X_j is of pure dimension p and Y_j is of pure dimension q and X_j yields a complete intersection with Y_j. Then, there exists a subset E contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k-1$ such that for all $a\in\Delta^k\smallsetminus E$, the slice of $\sum\limits_{1\leq j\leq N} U_{[X_j]}\wedge[Y_j]$ at point a is given by*

$$\langle \sum_{1\leq j\leq N} U_{[X_j]}\wedge[Y_j],\pi,a\rangle_\varphi = \sum_{1\leq j\leq N} j_a^* U_{[X_j]}\wedge\big[Y_j\cap\pi^{-1}(a)\big].$$

In the sequel, we show that the above results enjoy interesting properties like the integrability of coefficients of the above currents and their slices. For instance, the following Theorem 1.4 provides an important integrability property for a given psh function v not identically $-\infty$ on Δ^n. In fact, it describes the integrability of the function $\exp(-v)$ across an analytic subset X. Consequently, we get the following beautiful result in terms of slices, about the integrability of $\exp(-v)$ across the intersection $\pi^{-1}(a)\cap X$.

Theorem 1.4. *Assume we have a finite family $(v_j)_{1\leq j\leq N}$ of psh functions in Δ^n such that for all $1\leq j\leq N$ the set of singular points of v_j is contained in a hypersurface X that yields a complete intersection with another analytic subset Y in Δ^n. Then, there exist a constant $\alpha>0$ and a* *pluripolar subset* E *of* Δ^k*, such that for any point* $a \in \Delta^k \smallsetminus E$*, the function* $\sum_{1\leq j\leq N} \exp\left(-\alpha v_j\right)_{|Y\cap\pi^{-1}(a)}$ *lies in* $L^1_{loc}\big(Y \cap \pi^{-1}(a)\big)$.

Finally, as an other application we establish the following result valid for analytic subsets X and Y yielding a complete intersection in Ω.

Theorem 1.5. *Assume we have analytic subsets* X *and* Y *of pure dimensions* p *and* q *respectively, yielding a complete intersection in* Δ^n*. Then, there exist a constant* $\delta > 0$ *and a pluripolar subset* E *in* $\mathbb{C}^k$*, such that for all* $a \in \Delta^k \smallsetminus E$*, the coefficients of the current* $j_a^*(U_{[X]}) \wedge \big[Y \cap \pi^{-1}(a)\big]$ *lie in* $L^{1+\delta}_{loc}(Y \cap \pi^{-1}(a))$.

2. Preliminaries

Let Ω be a domain in $\mathbb{C}^n$, we use the standard notations for the operators $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$; the operator dd^c is then defined by $dd^c = 2i\partial\bar\partial$. The space $\mathscr{D}'_{p,p}(\Omega)$ of $(n-p, n-p)$-currents (of bidimension (p,p)) on Ω, is the dual of the space $\mathscr{D}_{p,p}(\Omega)$ of smooth compactly supported (p,p)-forms on Ω. We say that a current $T \in \mathscr{D}'_{p,p}(\Omega)$ is positive, if for all smooth $(1,0)$-forms $\alpha_1, \dots, \alpha_p$ on Ω, the product

$$T \wedge i\alpha_1 \wedge \bar\alpha_1 \wedge \ldots \wedge i\alpha_p \wedge \bar\alpha_p$$

is a positive measure. The current T is said to be closed if $dT = 0$ and psh if $dd^cT \geq 0$.

In particular, if $p = n$, then T is a psh function u on Ω, that is a distribution such that

$$\sum_{1\leq j,k\leq n} \frac{\partial^2 u}{\partial z_j \partial \bar z_k} i dz_j \wedge d\bar z_k$$

is a positive current of bidimension $(n-1, n-1)$, the derivatives are taken here in the sense of distributions.

We denote $Psh(\Omega)$ the set of psh functions on Ω and $L^\infty_{loc}(\Omega) \cap Psh(\Omega)$ the subset of elements in $Psh(\Omega)$ which are locally bounded. The Kähler form on $\mathbb{C}^n$ is denoted $\beta(t) = dd^c|t|^2$. It can be decomposed into the sum $\beta'(t') + \beta''(t'')$ where β' and β'' are Kähler forms on $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$ respectively. For a point $a \in \mathbb{C}^k$, we denote j_a the map defined by

$$j_a : \begin{array}{ccc} \mathbb{C}^{n-k} & \longrightarrow & \{a\} \times \mathbb{C}^{n-k} \\ z'' & \longmapsto & (a, z') \end{array}$$

3. Proof of Theorem 1.1: slicing of the current $(\log|f|)[X]$

First of all, we require the following well known propositions:

Proposition 3.1. *Let* X *be an analytic subset of* Δ^n *and let* m *be its complex dimension. Then, the following statements hold:*

(1) *if $m < k$, then $\pi(X)$ is contained in a countable union of analytic subsets of Δ^k of dimensions $\leq m$.*

(2) *If $m \geq k$, then the set*

$$Z = \{a \in \Delta^k \ / \ dim_{\mathbb{C}}(X \cap \pi^{-1}(X)) \geq m - k + 1\}$$

is contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k - 1$.

