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Some Properties on The [*p,q*]-Order of Meromorphic Solutions of Homogeneous and Non-homogeneous Linear Differential Equations With Meromorphic Coefficients

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Abstract. In the present paper, we investigate the $[p,q]$-order of solutions of higher order linear differential equations

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0$$

and

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z),$$

where $A_0(z), A_1(z), ..., A_k(z) \not\equiv 0$ and $F(z) \not\equiv 0$ are meromorphic functions of finite $[p,q]$-order. We improve and extend some results of the authors by using the concept $[p,q]$-order.

1. Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [7], [9], [14], [24]). In addition, for any integers $p \geq q \geq 1$ and a meromorphic function f in the whole complex plane, we will use $\rho_{[p,q]}(f)$, $\mu_{[p,q]}(f)$ to denote respectively the $[p,q]$-order and the lower $[p,q]$-order, $\overline{\lambda}_{[p,q]}(f-a)$ (or $\lambda_{[p,q]}(f-a)$) to denote the $[p,q]$-convergence exponent of the sequence of distinct a-points (or of a-points) and $\lambda_{[p,q]}\left(\frac{1}{f}\right)$ to denote the $[p,q]$-exponent of convergence of the poles, we refer the reader to see [12], [15], [16] and [25]. In particular for $q=1$, $\rho_{[p,1]}(f) = \rho_p(f)$ is the iterated p-order, $\mu_{[p,1]}(f) = \mu_p(f)$ is the iterated lower p-order, $\overline{\lambda}_{[p,1]}(f-a) = \overline{\lambda}_p(f,a)$ (or $\lambda_{[p,1]}(f-a) = \lambda_p(f,a)$) is the iterated convergence exponent of the sequence of distinct a-points (or of a-points), $\lambda_{[p,1]}\left(\frac{1}{f}\right) = \lambda_p\left(\frac{1}{f}\right)$ is the iterated exponent of convergence of the poles, see [7], [11], [13], [14] and [24] for notations and definitions.

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Several authors have investigated the growth of solutions of second order and higher order homogeneous and non-homogeneous linear differential equations with analytic, entire or meromorphic coefficients, see ([1−3], [6], [8], [11], [13−16], [18] , [20−21], [23], [25]). In the recent years, many authors have studied the complex linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.2}$$

where $A_0(z) \not\equiv 0$, $A_1(z), ..., A_{k-1}(z)$ and $F(z) \not\equiv 0$ are meromorphic functions of finite iterated p-order. In [2], Belaïdi considered the growth of meromorphic solutions of equations (1.1) and (1.2) with meromorphic coefficients of finite iterated $p-$order and obtained some results which improve and generalize some previous results.

Theorem A ([2]) *Let $H \subset [0, +\infty)$ be a set with a positive upper density, and let $A_j(z)$ $(j = 0, 1, ..., k-1)$ be meromorphic functions with finite iterated p-order. If there exist positive constants $\sigma > 0, \alpha > 0$ such that $\rho = \max\left\{\rho_p\left(A_j\right) : j = 1, ..., k-1\right\} < \sigma$ and $|A_0(z)| \geq \exp_p(\alpha r^\sigma)$ as $|z| = r \in H$, $r \to +\infty$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies*

$$\mu_p(f) = \rho_p(f) = +\infty,\ \rho_{p+1}(f) \geq \sigma.$$

Furthermore, if $\lambda_p\left(\frac{1}{f}\right) < \infty$, then $i(f) = p + 1$ and

$$\sigma \leq \rho_{p+1}(f) \leq \rho_p(A_0).$$

Theorem B ([2]) *Let $H \subset [0, +\infty)$ be a set with a positive upper density, and let $A_j(z)$ $(j = 0, 1, ..., k-1)$ and $F(z) \not\equiv 0$ be meromorphic functions with finite iterated p-order. If there exist positive constants $\sigma > 0, \alpha > 0$ such that $|A_0(z)| \geq \exp_p(\alpha r^\sigma)$ as $|z| = r \in H$, $r \to +\infty$, and $\rho = \max\left\{\rho_p\left(A_j\right)\ (j = 1, ..., k-1), \rho_p(F)\right\} < \sigma$, then every meromorphic solution of equation (1.2) with $\lambda_p\left(\frac{1}{f}\right) < \sigma$ satisfies*

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty,\ \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f).$$

Furthermore, if $\lambda_p\left(\frac{1}{f}\right) < \min\{\mu_p(f), \sigma\}$, then $i(f) = p + 1$ and

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) \leq \rho_p(A_0).$$

Recently, in [18] the authors have studied the growth of solutions of the equations (1.1) and (1.2) when $A_s(z)$ to dominate all other coefficients and they got some results about $\rho_{p+1}(f)$ as follows.

Theorem C ([18]) *Let $H \subset (1, +\infty)$ be a set with a positive upper logarithmic density (or $m_l(H) = +\infty$), and let $A_j(z)$ $(j = 0, 1, ..., k-1)$ be meromorphic functions with finite iterated p-order.*

If there exist positive constants $\sigma > 0, \alpha > 0$ and an integer s, $0 \leq s \leq k-1$, such that $|A_s(z)| \geq \exp_p(\alpha r^{\sigma})$ as $|z| = r \in H$, $r \to +\infty$, and $\rho = \max\left\{\rho_p\left(A_j\right)\ (j \neq s)\right\} < \sigma$, then every non-transcendental meromorphic solution $f \not\equiv 0$ of (1.1) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution f of (1.1) with $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ satisfies $i(f) = p+1$

$$\mu_p(f) = \rho_p(f) = +\infty$$

and

$$\sigma \leq \rho_{p+1}(f) \leq \rho_p(A_s).$$

Theorem D ([18]) *Let $H \subset (1, +\infty)$ be a set with a positive upper logarithmic density (or $m_l(H) = +\infty$), and let $A_j(z)$ $(j = 0, 1, ..., k-1)$ and $F(z) \not\equiv 0$ be meromorphic functions with finite iterated p-order. If there exist positive constants $\sigma > 0, \alpha > 0$ and an integer s, $0 \leq s \leq k-1$, such that $|A_s(z)| \geq \exp_p(\alpha r^{\sigma})$ as $|z| = r \in H$, $r \to +\infty$, and $\max\left\{\rho_p\left(A_j\right)\ (j \neq s),\ \rho_p(F)\right\} < \sigma$, then every non-transcendental meromorphic solution f of (1.2) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution f of (1.2) with $\lambda_p\left(\frac{1}{f}\right) < \min\{\sigma, \mu_p(f)\}$ satisfies $i(f) = p+1$*

$$\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \mu_p(f) = +\infty$$

and

$$\sigma \leq \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) \leq \rho_p(A_s).$$

Thus, the following question arises: can we have the same properties as in Theorems C and D for the solutions of equations

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.3}$$

and

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.4}$$

when the coefficients A_j $(j = 0, 1, ..., k)$ are of $[p, q]-$order? In this paper, we proceed this way and we obtain the following results.

Theorem 1.1 *Let $H \subset (1, +\infty)$ be a set with a positive upper logarithmic density (or $m_l(H) = +\infty$) and let $A_j(z)$ $(j = 0, 1, ..., k)$ with $A_k(z) \not\equiv 0$ be meromorphic functions with finite $[p, q]$-order. If there exist a positive constant $\sigma > 0$ and an integer s, $0 \leq s \leq k$, such that for sufficiently small $\varepsilon > 0$, we have $|A_s(z)| \geq \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q r\right\}$ as $|z| = r \in H$, $r \to +\infty$ and $\rho = \max\left\{\rho_{[p,q]}\left(A_j\right)\ (j \neq s)\right\} < \sigma$, then every non-transcendental meromorphic solution $f \not\equiv 0$ of (1.3) is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution f of (1.3) with $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$ satisfies*

$$\rho_{[p,q]}(f) = \mu_{[p,q]}(f) = +\infty,\ \sigma \leq \rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s).$$

Remark 1.1 Putting $A_k(z) \equiv 1$ and $q = 1$ in Theorem 1.1, we obtain Theorem C.

Corollary 1.1 *Under the hypotheses of Theorem 1.1, suppose further that φ is a transcendental meromorphic function satisfying $\rho_{[p+1,q]}(\varphi) < \sigma$. Then, every transcendental meromorphic solution f of equation* (1.3) *with $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$ satisfies*

$$\sigma \leq \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi)$$

$$= \rho_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s).$$

Considering the non-homogeneous linear differential equation (1.4), we obtain the following results.

Theorem 1.2 *Let $H \subset (1,+\infty)$ be a set with a positive upper logarithmic density (or $m_l(H) = +\infty$), and let $A_j(z)$ $(j = 0, 1, ..., k)$ with $A_k(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions with finite $[p,q]$-order. If there exist a positive constant $\sigma > 0$ and an integer s, $0 \leq s \leq k$, such that for sufficiently small $\varepsilon > 0$, we have $|A_s(z)| \geq \exp_{p+1}\left\{(\sigma-\varepsilon)\log_q r\right\}$ as $|z| = r \in H$, $r \to +\infty$ and $\max\left\{\rho_{[p,q]}\left(A_j\right) \ (j \neq s),\ \rho_{[p,q]}(F)\right\} < \sigma$, then every non-transcendental meromorphic solution f of* (1.4) *is a polynomial with $\deg f \leq s-1$ and every transcendental meromorphic solution f of* (1.4) *with $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \min\left\{\sigma, \mu_{[p,q]}(f)\right\}$ satisfies*

$$\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f) = \mu_{[p,q]}(f) = +\infty$$

and

$$\sigma \leq \overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s).$$

Remark 1.2 Putting $A_k(z) \equiv 1$ and $q = 1$ in Theorem 1.2, we obtain Theorem D.

Corollary 1.2 *Let $A_j(z)$ $(j = 0, 1, ..., k)$, $F(z)$, H satisfy all the hypotheses of Theorem 1.2, and let φ be a transcendental meromorphic function satisfying $\rho_{[p+1,q]}(\varphi) < \sigma$. Then, every transcendental meromorphic solution f with $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \min\{\sigma, \mu_{[p,q]}(f)\}$ of equation* (1.4) *satisfies* $\sigma \leq \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f-\varphi) \leq \rho_{[p,q]}(A_s)$.

Remark 1.3 In [17, 19], the authors have studied the growth and the oscillation of solutions of equations (1.3) and (1.4) when the coefficients $A_j(z)$ $(j = 0, 1, ..., k)$ and $F(z)$ are entire functions of iterated p-order or of $[p,q]$-order. However, in the present paper the coefficients $A_j(z)$ $(j = 0, 1, ..., k)$ and $F(z)$ are meromorphic functions with reduction of the hypotheses in Theorems 1.1 and 1.2. So, this article may be understood as an extension and an improvement of [17, 19].

2. Some auxiliary lemmas

In order to prove our theorems, we need the following definition, proposition and lemmas. The Lebesgue linear measure of a set $E \subset [0, +\infty)$ is $m(E) = \int_E dt$, and the logarithmic measure of a set $F \subset [1, +\infty)$ is $m_l(F) = \int_F \frac{dt}{t}$. The upper density of $E \subset [0, +\infty)$ is given by

$$\overline{dens}(E) = \limsup_{r\to\infty} \frac{m(E \cap [0, r])}{r}$$

and the upper logarithmic density of the set $F \subset [1, +\infty)$ is defined by

$$\overline{\log dens}(F) = \limsup_{r\longrightarrow+\infty} \frac{m_l(F \cap [1, r])}{\log r}.$$

Proposition 2.1 ([2]) For all $H \subset (1, +\infty)$ the following statements hold:
(i) If $m_l(H) = +\infty$, then $m(H) = +\infty$;
(ii) If $\overline{dens}(H) > 0$, then $m(H) = +\infty$;
(iii) If $\overline{\log dens}(H) > 0$, then $m_l(H) = +\infty$.

Lemma 2.1 ([5]) *Let f be a transcendental meromorphic function in the plane, and let $\alpha > 1$ be a given constant. Then, there exist a set $E_1 \subset (1, +\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ depending only on α and (i, j) ((i, j) positive integers with $i > j$) such that for all z with $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(i)}(z)}{f^{(j)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\right)^{i-j}.$$

Lemma 2.2 ([4]) *Let $p \geq q \geq 1$ be integers and g be an entire function such that $\rho_{[p,q]}(g) < +\infty$. Then, there exist entire functions $u(z)$ and $v(z)$ such that*

$$g(z) = u(z)e^{v(z)},$$

$$\rho_{[p,q]}(g) = \max\left\{\rho_{[p,q]}(u), \rho_{[p,q]}\left(e^{v(z)}\right)\right\}$$

and

$$\rho_{[p,q]}(u) = \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{g}\right)}{\log_q r}.$$

Moreover, for any given $\varepsilon > 0$, we have

$$|u(z)| \geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(u) + \varepsilon\right)\log_q r\right\}\right\} \quad (r \notin E_2),$$

where $E_2 \subset (1, +\infty)$ is a set of r of finite linear measure.

Lemma 2.3 *Let $p \geq q \geq 1$ be integers. Suppose that f is a meromorphic function such that $\rho_{[p,q]}(f) < +\infty$. Then, there exist entire functions $u_1(z)$, $u_2(z)$ and $v(z)$ such that*

$$f(z) = \frac{u_1(z)\, e^{v(z)}}{u_2(z)} \tag{2.1}$$

and

$$\rho_{[p,q]}(f) = \max\left\{\rho_{[p,q]}(u_1), \rho_{[p,q]}(u_2), \ \rho_{[p,q]}(e^{v(z)})\right\}. \tag{2.2}$$

Moreover, for any given $\varepsilon > 0$, we have

$$\exp\left\{-\exp_p\left\{(\rho_{(p,q)}(f) + \varepsilon)\log_q r\right\}\right\} \leq |f(z)|$$
$$\leq \exp_{p+1}\left\{(\rho_{(p,q)}(f) + \varepsilon)\log_q r\right\} \quad (r \notin E_3), \tag{2.3}$$

where $E_3 \subset (1, +\infty)$ is a set of r of finite linear measure.