As a consequence of the expansion of holomorphic functions in power series we have:

Proposition 3.2. *Let $f \not\equiv 0$ be a non vanishing holomorphic function on Δ^n such that for all $z'' \in \Delta^{n-k}, f(0, z'') = 0$. Then, there exist a natural number $m \geq 1$ and holomorphic functions $a_\mu(z'')$ on Δ^{n-k}, such that, for all $(z'; z'') \in \Delta^k \times \Delta^{n-k}$, we have $f(z', z'') = \sum_{j=m}^{\infty} Q_j(z'; z'')$ where $Q_m(z', z'') = \sum_{|\mu|=m} z'^{\mu} a_\mu(z'')$ and $|\mu| = \mu_1 + \cdots + \mu_k$.*

Now, given a psh function v on Δ^n such that v is locally integrable on the analytic subset X, then, for all test form Ψ, for all fixed $\varepsilon > 0$ and all $a \in S_\varphi$, we shall justify the well definition and the finiteness of the quantity

$$\int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} v[X] \wedge (dd^c\tilde{\varphi})^k \wedge \Psi.$$

Proposition 3.3. *Let $X \subset \Delta^n$ be analytic of pure dimension $p > k$, let $v \in Psh(\Delta^n) \cap L^1_{loc}(X)$ be negative and let $\varphi \in Psh(\Delta^k) \cap L^\infty_{loc}(\Delta^k)$ be given such that $S_\varphi = \Delta^k$. Then, for any point $a \in \Delta^k$, for any fixed $\varepsilon > 0$ and for any positive test form $\Psi \in \mathscr{D}_{(p-k,p-k)}(\Delta^n)$, we have*

$$\frac{1}{\mu_\varphi(B_k(a, \varepsilon))} \int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} -v[X] \wedge (dd^c\tilde{\varphi})^k \wedge \Psi < \infty.$$

Proof. The result is local. We may take $a = 0$, without loss of generality we may choose the test form Ψ such that $\Psi = \psi(z)\beta''^{n-k} \in \mathscr{D}_{(n-p,n-p)}(\Delta^{n-k})$ where ψ is a positive test function on Δ^{n-k}. Let denote

$$\Gamma_\varepsilon := \frac{1}{\mu_\varphi(B_k(a, \varepsilon))} \int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} -v[X] \wedge (dd^c\tilde{\varphi})^k \wedge \Psi. \tag{3.2}$$

There exists a neighborhood $U = U' \times U'' \subset \Delta^k \times \Delta^{p-k} \times \Delta^{n-p}$ of 0 and a coordinate system (t, ζ, z'') such that the projection

$$\pi_X : \begin{array}{l} X \cap U \to \Delta^p = \Delta^k \times \Delta^{p-k} \\ \big((t, \zeta), z''\big) \mapsto (t, \zeta) \end{array}$$

is a ramified covering of X. Let Z be the ramification locus of π_X and set

$$X_Z = X \cap ((U' \smallsetminus Z) \times U'') \subset X_{reg}.$$

The restriction of

$$\pi_{X_Z} : \ X_Z \to U' \smallsetminus Z$$

is then a covering with a finite sheet number. The expression of Γ_ε given by (3.2) will be transformed as

$$\Gamma_\varepsilon = \frac{1}{\mu_\varphi(B_k(0,\varepsilon))}\int_{X_Z\cap B_k(0,\varepsilon)\times\mathbb{C}^{n-k}} {\pi_{X_Z}}^*\big(-v(dd^c\varphi)^k\wedge\Psi\big). \tag{3.3}$$

We may choose $\varepsilon>0$ small enough, so that

$$U_\varepsilon := X_Z\cap B_k(0,\varepsilon)\times\mathbb{C}^{n-k}B_k(0,\varepsilon)\times\Delta^{p-k}\times\{0\}_{\mathbb{C}^{n-p}}.$$

Then, the expression given by (3.3) can be written as

$$\Gamma_\varepsilon = \frac{1}{\mu_\varphi(B_k(0,\varepsilon))}\int_{B_k(0,\varepsilon)} w(t)(dd^c\varphi)^k, \tag{3.4}$$

where

$$w(t) = \int_{\Delta^{p-k}} \pi^*_{X_Z}(-v\psi)(t,\zeta)d\lambda_{p-k}(\zeta).$$

Furthermore, following [6] we have

$$||w(dd^c\varphi)^k||_{B_k(0,\varepsilon)} \le ||w||_{L^1(B_k(0,2\varepsilon))}||\varphi||^k_{L^\infty(B_k(0,2\varepsilon))}, \tag{3.5}$$

the inequality (3.5) implies that w is a locally integrable function with respect to the positive measure μ_φ and hence the integral given by (3.4) is finite. □

Now we are ready to give the proof of Theorem 1.1.