Proof. When $p \geq q = 1$, the lemma is due to Tu and Long [21]. Thus, we assume that $p > q > 1$ or $p = q > 1$. By Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_{[p,q]}(g) = \mu_{[p,q]}(f) = \mu \leq \rho_{[p,q]}(f) = \rho_{[p,q]}(g) < +\infty$$

and

$$\lambda_{[p,q]}(d) = \rho_{[p,q]}(d) = \lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu.$$

By Lemma 2.2, there exist entire functions $u(z)$ and $v(z)$ such that

$$g(z) = u(z)e^{v(z)}, \ \ \rho_{[p,q]}(g) = \max\left\{\rho_{[p,q]}(u), \rho_{[p,q]}\left(e^{v(z)}\right)\right\}.$$

So, there exist entire functions $u(z)$, $v(z)$ and $d(z)$ such that

$$f(z) = \frac{u(z)e^{v(z)}}{d(z)}$$

and

$$\rho_{[p,q]}(f) = \max\left\{\rho_{[p,q]}(u), \rho_{[p,q]}(d), \rho_{[p,q]}\left(e^{v(z)}\right)\right\}.$$

Thus (2.1) and (2.2) hold. Set $f(z) = \frac{u_1(z)e^{v(z)}}{u_2(z)}$, where $u_1(z)$, $u_2(z)$ are the canonical products formed with the zeros and poles of f respectively. By the definition of $[p,q]$-order, for sufficiently large r and any given $\varepsilon > 0$, we have

$$\begin{aligned} |u_1(z)| &\leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(u_1) + \tfrac{\varepsilon}{3}\right)\log_q r\right\}, \\ |u_2(z)| &\leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(u_2) + \tfrac{\varepsilon}{3}\right)\log_q r\right\}. \end{aligned} \tag{2.4}$$

Since $\max\left\{\rho_{[p,q]}(u_1), \rho_{[p,q]}(u_2), \ \rho_{[p,q]}(e^{v(z)})\right\} = \rho_{[p,q]}(f)$, then we obtain

$$|u_1(z)| \leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(f) + \frac{\varepsilon}{3}\right)\log_q r\right\}, \tag{2.5}$$

$$|u_2(z)| \leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(f) + \frac{\varepsilon}{3}\right)\log_q r\right\}, \tag{2.6}$$

$$\left|e^{v(z)}\right| \leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}. \tag{2.7}$$

By Lemma 2.2, there exists a set $E_3 \subset (1,+\infty)$ of r with a finite linear measure such that for any given $\varepsilon > 0$, we have

$$|u_1(z)| \geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(u_1)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}$$

$$\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}, \quad (r \notin E_3), \tag{2.8}$$

$$|u_2(z)| \geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(u_2)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}$$

$$\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}, \quad (r \notin E_3). \tag{2.9}$$

Then, by using (2.5), (2.7) and (2.9), we obtain for sufficiently large $r \notin E_3$ and any given $\varepsilon > 0$

$$|f(z)| = \frac{|u_1(z)|\left|e^{v(z)}\right|}{|u_2(z)|}$$

$$\leq \frac{\exp_{p+1}\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}\exp_{p+1}\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}}{\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}}$$

$$\leq \exp_{p+1}\left\{\left(\rho_{[p,q]}(f)+\varepsilon\right)\log_q r\right\}. \tag{2.10}$$

On the other hand, we have $\rho_{[p-1,q]}(v) = \rho_{[p,q]}\left(e^{v(z)}\right) \leq \rho_{[p,q]}(f)$ and $\left|e^{v(z)}\right| \geq e^{-|v(z)|}$. Making use of the definition of $[p,q]$–order, we obtain

$$|v(z)| \leq M(r,v) \leq \exp_p\left\{\left(\rho_{(p-1,q)}(v)+\frac{\varepsilon}{3}\right)\log_q r\right\}$$

$$\leq \exp_p\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}.$$

Then, for sufficiently large r and any given $\varepsilon > 0$, we have

$$\left|e^{v(z)}\right| \geq e^{-|v(z)|} \geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}. \tag{2.11}$$

By (2.6), (2.8) and (2.11), we can easily obtain

$$|f(z)| = \frac{|u_1(z)|\left|e^{v(z)}\right|}{|u_2(z)|}$$

$$\geq \frac{\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}\exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}}{\exp_{p+1}\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}}.$$

$$= \exp\left\{-3\exp_p\left\{\left(\rho_{[p,q]}(f)+\frac{\varepsilon}{3}\right)\log_q r\right\}\right\}$$

$$\geq \exp\left\{-\exp_p\left\{\left(\rho_{[p,q]}(f)+\varepsilon\right)\log_q r\right\}\right\}.$$

Thus, we complete the proof of Lemma 2.3.

Lemma 2.4 *Under the assumptions of Theorem* 1.1 *or Theorem* 1.2, *we have* $\rho_{[p,q]}(A_s) = \beta \geq \sigma$.

Proof. Assume that $\rho_{[p,q]}(A_s) = \beta < \sigma$. According to the hypotheses of Theorems 1.1 or 1.2, there exists a positive constant $\sigma > 0$ such that for sufficiently small $\varepsilon > 0$, we have

$$|A_s(z)| \geq \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q r\right\} \tag{2.12}$$

as $|z| = r \in H$, $r \to +\infty$, where $H \subset (1, +\infty)$ is a set with a positive upper logarithmic density (by Proposition 2.1, we have $m_l(H) = +\infty$). By Lemma 2.3, we can find a set $E_3 \subset (1, +\infty)$ that has finite linear measure (and so of finite logarithmic measure) such that when $|z| = r \notin E_3$, we have for any given $\varepsilon\,(0 < 2\varepsilon < \sigma - \beta)$

$$|A_s(z)| \leq \exp_{p+1}\left\{(\beta + \varepsilon)\log_q r\right\}. \tag{2.13}$$

By (2.12) and (2.13), we obtain for $|z| = r \in H \setminus E_3$, $r \to +\infty$

$$\exp_{p+1}\left\{(\sigma - \varepsilon)\log_q r\right\} \leq |A_s(z)| \leq \exp_{p+1}\left\{(\beta + \varepsilon)\log_q r\right\}$$

and by ε $(0 < 2\varepsilon < \sigma - \beta)$ this is a contradiction. Hence $\rho_{[p,q]}(A_s) = \beta \geq \sigma$.

Lemma 2.5 (Wiman-Valiron, [10], [22]) *Let f be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then the estimation*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) \quad (j \geq 1 \text{ is an integer})$$

holds for all $|z|$ outside a set E_4 of r of finite logarithmic measure, where $\nu_f(r)$ is the central index of f.

Lemma 2.6 ([12]) *Let f be an entire function of $[p, q]$-order and let $\nu_f(r)$ be the central index of f. Then*

$$\rho_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p \nu_f(r)}{\log_q r}, \quad \mu_{[p,q]}(f) = \liminf_{r\to+\infty} \frac{\log_p \nu_f(r)}{\log_q r}.$$

The following two lemmas were given in [4] without proof, so for the convenience of the reader, we prove them.

Lemma 2.7 *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g) = \mu_{[p,q]}(f) = \mu \leq \rho_{[p,q]}(f) = \rho_{[p,q]}(g) \leq +\infty$ and $\lambda_{[p,q]}(d) = \rho_{[p,q]}(d) = \beta = \lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu$. Then, there exists a set $E_5 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_5$ and $|g(z)| = M(r, g)$, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1 + o(1)), \quad n \in \mathbb{N},$$

where $\nu_g(r)$ denote the central index of g.