Proof. The result is local. Without loss of generalities we may suppose $N=1$. Furthermore, for simplicity, we may consider the slice at point $a=0$. Put $T=\log|f|[Y]$ and

$$\Gamma_\varepsilon(T) = \frac{1}{\mu_\varphi(B_k(0,\varepsilon))}\int_{B_k(0,\varepsilon)\times\mathbb{C}^{n-k}} T\wedge(dd^c\tilde{\varphi})^k\wedge\Psi, \tag{3.6}$$

where Ψ is a test form such that $\Psi = h(z)\beta''^{n-k}\in\mathscr{D}_{(n-q,n-q)}(\Delta^{n-k})$ where h is a positive smooth function with compact support. We may find a neighborhood $U=U'\times U''$ of 0 in $\Delta^k\times\Delta^{q-k}\times\Delta^{n-q}$ and a coordinate system (t,ζ,z'') such that the projection

$$\pi_Y: \begin{array}{l} Y\cap U\to\Delta^q \\ \big((t,\zeta),z''\big)\mapsto(t,\zeta)\end{array}$$

defines a ramified covering of Y. Let Z be the ramification locus of π and $Y_Z = Y\cap((U'\smallsetminus Z)\times U'')\subset Y_{reg}$. The restriction of $\pi_{Y_Z}: Y_Z\to U'\smallsetminus Z$ is then a covering with a finite sheet number. Equality (3.6) can be written as

$$\Gamma_\varepsilon(T) = \frac{1}{\mu_\varphi(B_k(0,\varepsilon))}\int_{Y_Z\cap B_k(0,\varepsilon)\times\mathbb{C}^{n-k}} \pi^*\big(\log|f|(dd^c\varphi)^k\wedge\Psi\big). \tag{3.7}$$

We have to find the limit, as $\varepsilon\to 0$, of (6.20). For $\varepsilon>0$ small enough, the set $U_\varepsilon := Y_Z\cap B_k(0,\varepsilon)\times\mathbb{C}^{n-k}$ can be viewed as $B_k(0,\varepsilon)\times\Delta^{q-k}\times\{0\}_{\mathbb{C}^{n-q}}$. The integral in the right hand side of (6.20) can be written such that

$$\Gamma_\varepsilon(T) = \frac{1}{\mu_\varphi(B_k(0,\varepsilon))}\int_{B_k(0,\varepsilon)} w_h(t)(dd^c\varphi)^k, \tag{3.8}$$

where

$$\begin{aligned} w_h(t) &= \int_{\Delta^{q-k}} \log|f(t,\zeta,0)| h(\zeta,0) d\lambda_{q-k}(\zeta) \\ &= \int_{Y_Z\cap\pi^{-1}(0)} \log|i^*_{Y_Z\cap\pi^{-1}(0)} f(t,\zeta,z'')| i^*_{Y_Z\cap\pi^{-1}(0)} h(\zeta,z''), \end{aligned}$$

and $i_{Y_Z\cap\pi^{-1}(0)} : Y_Z\cap\pi^{-1}(0)\to\Delta^n$ is the canonical injection. Let E be the closed pluripolar subset of Δ^k defined by

$$E = \{a\in\Delta^k \ : w(a) = -\infty \ \text{ or } \ w\notin L^1_{loc}(\mu_\varphi) \ \text{ near } a\}.$$

By Theorem 1.1 in [11], for all $a\in\Delta^k\smallsetminus E$, we have:

(1) if $(i_{Y_Z\cap\pi^{-1}(0)})^* f\not\equiv 0$, then, as $\varepsilon\to 0$, the limit of the integral given by (3.8) will be

$$\begin{aligned} \lim_{\varepsilon\to 0}\Gamma_\varepsilon &= w_h(0) \\ &= \int_{\Delta^{q-k}} \log|f(0,\zeta,0)| h(\zeta,0) d\lambda_{q-k}(\zeta) \\ &= \int_{Y\cap\pi^{-1}(0)} (i_{Y_Z\cap\pi^{-1}(0)})^*[\log|f|\wedge\Psi] \\ &= \langle (i_{Y_Z\cap\pi^{-1}(0)})^*(\log|f|)[Y\cap\pi^{-1}(0)], (i_{Y_Z\cap\pi^{-1}(0)})^*(\Psi)\rangle_\varphi. \end{aligned}$$

(2) If $(i_{Y_Z\cap\pi^{-1}(0)})^* f\equiv 0$, then by Proposition 3.2 applied with the function $(i_{Y_Z\cap\pi^{-1}(0)})^* f$ on $\Delta^q = \Delta^k\times\Delta^{q-k}$, there exist an integer $m\in\mathbb{N}$, a non vanishing homogenous polynomial $Q_m(t,\zeta)$ and smooth functions $(t,\zeta)\mapsto R(t,\zeta)\in\mathscr{C}^\infty(\Delta^k\times\Delta^{q-k})$ such that

$$f(t,\zeta) = Q_m(t,\zeta) + tR(t,\zeta) \quad \text{on} \quad B_k(0,\varepsilon)\times\Delta^{p-k}.$$

Thanks to Taylor's formula, the rest $R(t,\zeta)$ can be chosen such that $|R(t,\zeta)|\le 1$. This with the triangle inequality legitimate the following inequalities

$$\begin{aligned} \log||Q_m| - |tR|| \le \log||f(t,\zeta)|| &= \log|(i_{Y_Z\cap\pi^{-1}(0)})^* f(t,w,z'')| \\ &\le \log(|Q_m| + |tR|). \end{aligned}$$

Since $(t,\zeta)\in B_k(0,\varepsilon)\times\Delta^{p-k}$ and $|R(t,\zeta)|\le 1$, then, as $\varepsilon\to 0$, $|t|\to 0$ and $|tR|\to 0$. Hence, we get

$$\begin{aligned} \lim_{\varepsilon\to 0}\Gamma_\varepsilon &= w_h(0) \\ &= \int_{\Delta^{q-k}} \log|Q_m(0,\zeta)| h(\zeta,z'') d\lambda_{q-k}(\zeta) \\ &= \int_{Y\cap\pi^{-1}(0)} (i_{Y_Z\cap\pi^{-1}(0)})^*[\log|Q_m(0,\zeta|\wedge\Psi] \\ &= \langle (\log|(i_{Y_Z\cap\pi^{-1}(0)})^* Q_m|)[Y\cap\pi^{-1}(0)], (i_{Y_Z\cap\pi^{-1}(0)})^*(\Psi)\rangle_\varphi. \end{aligned}$$

The proof is completed. □

The following example illustrates Theorem 1.1.

Example 3.4. In $\mathbb{C}^3$ we let $f(z) = z_1^2 + z_2^2 - z_3$, $Y = \{z_1 = -z_3\}$, $k = 1$ and $\varphi(z') = |z'|^2 = |z_1|^2$. It is clear that the subsets $X = \{f = 0\}$ and Y yield a complete intersection near 0. Hence, according to [9], we deduce that $\log|f|\in L^1_{loc}(Y)$. By Theorem 1.1, we have

$$\langle \log|f|.[Y], \pi, 0\rangle = \log|z_2^2 - z_3|.[z_1 = z_3 = 0].$$

4. Proof of Theorem 1.2: slicing of the current $v[X]$

Proof. We use notations and arguments as in the proof of Theorem 1.1 and we may suppose $N = 1$.

We have

$$\Gamma_\varepsilon(v[X]) = \frac{1}{\mu_\varphi(B_k(0,\varepsilon))}\int_{B_k(0,\varepsilon)\times\mathbb{C}^{n-k}} v[X]\wedge(dd^c\tilde{\varphi})^k\wedge\Psi. \tag{4.9}$$

Let X_{reg} be the set of regular points of X. Assume that $0\in X_{reg}$ and put

$$Z_1 = \{a\in\Delta^k : dim_{\mathbb{C}}(X\cap\pi^{-1}(a)) > p-k\}.$$

Then, by Proposition 3.1, Z_1 is contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k-1$. As the dimension m of the set X_{sing} of singular points, satisfies $m\leq p-1$, then, by Proposition 3.1, there exists a set Z_2 contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k-1$, such that for all $a\in\Delta^k\smallsetminus Z_2$, $X_{sing}\cap\pi^{-1}(a)$ is an analytic subset of Δ^k of dimension $m-k$ (otherwise is empty). Put $Z = Z_1\cup Z_2$ and denote $\bar{\pi} := \pi_{|X_{reg}}$. The expression (4.9) can be transformed to

$$\Gamma_\varepsilon(v[X]) = \frac{1}{\mu_\varphi(B_k(a,\varepsilon))}\int_{X_{reg}\smallsetminus(\bar{\pi}^{-1}(Z))\cap(B_k(a,\varepsilon)\times\mathbb{C}^{n-k})} \bar{\pi}^*[v(dd^c\varphi)^k\wedge\Psi]. \tag{4.10}$$

In a local chart of coordinates $(z_1,\dots,z_k,w_1,\dots,w_{n-k})$, such that

$$X_{reg}\smallsetminus(\bar{\pi}^{-1}(Z))\cap(B_k(a,\varepsilon)\times\mathbb{C}^{n-k}) = B_k(a,\varepsilon)\times\mathbb{C}^{p-k}\times\{0\}_{\mathbb{C}^{n-p}},$$

when $\varepsilon > 0$ is small enough, the integral (4.10), can be written as

$$\Gamma_\varepsilon = \frac{1}{\mu_\varphi(B_k(a,\varepsilon))}\int_{\Delta^p\smallsetminus(\bar{\pi}^{-1}(Z))\cap(B_k(a,\varepsilon)\times\mathbb{C}^{n-k})} v(z',w)(dd^c\varphi)^k\wedge\Psi(z',w). \tag{4.11}$$

According to [11], since $z'\mapsto v(z',w)$ is locally integrable on $X_{reg}\smallsetminus(\bar{\pi}^{-1}(Z)$, then when $\varepsilon\to 0$, we get the limit of (4.11), as follows

$$\begin{aligned}\lim_{\varepsilon\to 0}\Gamma_\varepsilon(v[X]) &= \int_{\Delta^{p-k}} v(a,.)\psi(a,w)\beta''^{p-k}\\ &= \int_{X_{reg}\setminus(\bar{\pi}^{-1}(Z))\cap\pi^{-1}\{a\}} v(a,.)\psi(a,w)\beta''^{p-k}\\ &= \langle v_{|X\cap\pi^{-1}(a)}.\big[X\cap\pi^{-1}(a)\big],(i_{X\cap\pi^{-1}(a)})^*\Psi\rangle.\end{aligned}$$

This finishes the proof. ☐

We illustrate Theorem 1.2 by the following example.

Example 4.1. In $\mathbb{C}^4 = \mathbb{C}\times\mathbb{C}^3$, we take

$$v(z) = \log(|z_1|^3+|z_3|^5),\ Y = \{z_4 = z_1^2z_2^4\}\ , k = 1 \text{ and } \varphi(z') = |z'|^2 = |z_1|^2.$$

The subset $X = \{z_1 = z_3 = 0\}$ that contains $\{v = -\infty\}$, yields a complete intersection with the subset Y near 0. By Theorem 1.2, up to a constant, we have

$$\langle v[Y],\pi,0\rangle = \log|z_3|\big[z_1 = z_2 = z_4 = 0\big].$$

For the next section, we will be concerned with points $a \in \Delta^k$ at which the slice $\langle U_{[X]} \wedge [Y], \pi, a\rangle$ of the current $U_{[X]} \wedge [Y]$, is well defined and how it can be expressed.