Proof. By mathematical induction, we obtain

$$f^{(n)}=\frac{g^{(n)}}{d}+\sum_{j=0}^{n-1}\frac{g^{(j)}}{d}\sum_{(j_1\ldots j_n)}C_{jj_1\ldots j_n}\left(\frac{d'}{d}\right)^{j_1}\times\cdots\times\left(\frac{d^{(n)}}{d}\right)^{j_n}, \tag{2.14}$$

where $C_{jj_1\ldots j_n}$ are constants and $j+j_1+2j_2+\cdots+nj_n=n$. Hence

$$\frac{f^{(n)}}{f}=\frac{g^{(n)}}{g}+\sum_{j=0}^{n-1}\frac{g^{(j)}}{g}\sum_{(j_1\ldots j_n)}C_{jj_1\ldots j_n}\left(\frac{d'}{d}\right)^{j_1}\times\cdots\times\left(\frac{d^{(n)}}{d}\right)^{j_n}. \tag{2.15}$$

From Lemma 2.5, there exists a set $E_4\subset(1,+\infty)$ with finite logarithmic measure such that for a point z satisfying $|z|=r\notin E_4$ and $|g(z)|=M(r,g)$, we have

$$\frac{g^{(j)}(z)}{g(z)}=\left(\frac{\nu_g(r)}{z}\right)^j(1+o(1))\quad(j=1,2,...,n), \tag{2.16}$$

where $\nu_g(r)$ is the central index of g. Substituting (2.16) into (2.15) yields

$$\frac{f^{(n)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^n\Bigg[(1+o(1))$$
$$+\sum_{j=0}^{n-1}\left(\frac{\nu_g(r)}{z}\right)^{j-n}(1+o(1))\sum_{(j_1\ldots j_n)}C_{jj_1\ldots j_n}\left(\frac{d'}{d}\right)^{j_1}\times\cdots\times\left(\frac{d^{(n)}}{d}\right)^{j_n}\Bigg]. \tag{2.17}$$

Since $\rho_{[p,q]}(d)=\beta<\mu$, then for any given ε $(0<2\varepsilon<\mu-\beta)$ and sufficiently large r, we have

$$T(r,d)\leq\exp_p\left\{\left(\beta+\frac{\varepsilon}{2}\right)\log_q r\right\}$$

By using Lemma 2.1, for $\alpha=2$, there exist a set $E_1\subset(1,+\infty)$ with $m_l(E_1)<\infty$ and a constant $B>0$, such that for all z satisfying $|z|=r\notin[0,1]\cup E_1$, we have

$$\left|\frac{d^{(m)}(z)}{d(z)}\right|\leq B\left[T(2r,d)\right]^{m+1}\leq B\left[\exp_p\left\{\left(\beta+\frac{\varepsilon}{2}\right)\log_q(2r)\right\}\right]^{m+1}$$
$$\leq\exp_p\left\{(\beta+\varepsilon)\log_q r\right\}^m,\ \ m=1,2,...,n. \tag{2.18}$$

By Lemma 2.6 and $\mu_{[p,q]}(g)=\mu_{[p,q]}(f)=\mu$, it follows that

$$\nu_g(r)>\exp_p\left\{(\mu-\varepsilon)\log_q r\right\}$$

for sufficiently large r. Thus, by using $j_1+2j_2+\cdots+nj_n=n-j$, we obtain

$$\left|\left(\frac{\nu_g(r)}{z}\right)^{j-n}\left(\frac{d'}{d}\right)^{j_1}\times\cdots\times\left(\frac{d^{(n)}}{d}\right)^{j_n}\right|\leq\left[\frac{\exp_p\left\{(\mu-\varepsilon)\log_q r\right\}}{r}\right]^{j-n}$$
$$\times\left[\exp_p\left\{(\beta+\varepsilon)\log_q r\right\}\right]^{n-j}$$
$$=\left[\frac{r\exp_p\left\{(\beta+\varepsilon)\log_q r\right\}}{\exp_p\left\{(\mu-\varepsilon)\log_q r\right\}}\right]^{n-j}\to0 \tag{2.19}$$

as $r\to+\infty$, where $|z|=r\notin[0,1]\cup E_5,\ E_5=E_1\cup E_4$ and $|g(z)|=M(r,g)$. From (2.17) and (2.19), we obtain our assertion.

Lemma 2.8 *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g) = \mu_{[p,q]}(f) = \mu \le \rho_{[p,q]}(f) = \rho_{[p,q]}(g) \le +\infty$ and $\lambda_{[p,q]}(d) = \rho_{[p,q]}(d) = \lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu$. Then, there exists a set $E_6 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_6$ and $|g(z)| = M(r, g)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \le r^{2s}, \quad (s \in \mathbb{N}).$$

Proof. By Lemma 2.7, there exists a set E_5 of finite logarithmic measure such that the estimation

$$\frac{f^{(s)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^s (1 + o(1)) \ \ (s \ge 1 \text{ is an integer}) \tag{2.20}$$

holds for all $|z| = r \notin [0,1] \cup E_5$ and $|g(z)| = M(r, g)$, where $\nu_g(r)$ is the central index of g. On the other hand, by Lemma 2.6, for any given ε $(0 < \varepsilon < 1)$, there exists $R > 1$ such that for all $r > R$, we have

$$\nu_g(r) > \exp_p\left\{(\mu - \varepsilon)\log_q(r)\right\}. \tag{2.21}$$

If $\mu = +\infty$, then $\mu - \varepsilon$ can be replaced by a large enough real number M. Set $E_6 = [1, R] \cup E_5$, $lm(E_6) < +\infty$. Hence from (2.20) and (2.21), we obtain

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| = \left|\frac{z}{\nu_g(r)}\right|^s \frac{1}{|1 + o(1)|} \le \frac{r^s}{\left(\exp_p\left\{(\mu - \varepsilon)\log_q(r)\right\}\right)^s} \le r^{2s},$$

where $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $|g(z)| = M(r, g)$.

Lemma 2.9 ([6]) *Let $\varphi : [0, +\infty) \to \mathbb{R}$ and $\psi : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $\varphi(r) \le \psi(r)$ for all $r \notin (E_7 \cup [0,1])$, where E_7 is a set of finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then, there exists an $r_1 = r_1(\alpha) > 0$ such that $\varphi(r) \le \psi(\alpha r)$ for all $r > r_1$.*

Lemma 2.10 ([19]) *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions. If $0 \le \rho_{[p,q]}(d) < \mu_{[p,q]}(f)$, then $\mu_{[p,q]}(g) = \mu_{[p,q]}(f)$ and $\rho_{[p,q]}(g) = \rho_{[p,q]}(f)$. Moreover, if $\rho_{[p,q]}(f) = +\infty$, then $\rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f)$.*

Lemma 2.11 *Assume that $k \ge 2$ and $A_0, A_1, ..., A_k \not\equiv 0$, F are meromorphic functions. Let $\rho = \max\left\{\rho_{[p,q]}(A_j) \ (j = 0, 1, ..., k), \rho_{[p,q]}(F)\right\} < \infty$ and let f be a meromorphic solution of infinite $[p,q]$-order of equation (1.4) with $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$. Then, $\rho_{[p+1,q]}(f) \le \rho$.*