Remember that the wedge product $U_{[X]} \wedge [Y]$ was defined in [10] as a current on Y, as well as, the analytic subsets X and Y yield a complete intersection in Ω. Actually a necessary and sufficient condition was established, by expressing that the current $U_{[X]}$ has integrable coefficients with respect to the trace measure of the current $[Y]$, if and only if X and Y yield a complete intersection in Ω. The condition of complete intersection is optimal for the definition of this wedge product in the weak sense of currents. Recall that, X and Y yield a complete intersection, if for any irreducible components X_j of X and Y_k of Y, we have $Codim(X_j \cap Y_k) = Codim X_j + Codim Y_k$.

5. Proof of Theorem 1.3: slicing of the current $U_{[X]} \wedge [Y]$

In order to avoid complications, we assume here that the function φ is smooth and hence the measure μ_φ can be considered as the Lebesgue measure on $\mathbb{C}^k$, with density a smooth function denoted $\mu_\varphi(z')$.

Proof. Let $\Psi \in \mathscr{D}_{(p+q-n+1-k,p+q-n+1-k)}(\Delta^n)$ be a test form. We have to prove the existence of a pluripolar subset E of Δ^k such that for all $a \notin E$, we have

$$\lim_{\varepsilon\to 0} \Gamma_\varepsilon = \int_{\Delta^n} J_a^* U_{[X]} \wedge \left[Y \cap \pi^{-1}(a)\right] \wedge j_a^*(\Psi)$$

where

$$\Gamma_\varepsilon = \frac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} U_{[X]} \wedge [Y] \wedge (dd^c\tilde{\varphi})^k \wedge \Psi. \tag{5.12}$$

Since the potential $U_{[X]}$ involved in (5.12), is a finite linear combination of forms with coefficients $B_{I,J}(s,z)$ given by the following expression

$$B_{I,J}(s,z) = \int_{\xi\in\mathbb{C}^n} \eta(\xi) N(z-\xi)[X](\xi) \wedge \beta^{p-s}(\xi) \wedge 2^s i^{s^2} d\xi_I \wedge d\bar{\xi}_J \ (|I| = |J| = s). \tag{5.13}$$

We need to find $\lim_{\varepsilon\to 0} \Gamma_\varepsilon(B_{I,J}(s,z))$, where $\Gamma_\varepsilon(B_{I,J}(s,z))$ is given by

$$\Gamma_\varepsilon(B_{I,J}(s,z)) := \frac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{B_k(a,\varepsilon)\times\mathbb{C}^{n-k}} B_{I,J}(s,z) \wedge [Y] \wedge (dd^c\tilde{\varphi})^k \wedge \Psi. \tag{5.14}$$

Since the current $U_{[X]} \wedge [Y]$ is well defined then, the functions $z \mapsto B_{I,J}(s,z)$ are locally integrable on Y, and hence $\Gamma_\varepsilon(B_{I,J}(s,z))$ given by (5.14), can be written as

$$\begin{aligned} \Gamma_\varepsilon &:= \Gamma_\varepsilon(B_{I,J}(s,z)) \\ &= \tfrac{1}{\mu_\varphi(B_k(a,\varepsilon))} \int_{(\xi,z)\in V_\varepsilon} \eta(\xi) N(z-\xi)\left[X\times Y\right](\xi,z) \wedge (dd^c\varphi)^k \wedge F(\xi,z), \end{aligned}$$

where

$$F(\xi,z) = \beta^{p-s}(\xi) \wedge 2^s i^{s^2} d\xi_I \wedge d\bar{\xi}_J \wedge \Psi(z) \ \text{ and } V_\varepsilon = \mathbb{C}^n \times \left[B_k(a,\varepsilon)\times\mathbb{C}^{n-k}\right].$$

Let $[X \times Y]_{reg}$ be the set of regular points of $X \times Y$. We may assume that $0 \in [X \times Y]_{reg}$. Put

$$\begin{aligned} Z_1 &= \{a \in \Delta^k : dim_{\mathbb{C}}((X \times Y) \cap (\pi^{-1}(a) \times \pi^{-1}(a)) > p + q - k\} \\ &= \{a \in \Delta^k : dim_{\mathbb{C}}(X \cap \pi^{-1}(a)) \times (Y \cap \pi^{-1}(a)) > p + q - k\}. \end{aligned}$$

Following Proposition 3.1, Z_1 is contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k - 1$. As the dimension m of the subset $(X \times Y)_{sing}$ of singular points, satisfies $m \leq p + q - 1$, then, again by Proposition 3.1, there exists a subset Z_2 contained in a countable union of analytic subsets of Δ^k of dimensions $\leq k-1$, such that for any point $a \in \Delta^k \smallsetminus Z_2$, the set $(X \times Y)_{sing} \cap (\pi^{-1}(a) \times \pi^{-1}(a)$ is analytic in Δ^k of dimension $m - k$ (otherwise is empty). Put $Z = Z_1 \cup Z_2$ and denote $\bar{\pi} := \pi_{|(X \times Y)_{reg}}$, we have

$$\Gamma_\varepsilon = \int_{(X\times Y)_{reg} \smallsetminus (\bar{\pi}^{-1}(Z)) \cap V_\varepsilon} \bar{\pi}^* \big[\eta(\xi) N(z - \xi)(dd^c\varphi)^k \wedge F(\xi, z)\big]. \tag{5.15}$$

We can find local coordinates

$$(\xi, z) = (\xi, (z', w)) = \big(\xi, (z_1, \ldots, z_k, w_1, \ldots, w_{n-k})\big),$$

so that, for all $\varepsilon > 0$ small enough,

$$(X \times Y)_{reg} \smallsetminus (\bar{\pi}^{-1}(Z)) \cap V_\varepsilon = \mathbb{C}^p \times \{0\}_{\mathbb{C}^{n-p}} \times B_k(a, \varepsilon) \times \mathbb{C}^{q-k} \times \{0\}_{\mathbb{C}^{n-q}}.$$

Since $\mu_\varphi(B_k(a, \varepsilon)) \sim \omega_{2k} \varepsilon^{2k} \mu_\varphi(a)$ as $\varepsilon \to 0$. Then, by an application of Fubini's theorem and by the change of variables $(\xi, z') \leftrightarrow (\xi, \frac{z'-a}{\varepsilon})$, when $\varepsilon > 0$ is small enough, the equality (5.15) can be transformed to the following

$$\Gamma_\varepsilon = \frac{1}{\omega_{2k} \mu_\varphi(a)} \int_V \mu_\varphi(a + \varepsilon t) \eta(\xi) N((a + \varepsilon t, w) - \xi) F(\xi, (a + \varepsilon t, w)) d\nu(t, w), \tag{5.16}$$

where $V = \mathbb{C}^{p+q-k} \times B_k(0, 1)$ and $d\nu(t, w) = d\lambda_k(t) \otimes d\lambda_{p+q-k}(w)$. By letting $\varepsilon \to 0$ in (5.16), we get

$$\begin{aligned} \lim_{\varepsilon \to 0} \Gamma_\varepsilon &= \int_{\mathbb{C}^{p+q-k}} \eta(\xi) N((a, w) - \xi) F(\xi, (a, w)) d\lambda_{p+q-k}(w) \\ &= \int_{(X\times Y)_{reg} \cap (\pi^{-1}(a) \times \pi^{-1}(a))} \eta(\xi) N((a, w) - \xi) \wedge F(\xi, (a, w)) \\ &= \int_{(X \cap \pi^{-1}(a))_{reg} \times (Y \cap \pi^{-1}(a))_{reg}} \eta(\xi) N((a, w) - \xi) \wedge F(\xi, (a, w)) \\ &= \int_{\mathbb{C}^{n-k}} B_{I,J}(s, (a, w)) \wedge [Y \cap \pi^{-1}(a)] \wedge \Psi(a, z'') \\ &= \langle J_a^*(B_{I,J}(s, z)) \wedge [Y \cap \pi^{-1}(a)], j_a^*(\Psi) \rangle. \end{aligned}$$

The last equality holds true since we have

$$\dim \big((X \times Y)_{sing} \cap (\pi^{-1}(a) \times \pi^{-1}(a))\big) < p + q - k.$$

This achieves the proof of Theorem 1.3. □

Let illustrate Theorem 1.3 with the following example.

Example 5.1. Consider in $\mathbb{C}^5 = \mathbb{C} \times \mathbb{C}^4$, $X = \{z_2 = z_3 = z_4 = 0\}$ and $Y = \{z_4 = z_2^2 z_3^2\}$ take $k = 1$ and $\varphi(z') = |z'|^2 = |z_1|^2$. Since X is smooth, then a potential of $[X]$ is

$$U = \frac{\log(|z_2|^2 + |z_3|^2 + |z_4|^2) dd^c \log(|z_2|^2 + |z_3|^2 + |z_4|^2)}{|z_2|^2 + |z_3|^2 + |z_4|^2}.$$

As X and Y yield a complete intersection, then by [9], the current $U \wedge [Y]$ is well defined. The current $U \wedge [Y]$ is given by the following

$$U \wedge [Y] = i_Y^* \left[\frac{\log(|z_2|^2 + |z_3|^2 + |z_2 z_3|^4) dd^c \log(|z_2|^2 + |z_3|^2 + |z_2 z_3|^4)}{|z_2|^2 + |z_3|^2 + |z_2 z_3|^4} \right].$$

We have

$$X \cap \pi^{-1}(0) = \{z_1 = z_2 = z_3 = z_4 = 0\} \text{ and } Y \cap \pi^{-1}(0) = \{z_1 = z_4^2 - z_2 z_3 = 0\}.$$

It is clear that $X \cap \pi^{-1}(0)$ and $Y \cap \pi^{-1}(0)$ yield a complete intersection in $\{0\} \times \mathbb{C}^4$. If $\sigma = i_{Y \cap \pi^{-1}(0)}$, then,

$$\langle U \wedge [Y], \pi, 0 \rangle = \sigma^* \left[\frac{\log(|z_2|^2 + |z_3|^2 + |z_2 z_3|^4) dd^c \log(|z_2|^2 + |z_3|^2 + |z_2 z_3|^4)}{|z_2|^2 + |z_3|^2 + |z_2 z_3|^4} \right].$$