Proof. Let f be a meromorphic solution of infinite $[p,q]$-order of equation (1.4) with $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$. So, we can use Hadamard factorization theorem and write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying $\mu_{[p,q]}(g) = \mu_{[p,q]}(f) = \mu \le \rho_{[p,q]}(f) = \rho_{[p,q]}(g) \le +\infty$

and $\lambda_{[p,q]}(d) = \rho_{[p,q]}(d) = \lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu$. By Lemma 2.3, there exists a set $E_3 \subset (1, +\infty)$ of r with a finite linear measure such that for all $|z| = r \notin E_3$ and any given ε $(0 < 2\varepsilon < \mu_{[p,q]}(f) - \rho_{[p,q]}(d))$, we have

$$\begin{aligned} |A_j(z)| &\leq \exp_{p+1}\left\{\left(\rho_{(p,q)}\left(A_j\right) + \varepsilon\right)\log_q r\right\} \\ &\leq \exp_{p+1}\left\{(\rho + \varepsilon)\log_q r\right\}, \ j = 0, 1, ..., k-1, \end{aligned} \tag{2.22}$$

$$\begin{aligned} |A_k(z)| &\geq \exp\left\{-\exp_p\left\{\left(\rho_{(p,q)}(A_k) + \varepsilon\right)\log_q r\right\}\right\} \\ &\geq \exp\left\{-\exp_p\left\{(\rho + \varepsilon)\log_q r\right\}\right\} \end{aligned} \tag{2.23}$$

and

$$|F(z)| \leq \exp_{p+1}\left\{\left(\rho_{(p,q)}(F) + \varepsilon\right)\log_q r\right\} \leq \exp_{p+1}\left\{(\rho + \varepsilon)\log_q r\right\}. \tag{2.24}$$

By (2.24), for all z satisfying $|z| = r \notin E_3$ at which $|g(z)| = M(r, g)$ and any given $\varepsilon\left(0 < 2\varepsilon < \mu_{[p,q]}(f) - \rho_{[p,q]}(d)\right)$, we obtain

$$\begin{aligned} \left|\frac{F(z)}{f(z)}\right| &= \frac{|F(z)|}{|g(z)|}|d(z)| \\ &\leq \frac{\exp_{p+1}\left\{\left(\rho_{[p,q]}(d) + \varepsilon\right)\log_q r\right\}\exp_{p+1}\left\{(\rho + \varepsilon)\log_q r\right\}}{\exp_{p+1}\left\{\left(\mu_{[p,q]}(f) - \varepsilon\right)\log_q r\right\}} \\ &\leq \exp_{p+1}\left\{(\rho + \varepsilon)\log_q r\right\}. \end{aligned} \tag{2.25}$$

By Lemma 2.7, there exists a set $E_5 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_5$ and $|g(z)| = M(r, g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)), \ j = 1, ..., k. \tag{2.26}$$

We can rewrite (1.4) as

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \frac{1}{|A_k(z)|}\left(|A_0(z)| + \left|\frac{F(z)}{f(z)}\right| + \sum_{j=1}^{k-1}|A_j(z)|\left|\frac{f^{(j)}(z)}{f(z)}\right|\right). \tag{2.27}$$

By substituting $(2.22), (2.23), (2.25)$ and (2.26) into (2.27), we obtain

$$\begin{aligned} \left|\frac{\nu_g(r)}{z}\right|^k |1 + o(1)| &\leq \frac{1}{\exp\left\{-\exp_p\left\{(\rho + \varepsilon)\log_q r\right\}\right\}} \times \\ &\left(\left\{1 + \sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1 + o(1)|\right\}\exp_{p+1}\left\{(\rho + \varepsilon)\log_q r\right\}\right. \\ &\left. + \exp_{p+1}\left\{(\rho + \varepsilon)\log_q r\right\}\right) \\ &= \left\{2 + \sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1 + o(1)|\right\}\exp\left\{2\exp_p\left\{(\rho + \varepsilon)\log_q r\right\}\right\}. \end{aligned}$$

Hence

$$|\nu_g(r)||1 + o(1)| \leq (k + 1) r |1 + o(1)| \exp\left\{2\exp_p\left\{(\rho + \varepsilon)\log_q r\right\}\right\} \tag{2.28}$$

holds for all z satisfying $|z| = r \notin [0,1] \cup E_3 \cup E_5$ and $|g(z)| = M(r,g)$, $r \to +\infty$. By (2.28), we get

$$\limsup_{r\to+\infty} \frac{\log_{p+1} \nu_g(r)}{\log_q r} \leq \rho + \varepsilon. \tag{2.29}$$

Since $\varepsilon > 0$ is arbitrary, by (2.29) and Lemma 2.6, we obtain $\rho_{[p+1,q]}(g) \leq \rho$. Since $\rho_{[p,q]}(d) < \mu_{[p,q]}(f)$, so by Lemma 2.10, we have $\rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f)$. Thus, $\rho_{[p+1,q]}(f) \leq \rho$. Therefore, Lemma 2.11 is proved.

Lemma 2.12 ([19]) *Let $A_j(z)$ $(j = 0, 1, \dots, k)$, $A_k(z) (\not\equiv 0)$, $F(z) (\not\equiv 0)$ be meromorphic functions and let f be a meromorphic solution of* (1.4) *of infinite $[p,q]$-order satisfying the following condition*

$$b = \max\left\{\rho_{[p+1,q]}(F),\ \rho_{[p+1,q]}(A_j)\ \ (j = 0, 1, \dots, k)\right\} < \rho_{[p+1,q]}(f).$$

Then

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f).$$

Lemma 2.13 *Let $H \subset (1, +\infty)$ be a set with a positive upper logarithmic density (or infinite logarithmic measure), and let $A_j(z)$ $(j = 0, 1, \dots, k)$ with $A_k(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions with finite $[p,q]$-order. If there exist a positive constant $\sigma > 0$ and an integer s, $0 \leq s \leq k$, such that for sufficiently small $\varepsilon > 0$, we have $|A_s(z)| \geq \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q r\right\}$ as $|z| = r \in H$, $r \to +\infty$ and*

$$\max\left\{\rho_{[p,q]}(A_j)\ \ (j \neq s),\ \rho_{[p,q]}(F)\right\} < \sigma,$$

then every transcendental meromorphic solution f of equation (1.4) *satisfies $\rho_{[p,q]}(f) \geq \sigma$.*

Proof. Assume that f is a transcendental meromorphic solution of equation (1.4) with $\rho_{[p,q]}(f) < \sigma$. From (1.4), we have

$$A_s = \frac{F}{f^{(s)}} - \sum_{\substack{j=0\\ j\neq s}}^{k} A_j \frac{f^{(j)}}{f^{(s)}}. \tag{2.30}$$

Since $\max\left\{\rho_{[p,q]}(A_j)\ \ (j \neq s),\ \rho_{[p,q]}(F)\right\} < \sigma$ and $\rho_{[p,q]}(f) < \sigma$, then from (2.30) we obtain that

$$\rho_1 = \rho_{[p,q]}(A_s) \leq \max\left\{\rho_{[p,q]}(A_j)\ \ (j \neq s),\ \rho_{[p,q]}(F),\ \rho_{[p,q]}(f)\right\} < \sigma.$$

By Lemma 2.3, for any ε $(0 < 2\varepsilon < \sigma - \rho_1)$, there exists a set $E_3 \subset (1, +\infty)$ with a finite linear measure such that

$$|A_s(z)| \leq \exp_{p+1}\left\{(\rho_{(p,q)}(A_s) + \varepsilon)\log_q r\right\} = \exp_{p+1}\left\{(\rho_1 + \varepsilon)\log_q r\right\} \tag{2.31}$$

holds for all z satisfying $|z| = r \notin E_3$. From the hypotheses of Lemma 2.13, there exists a set H with $\overline{\log dens} H > 0$ (or $m_l(H) = +\infty$) such that

$$|A_s(z)| \geq \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q r\right\} \tag{2.32}$$

holds for all z satisfying $|z| = r \in H$, $r \to +\infty$. By (2.31) and (2.32), we conclude that for all z satisfying $|z| = r \in H \setminus E_3$, $r \to +\infty$, we have

$$\exp_{p+1}\left\{(\sigma - \varepsilon)\log_q r\right\} \le \exp_{p+1}\left\{(\rho_1 + \varepsilon)\log_q r\right\}$$

and by ε $(0 < 2\varepsilon < \sigma - \rho_1)$ this is a contradiction as $r \to +\infty$. Consequently, any transcendental meromorphic solution f of equation (1.4) satisfies $\rho_{[p,q]}(f) \ge \sigma$.

Lemma 2.14 ([23]) *Let $p \ge q \ge 1$ be integers. Let f be a meromorphic function for which $\rho_{[p,q]}(f) = \beta < +\infty$, and let $k \ge 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{(\beta + \varepsilon)\log_q r\right\}\right),$$

holds outside of a possible exceptional set E_8 of finite linear measure.

Lemma 2.15 *Let $A_0, A_1, ..., A_k \not\equiv 0, F \not\equiv 0$ be finite $[p,q]$-order meromorphic functions. If f is a meromorphic solution with $\rho_{[p,q]}(f) = +\infty$ and $\rho_{[p+1,q]}(f) = \rho < +\infty$ of equation (1.4), then $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f) = +\infty$ and $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho$.*

Proof Let f be a meromorphic solution of (1.4) with infinite $[p,q]$-order and $\rho_{[p+1,q]}(f) = \rho < +\infty$. Note first that by definition, we have $\overline{\lambda}_{[p+1,q]}(f) \le \lambda_{[p+1,q]}(f) \le \rho_{[p+1,q]}(f)$. Then, it remains to show that

$$\rho_{[p+1,q]}(f) \le \overline{\lambda}_{[p+1,q]}(f) \le \lambda_{[p+1,q]}(f).$$

We rewrite (1.4) as

$$\frac{1}{f} = \frac{1}{F}\left(A_k(z)\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\right). \tag{2.33}$$

By using Lemma 2.14 and (2.33), for $|z| = r$ outside a set E_8 of a finite linear measure and any given $\varepsilon > 0$, we get

$$m\left(r, \frac{1}{f}\right) \le m\left(r, \frac{1}{F}\right) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=0}^{k} m\left(r, A_j\right) + O(1)$$

$$\le m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k} m\left(r, A_j\right) + O\left(\exp_p\left\{(\rho + \varepsilon)\log_q r\right\}\right). \tag{2.34}$$

On the other hand, by (1.4), if f has a zero at z_0 of order $\alpha\,(\alpha > k)$, and A_0, $A_1, ...,$ A_k are all analytic at z_0, then F must have a zero at z_0 of order at least $\alpha - k$. Hence,

$$n\left(r, \frac{1}{f}\right) \le k\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k} n\left(r, A_j\right)$$

and

$$N\left(r, \frac{1}{f}\right) \leq k\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k} N\left(r, A_j\right). \tag{2.35}$$

Therefore, by (2.34) and (2.35), for all sufficiently large $r \notin E_8$ and any given $\varepsilon > 0$, we have

$$T(r, f) = T(r, \frac{1}{f}) + O(1) \leq T(r, F) + \sum_{j=0}^{k} T\left(r, A_j\right)$$

$$+ k\overline{N}\left(r, \frac{1}{f}\right) + O\left(\exp_p\left\{(\rho + \varepsilon)\log_q r\right\}\right). \tag{2.36}$$

Noting $c = \max\left\{\rho_{[p,q]}\left(A_j\right) \ (j = 0, 1, ..., k), \rho_{[p,q]}(F)\right\}$. Then, by using the definition of the $[p, q]$-order, for the above ε and sufficiently large r, we have

$$T(r, F) \leq \exp_p\left\{(c + \varepsilon)\log_q r\right\}, \tag{2.37}$$

$$T\left(r, A_j\right) \leq \exp_p\left\{(c + \varepsilon)\log_q r\right\}, \ j = 0, 1, ..., k. \tag{2.38}$$

Replacing (2.37) and (2.38) into (2.36), for $r \notin E_8$ sufficiently large and any given $\varepsilon > 0$, we obtain

$$T(r, f) \leq k\overline{N}\left(r, \frac{1}{f}\right) + (k + 2)\exp_p\left\{(c + \varepsilon)\log_q r\right\} + O\left(\exp_p\left\{(\rho + \varepsilon)\log_q r\right\}\right). \tag{2.39}$$

Hence, for any f with $\rho_{[p,q]}(f) = +\infty$ and $\rho_{[p+1,q]}(f) = \rho$, by (2.39), we have

$$\overline{\lambda}_{[p,q]}(f) \geq \rho_{[p,q]}(f) = +\infty, \ \overline{\lambda}_{[p+1,q]}(f) \geq \rho_{[p+1,q]}(f),$$

so

$$\rho_{[p+1,q]}(f) \leq \overline{\lambda}_{[p+1,q]}(f) \leq \lambda_{[p+1,q]}(f).$$

And the fact that $\overline{\lambda}_{[p+1,q]}(f) \leq \lambda_{[p+1,q]}(f) \leq \rho_{[p+1,q]}(f)$, we obtain

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho.$$

3. Proof of Theorem 1.1

Assume that $f \not\equiv 0$ is a rational solution of (1.3). First, we will prove that f must be a polynomial with $\deg f \leq s - 1$. For this, if f is a rational function, which has a pole at z_0 of degree $m \geq 1$, or f is a polynomial with $\deg f \geq s$, then $f^{(s)}(z) \not\equiv 0$. By (1.3) and Lemma 2.4, we obtain

$$\sigma \leq \rho_{[p,q]}(A_s) = \rho_{[p,q]}(A_s f^{(s)}) = \rho_{[p,q]}\left(-\left(\sum_{j=0,\ j\neq s}^{k} A_j f^{(j)}\right)\right)$$

$$\leq \max_{j=0,1,\dots,k,\ j\neq s}\left\{\rho_{[p,q]}\left(A_j\right)\right\}$$

which is a contradiction. Therefore, f must be a polynomial with $\deg f \leq s - 1$.