6. Proofs of Theorem 1.4 and Theorem 1.5 with applications

This section aims to give the proof of Theorem 1.4 and the proof of Theorem 1.5. We may suppose $N = 1$. Indeed, for the proof of Theorem 1.5, we use the fact that a finite union of pluripolar subsets is pluripolar. In addition, Theorem 1.4 is an immediate application of the following Theorem 6.1. In fact, if for all $1 \leq j \leq N$, there exists $\alpha_j > 0$ for which Theorem 6.1 holds for the psh function v_j, then for $\alpha = \min_{1 \leq j \leq N}(\alpha_j)$, Theorem 1.4 can be deduced from Theorem 6.1 in terms of slices and it works with the psh function $v = \sum_{1 \leq j \leq N} \exp(-\alpha v_j)$.

Theorem 6.1. *Assume we have a psh function v on Δ^n such that its set of singular points is contained in a hypersurface X forming a complete intersection with the analytic subset Y of Δ^n. Then, there exists $\alpha > 0$, such that the function $\exp(-\alpha v)_{|Y}$ lies in $L^1_{loc}(Y)$.*

Proof. The result is local, so it's enough to prove it near a singular point $z_0 = 0 \in \{v = -\infty\} \subset X$. We may suppose z_0 is a regular point of $X \cap Y$. Let q be the dimension of Y at z_0 $(1 \leq q \leq n-1)$, since X and Y yield a complete intersection near z_0, then we may find a neighborhood $V(z_0)$ of z_0 and local coordinates $(z_1, \ldots, z_n)$ such that

$$K \cap X \cap Y \subset K \cap Y \cap \{z_q = z_{q+1} = \cdots = z_n = 0\}, \quad K = \overline{V}(z_0). \tag{6.17}$$

Let $(v_\delta)_{\delta > 0}$ be a decreasing sequence of continuous psh functions such that $\lim_{\delta \to 0} v_\delta = v$ pointwise. We choose a fixed $\delta_0 > 0$ small enough, such that

$$v_{\delta_0}(z) - 1 \leq v(z) \leq v_{\delta_0}(z), \quad \forall\, z \in K \smallsetminus X \cap Y. \tag{6.18}$$

According to a classical result due to H. J. Bremermann and P. Lelong [4, 14], there exists $(f_j)_j$ a sequence of holomorphic functions on Δ^n such that the function v_{δ_0} is the regularized supremum limit on K of $\left(\frac{1}{j}\log|f_j|\right)_j$. Which means that

$$\begin{aligned} v_{\delta_0}(z) &= \left[\lim_{j\to\infty}\sup \tfrac{1}{j}\log|f_j(z)|\right]^* \\ &= \lim_{\varepsilon\to 0}\sup_{\zeta\in B(z,\varepsilon)}[\lim_{j\to\infty}\tfrac{1}{j}\log|f_j(\zeta)|], \quad z\in K. \end{aligned} \tag{6.19}$$

For the compact K, in view of (6.18) and (6.19), we can find a natural number N that depends on K, and analytic functions on Δ^n, $f_1,\ldots,f_N$, such that the function v and the function v_N defined by

$$v_N(z) = \max_{1\le j\le N}\frac{1}{j}\log|f_j(z)|, \tag{6.20}$$

satisfy the following inequalities

$$v_N - 1 \le v \le v_N \le 0 \quad \text{on} \quad K\setminus X\cap Y. \tag{6.21}$$

We may suppose the compact K is sufficiently small so that

$$K\cap X\cap Y \subset K\cap Y\cap\{f_j=0,\ j=1,\ldots,N\}.$$

By the Weierstrass preparation theorem (see [6]), for each indice $1\le j\le N$, one can write f_j such that

$$f_j(z) = h_j(z)P_j(z',z_n), \tag{6.22}$$

where h_j is an invertible holomorphic function on Δ^n and $P_j(z',z_n)$ is a Weierstrass polynomial in z_n, of the form

$$P_j(z',z_n) = z_n^{m_j} + a_{1,j}(z')z_n^{m_j-1} + \cdots + a_{\nu,j}(z')z_n^{m_j-\nu} + \cdots + a_{m_j,j,}(z'),\ a_{\nu,j}(0)=0,$$

with $m_j\ge 1$ is the vanishing order of f_j at $z_0=0$ and $(a_{\nu,j,}(z'))_{1\le\mu\le m_j}$ are holomorphic coefficients on the polydisc Δ^{n-1} in $\mathbb{C}^{n-1}$. Furthermore, for all $1\le j\le N$ and all $1\le\nu\le m_j$, there is a positive constant C_1 such that for all $z'\in\Delta^{n-1}$, the following inequality holds

$$|a_{\nu,j}(z')| \le C_1||z'||^{\nu}. \tag{6.23}$$

Therefore, by the triangle inequality, (6.23) provides the existence of a constant $C_2 = C_3(K) > 0$ such that for all $1\le j\le N$ and for all $z=(z',z_n)\in K$, the following inequality holds