Now, we assume that f is a transcendental meromorphic solution of (1.3) such that $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$. By Lemma 2.3, for any given ε $(0 < 2\varepsilon < \sigma - \rho)$, there exists a set $E_3 \subset (1, +\infty)$ with a finite linear measure (and so of finite logarithmic measure) such that

$$|A_j(z)| \leq \exp_{p+1}\left\{(\rho+\varepsilon)\log_q r\right\}, \; j = 0, 1, ..., k, \; j \neq s \tag{3.1}$$

holds for all z satisfying $|z| = r \notin E_3$. In view of Lemma 2.8, there exists a set $E_6 \subset (1, +\infty)$ of finite logarithmic measure such that $|z| = r \notin [0, 1] \cup E_6$, $|g(z)| = M(r, g)$ and for r sufficiently large, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s} \; (s \geq 1 \text{ is an integer}). \tag{3.2}$$

According to Lemma 2.1, there exist a set $E_1 \subset (1, +\infty)$ with $m_l(E_1) < \infty$ and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r, f)\right]^{k+1}, \; j = 1, 2, ..., k, \; j \neq s. \tag{3.3}$$

From the hypotheses of Theorem 1.1, there exists a set $H \subset (1, +\infty)$ with $m_l(H) = +\infty$, such that for all z satisfying $|z| = r \in H$, $r \to +\infty$ and sufficiently small $\varepsilon > 0$, we have

$$|A_s(z)| \geq \exp_{p+1}\left\{(\sigma - \varepsilon)\log_q r\right\}. \tag{3.4}$$

Now, by rewriting equation (1.3) in the form

$$|A_s| \leq \left|\frac{f}{f^{(s)}}\right|\left(|A_0| + \sum_{\substack{j=1 \\ j\neq s}}^{k} |A_j| \left|\frac{f^{(j)}}{f}\right|\right) \tag{3.5}$$

and substituting (3.1), (3.2), (3.3) and (3.4) into (3.5), for all z satisfying $|z| = r \in H \setminus ([0, 1] \cup E_1 \cup E_3 \cup E_6)$, $r \to +\infty$, we obtain

$$\exp_{p+1}\left\{(\sigma-\varepsilon)\log_q r\right\} \leq Bkr^{2s}\exp_{p+1}\left\{(\rho+\varepsilon)\log_q r\right\}\left[T(2r, f)\right]^{k+1}.$$

Since $0 < 2\varepsilon < \sigma - \rho$, then we have

$$\exp\left\{(1 - o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q r\right\}\right\} \leq Bkr^{2s}\left[T(2r, f)\right]^{k+1}. \tag{3.6}$$

From (3.6) and Lemma 2.9, for any given $\gamma > 1$ and sufficiently large $r > R$, we get

$$\exp\left\{(1 - o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q r\right\}\right\} \leq Bk(\gamma r)^{2s}\left[T(2\gamma r, f)\right]^{k+1}$$

which gives

$$\rho_{[p,q]}(f) = \mu_{[p,q]}(f) = +\infty, \; \sigma \leq \rho_{[p+1,q]}(f). \tag{3.7}$$

By using Lemma 2.4, we have

$$\max\left\{\rho_{[p,q]}\left(A_j\right) : j = 0, 1, ..., k\right\} = \rho_{[p,q]}(A_s) = \beta < +\infty.$$

Since f is of infinite $[p,q]$-order meromorphic solution of equation (1.3) satisfying $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$, then by Lemma 2.11, we obtain

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p,q]}\left(A_j\right) : j = 0, 1, ..., k\right\} = \rho_{[p,q]}(A_s). \tag{3.8}$$

By (3.7) and (3.8), we conclude that $\mu_{[p,q]}(f) = \rho_{[p,q]}(f) = +\infty$ and $\sigma \leq \rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s)$.

4. Proof of Corollary 1.1

Assume that φ is a transcendental meromorphic function such that $\rho_{[p+1,q]}(\varphi) < \sigma$. Noting $g = f - \varphi$, then $\rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f)$, so by Theorem 1.1, $\sigma \leq \rho_{[p+1,q]}(g) \leq \rho_{[p,q]}(A_s)$. By substituting $f = g + \varphi$ into (1.3), we obtain

$$\begin{aligned} &A_k(z)\, g^{(k)} + A_{k-1}(z)\, g^{(k-1)} + \cdots + A_1(z)\, g' + A_0(z)\, g \\ &= -\left(A_k(z)\,\varphi^{(k)} + A_{k-1}(z)\,\varphi^{(k-1)} + \cdots + A_1(z)\,\varphi' + A_0(z)\,\varphi\right) = G(z). \end{aligned} \tag{4.1}$$

It is clear that the right side G of equation (4.1) is non-zero, because by Theorem 1.1, φ is not a solution of equation (1.3). Moreover, the $[p+1,q]$-order of G satisfies

$$\rho_{[p+1,q]}(G) \leq \max\left\{\rho_{[p+1,q]}(\varphi),\ \rho_{[p+1,q]}\left(A_j\right)\ \ (j = 0, 1, ..., k)\right\} < \sigma,$$

which implies

$$\max\left\{\rho_{[p+1,q]}(G),\ \rho_{[p+1,q]}\left(A_j\right)\ \ (j = 0, 1, ..., k)\right\} < \sigma \leq \rho_{[p+1,q]}(g).$$

Then by Lemma 2.12, we obtain

$$\begin{aligned} \sigma &\leq \overline{\lambda}_{[p+1,q]}(g) = \lambda_{[p+1,q]}(g) \\ &= \rho_{[p+1,q]}(g) = \rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s), \end{aligned}$$

that is

$$\begin{aligned} \sigma &\leq \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) \\ &= \rho_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s). \end{aligned}$$

5. Proof of Theorem 1.2

Assume that f is a rational solution of (1.4). First, we will prove that f must be a polynomial with $\deg f \leq s-1$. For this, if f is a rational function, which has a pole at z_0 of degree $m \geq 1$, or f is a polynomial with $\deg f \geq s$, then $f^{(s)}(z) \not\equiv 0$. By (1.4) and Lemma 2.4, we obtain

$$\begin{aligned} \sigma \leq \rho_{[p,q]}(A_s) = \rho_{[p,q]}(A_s f^{(s)}) &= \rho_{[p,q]}\left(F - \sum_{\substack{j=0\\ j\neq s}}^{k} A_j(z)\, f^{(j)}\right) \\ &\leq \max_{j=0,1,\dots,k,\ j\neq s}\left\{\rho_{[p,q]}\left(A_j\right),\ \rho_{[p,q]}(F)\right\}, \end{aligned}$$

which is a contradiction. Therefore, f must be a polynomial with $\deg f \leq s-1$.

Now, we assume that f is a transcendental meromorphic solution of (1.4) such that $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \mu_{[p,q]}(f)$. From Lemma 2.13, we know that f satisfies $\rho_{[p,q]}(f) \geq \sigma$. By the hypothesis $\lambda_{[p,q]}\left(\frac{1}{f}\right) < \min\{\mu_{[p,q]}(f), \sigma\}$ and Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_{[p,q]}(g) = \mu_{[p,q]}(f) = \mu \leq \rho_{[p,q]}(g) = \rho_{[p,q]}(f),$$
$$\rho_{[p,q]}(d) = \lambda_{[p,q]}\left(\frac{1}{f}\right) = \beta < \min\{\mu_{[p,q]}(f), \sigma\}.$$

The definition of the lower $[p,q]$-order assures us that

$$|g(z)| = M(r,g) \geq \exp_{p+1}\left\{\left(\mu_{[p,q]}(g) - \varepsilon\right)\log_q r\right\}. \tag{5.1}$$

Putting

$$\rho_1 = \max\left\{\rho_{[p,q]}\left(A_j\right) \ \ (j \neq s), \rho_{[p,q]}(F)\right\} < \sigma.$$

Then, by Lemma 2.3 and (5.1), for any given ε satisfying

$$0 < 2\varepsilon < \min\{\sigma - \rho_1, \mu_{[p,q]}(g) - \rho_{[p,q]}(d)\},$$

there exists a set $E_3 \subset (1, +\infty)$ with a finite logarithmic measure such that for all z satisfying $|z| = r \notin E_3$ at which $|g(z)| = M(r,g)$, we obtain

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|}{|g(z)|}|d(z)|$$
$$\leq \frac{\exp_{p+1}\left\{\left(\rho_{[p,q]}(d) + \varepsilon\right)\log_q r\right\}\exp_{p+1}\left\{\left(\rho_1 + \varepsilon\right)\log_q r\right\}}{\exp_{p+1}\left\{\left(\mu_{[p,q]}(g) - \varepsilon\right)\log_q r\right\}}$$
$$\leq \exp_{p+1}\left\{\left(\rho_1 + \varepsilon\right)\log_q r\right\}. \tag{5.2}$$

By using the same arguments as in the proof of Theorem 1.1, for any given $\varepsilon\left(0 < 2\varepsilon < \min\{\sigma - \rho_1, \mu_{[p,q]}(g) - \rho_{[p,q]}(d)\}\right)$ and all z satisfying $|z| = r \in H \setminus (E_1 \cup E_3 \cup E_6)$, $r \to +\infty$ at which $|g(z)| = M(r,g)$, we have (3.2), (3.3), (3.4) hold and

$$|A_j(z)| \leq \exp_{p+1}\left\{\left(\rho_1 + \varepsilon\right)\log_q r\right\}, \ j = 0, 1, ..., k, \ j \neq s. \tag{5.3}$$

By (1.4), we have

$$|A_s| \leq \left|\frac{f}{f^{(s)}}\right|\left(|A_0| + \sum_{\substack{j=1 \\ j\neq s}}^{k}\left|A_j\right|\left|\frac{f^{(j)}}{f}\right| + \left|\frac{F}{f}\right|\right). \tag{5.4}$$

Hence, by substituting (3.2), (3.3), (3.4), (5.2) and (5.3) into (5.4), for all z satisfying $|z| = r \in H \setminus (E_1 \cup E_3 \cup E_6)$, $r \to +\infty$, at which $|g(z)| = M(r,g)$ and any given $\varepsilon\left(0 < 2\varepsilon < \min\{\sigma - \rho_1, \mu_{[p,q]}(g) - \rho_{[p,q]}(d)\}\right)$, we obtain

$$\exp_{p+1}\left\{\left(\sigma - \varepsilon\right)\log_q r\right\} \leq r^{2s}\left(\exp_{p+1}\left\{\left(\rho_1 + \varepsilon\right)\log_q r\right\}\right.$$

$$+\sum_{j=1,j\neq s}^{k} \exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q r\right\} B\left[T(2r,f)\right]^{k+1}$$

$$+\exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q r\right\}\Big)$$

$$\leq B(k+1)\,r^{2s}\left[T(2r,f)\right]^{k+1}\exp_{p+1}\left\{(\rho_1+\varepsilon)\log_q r\right\}. \tag{5.5}$$

Since $0<2\varepsilon<\sigma-\rho_1$, then we can use Lemma 2.9 with (5.5) such that for any given $\gamma>1$ and sufficiently large $r>R$, we obtain

$$\exp\left\{(1-o(1))\exp_p\left\{(\sigma-\varepsilon)\log_q r\right\}\right\}\leq B(k+1)(\gamma r)^{2s}\left[T(2\gamma r,f)\right]^{k+1}$$

which gives

$$\rho_{[p,q]}(f)=\mu_{[p,q]}(f)=+\infty,\;\; \rho_{[p+1,q]}(f)\geq\sigma. \tag{5.6}$$

Making use of Lemma 2.4, we have

$$\max\left\{\rho_{[p,q]}\left(A_j\right)\;\;(j=0,1,\ldots,k),\rho_{[p,q]}(F)\right\}=\rho_{[p,q]}(A_s)=\beta<+\infty.$$

By Lemma 2.11 and since f is of infinite $[p,q]$-order meromorphic solution of equation (1.4) satisfying $\lambda_{[p,q]}\left(\frac{1}{f}\right)<\mu_{[p,q]}(f)$, we get

$$\rho_{[p+1,q]}(f)\leq\max\left\{\rho_{[p,q]}\left(A_j\right)\;\;(j=0,1,\ldots,k),\rho_{[p,q]}(F)\right\}=\rho_{[p,q]}(A_s). \tag{5.7}$$

Since $F\not\equiv 0$, then by Lemma 2.15, we have

$$\overline{\lambda}_{[p,q]}(f)=\lambda_{[p,q]}(f)=\mu_{[p,q]}(f)=\rho_{[p,q]}(f)=+\infty \tag{5.8}$$

and

$$\sigma\leq\;\overline{\lambda}_{[p+1,q]}(f)=\lambda_{[p+1,q]}(f)=\rho_{[p+1,q]}(f). \tag{5.9}$$

By (5.7), (5.8) and (5.9), we conclude that

$$\overline{\lambda}_{[p,q]}(f)=\lambda_{[p,q]}(f)=\mu_{[p,q]}(f)=\rho_{[p,q]}(f)=+\infty$$

and

$$\sigma\leq\;\overline{\lambda}_{[p+1,q]}(f)=\lambda_{[p+1,q]}(f)=\rho_{[p+1,q]}(f)\leq\rho_{[p,q]}(A_s).$$

6. Proof of Corollary 1.2

Assume that φ is a transcendental meromorphic function such that $\rho_{[p+1,q]}(\varphi)<\sigma$. Noting $h=f-\varphi$, then $\rho_{[p+1,q]}(h)=\rho_{[p+1,q]}(f)$, so by Theorem 1.2, $\sigma\leq\rho_{[p+1,q]}(h)\leq\rho_{[p,q]}(A_s)$. By substituting $f=h+\varphi$ into (1.4), we obtain

$$A_k(z)\,h^{(k)}+A_{k-1}(z)\,h^{(k-1)}+\cdots+A_1(z)\,h'+A_0(z)\,h$$

$$=F(z)-\left(A_k(z)\,\varphi^{(k)}+A_{k-1}(z)\,\varphi^{(k-1)}+\cdots+A_1(z)\,\varphi'+A_0(z)\,\varphi\right)=\Psi(z). \tag{6.1}$$

It is clear that the right side Ψ of the equation (6.1) is non-zero, because by Theorem 1.2, φ is not a solution of equation (1.4). Moreover, the $[p+1,q]$-order of Ψ verifies

$$\rho_{[p+1,q]}(\Psi) \leq \max\left\{\rho_{[p+1,q]}(\varphi),\ \rho_{[p+1,q]}\left(A_j\right)\ \ (j=0,1,\ldots,k)\right\} < \sigma,$$

which leads to

$$\max\left\{\rho_{[p+1,q]}(\Psi),\ \rho_{[p+1,q]}\left(A_j\right)\ \ (j=0,1,\ldots,k)\right\} < \sigma \leq \rho_{[p+1,q]}(h).$$

Therefore, by Lemma 2.12, we obtain

$$\sigma \leq \overline{\lambda}_{[p+1,q]}(h) = \lambda_{[p+1,q]}(h)$$

$$= \rho_{[p+1,q]}(h) = \rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s),$$

that is

$$\sigma \leq \overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi)$$

$$= \rho_{[p+1,q]}(f-\varphi) = \rho_{[p+1,q]}(f) \leq \rho_{[p,q]}(A_s).$$

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