$$|P_j(z',z_n)| \le C_2||z'||^{m_j}. \tag{6.24}$$

On the other hand, since $\{f_j=0,\ j=1,\ldots,N\} = \{P_j=0,\ j=1,\ldots,N\}$ and since the polynomial P_j vanishes at $z_0=0$, then there is a constant $C_3>0$, such that for any point $z=(z',z_n)\in K$ sufficiently close to 0, the following inequality holds

$$|z_n| \le C_3||z'||. \tag{6.25}$$

Hence, using the triangle inequality with (6.25) and reasoning by induction on the degree m_j of P_j, $1 \leq j \leq N$, we can find a constant $C_4 = C_4(K) > 0$, such that for all $1 \leq j \leq N$ and for all $z = (z', z_n) \in K$, the polynomial $P_j(z', z_n)$ satisfies the following inequality

$$C_4||z'||^{m_j} \leq |P_j(z', z_n)|. \tag{6.26}$$

In addition, since for all $1 \leq j \leq N$, the holomorphic function h_j is invertible and N depends only on K, then we can find a constant $C_5 = C_5(K) > 0$ such that for all $z \in K$, the function defined by

$$v'_N(z) = \max_{1 \leq j \leq N} \frac{1}{j} \log |h_j(z)|$$

satisfies the following inequality

$$-C_5 \leq v'_N(z) \leq 0. \tag{6.27}$$

If we denote $\beta_N = \min_{1 \leq j \leq N} \frac{m_j}{j}$ and $w_N(z) = w_N(z', z_n) = \beta_N \log ||z'||$, then following (6.22) the function v_N defined by (6.20) is such that

$$\begin{aligned} v_N(z) &= \max_{1 \leq j \leq N} \tfrac{1}{j} \log |P_j(z', z,_n)| + \max_{1 \leq j \leq N} \tfrac{1}{j} \log |h_{j,\delta>0}(z)| \\ &\leq \beta_N \log ||z'|| + \max_{1 \leq j \leq N} \tfrac{1}{j} \log |h_j(z)| \\ &= w_N(z) + v'_N(z) \leq 0. \end{aligned} \tag{6.28}$$

Hence, in view of (6.21), (6.24), (6.26) and (6.28), we can find constants $C_6 = C_6(K) > 0$ and $C_7 = C_7(K) > 0$ such that for all $z \in K \setminus Y \cap X$, the functions v and w_N satisfy the following inequalities

$$C_6 w_N(z) - C_7 \leq v(z) \leq w_N(z) \leq 0. \tag{6.29}$$

It is clear that $(\beta_N)_N$ is a positive and decreasing sequence. Hence it has a limit $\beta \geq 0$ as $N \to +\infty$. In view of (6.17) and (6.29) we have $\beta > 0$. Indeed, if not, the function v will be bounded near 0 and then the point z_0 is not a singular point of v. Therefore, by letting $N \to +\infty$ in (6.29), we can find constants $\beta > 0$ and $C_8 = C_8(K) > 0$ such that for all $z = (z', z_n) \in K \setminus Y \cap X$, the function v satisfies the following inequality

$$\exp(-v(z)) \leq C_8 ||z'||^{-\beta}. \tag{6.30}$$

Taking $\alpha > 0$ so that $\alpha\beta \in (0, 2q-2)$, hence $z \mapsto \frac{1}{||z||^{\alpha\beta}} \in L^1_{loc}(\mathbb{C}^{q-1})$, and investigating (6.17) with (6.30), we get the following

$$\int_{z \in K \setminus X \cap Y} \exp(-\alpha v(z)) \leq C_8 \int_{z \in \Delta^{q-1}} \frac{d\lambda_{q-1}(z)}{||z||^{\alpha\beta}} < \infty. \tag{6.31}$$

Consequently, our proof is achieved thanks to (6.31). □

Remark 6.2. Theorem 6.1 generalizes a result in [9] showing that, if $Y = \{f = 0\}$ is a hypersurface in Ω given by a holomorphic function not identically vanishing and if Y yields a complete intersection with another analytic subset $X \subset \Omega$, then there exists $\delta > 0$ such that the coefficients of the current $\log |f|[X]$ lies in $L^{1+\delta}_{loc}(X)$. Moreover, there exists $\alpha > 0$, such that the coefficients of the current $|f|^{-\alpha}[X]$ lie in $L^1_{loc}(X)$. If we replace the hypersurface $Y = \{f = 0\}$ by a positive $(1,1)$-closed current of the form $T = dd^c v$ for some psh function v, then we provide Theorem 1.4. Note that we invite interested readers to proceed defining and investigating quaternionic hypersurfaces and quaternionic analytic sets in the space $\mathbb{H}_{\mathbb{C}}$ recently studied in [13] and find similar results as given in this paper.

In the direction of Theorem 1.3 and proceeding as in the proof of Theorem 3.3 in [9], we finally give the proof of Theorem 1.5.

Proof. We consider the analytic subsets $X \cap \pi^{-1}(a)$, $Y \cap \pi^{-1}(a)$ and

$$E = \{a \in \Delta^k : dim_{\mathbb{C}}(X \cap \pi^{-1}(a)) \times (Y \cap \pi^{-1}(a)) > p + q - k\}.$$

By Proposition 3.1, E is contained in a countable union of analytic subsets of Δ^k of dimension $\leq k-1$. Therefore the set E is pluripolar. We work as in [9] (proof of Theorem 3.3) around a point $a \notin E$. □

Competing interests

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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