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The Rellich-Kondrachov Theorem for Gelfand Pairs Over Hypergroups

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Abstract. Embedding results play important rôles in mathematical analysis. This paper addresses some embedding theorems in the context of Sobolev spaces theory on Gelfand pairs over hypergroups. Mainly, the analogue of the Rellich-Kondrachov theorem is proved.

1. Introduction

Sobolev spaces are well studied on subsets of $\mathbb{R}^n$ [1, 4] and on other classical spaces such as Riemannian manifolds [13, 14], metric measure spaces [12], etc. More recently, these studies are extended to other topological algebraic structures such as topological abelian groups, locally compact groups, Gelfand pairs over locally compact groups, locally compact commutative hypergroups, etc. More precisely, in [10, 11], Górka et al. constructed a class of Sobolev spaces on Hausdorff locally compact abelian groups by the means of the Fourier transform. This construction is generalized to Gelfand pairs over locally compact groups by Krukowski [16], to compact groups by Kumar and Kumar [17], to noncommutative locally compact groups by Mensah [18] and to noncommutative hypergroups by Bataka et al. [2].

In Sobolev spaces theory, embedding theorems are among the useful results that one may expect. They appear as support points in the analysis of partial differential equations and integral equations. Among such embedding theorems is the Rellich-Kondrachov theorem. It is a compact embedding theorem in Sobolev spaces theory which intervenes for instance in the proof of the Poincaré inequality. The Rellich-Kondrachov theorem took it origin in a special result by

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Rellich [19] and the general case was obtained by Kondrachov [15]. Such compact embedding theorem has many important applications in analysis for instance in linear elliptic partial differential equations defined over bounded domains [8, 9], in engineering applications [20], etc.

In this paper, we are mainly concerned with a generalization of the Rellich-Kondrachov theorem to a class of Sobolev spaces on Gelfand pairs associated with compact hypergroups. We paved the way with some results which amount to the proof of the Rellich-Kondrachov theorem in the present framework.

The paper is organized as follows. In Section 2, we recall some results which we may need. In Section 3, we present the main results, the culmination of which is the analogue of the Rellich-Kondrachov theorem.

2. Preliminaries

The important ingredients which constitute this section are borrowed from [3, 5, 6]. Let G be a locally compact space. Denote by

- $C(G)$, the set of complex-valued continuous functions on G,
- $M(G)$, the set of Radon measures on G,
- $M_b(G)$, the subset of $M(G)$ consisting of bounded measures,
- $M_1(G)$, the subset of $M_b(G)$ consisting of probability measures,
- $\mathfrak{C}(G)$, the set of compact subspaces of G,
- δ_x, the point measure at the element x.

The set $M(G)$ is endowed with the cône topology while $\mathfrak{C}(G)$ is endowed with the Michael topology.

Definition 2.1. *A locally compact space G is called a hypergroup if the following properties hold.*

(1) *There exists a binary operation $*$ (the convolution) on $M_b(G)$ which turns it into an associative algebra such that*
   (a) *the mapping $(\mu, \nu) \mapsto \mu * \nu$ is continuous from $M_b(G) \times M_b(G)$ into $M_b(G)$,*
   (b) *$\forall x, y \in G$, $\delta_x * \delta_y$ is a probability measure such that $supp(\delta_x * \delta_y)$ is compact.*
   (c) *The mapping $(x, y) \to supp(\delta_x * \delta_y)$ is continuous from $G \times G$ into $\mathfrak{C}(G)$.*

(2) *There exists a unique element e in G (the neutral element) such that*

$$\forall x \in G, \delta_x * \delta_e = \delta_e * \delta_x = \delta_x.$$

(3) *There exists an involutive homeomorphism $^\diamond : G \to G$ such that for all $x, y \in G$,*

$$(\delta_x * \delta_y)^\diamond = \delta_{y^\diamond} * \delta_{x^\diamond}.$$

(4) *$\forall x, y, z \in G$, $z \in supp(\delta_x * \delta_y) \Longleftrightarrow x \in supp(\delta_z * \delta_{y^\diamond})$.*

Definition 2.2. *A closed nonempty subset H of a hypergroup G is called a subhypergroup of G if*

(1) *$\forall x \in H$, $x^\diamond \in H$,*

(2) $\forall x, y \in H$, $supp(\delta_x * \delta_y) \subset H$.

Let G be a hypergroup and let K be a compact subhypergroup of G. For $x, y \in G$, $x * y$ stands for the support of $\delta_x * \delta_y$. The double coset of x with respect to K is

$$KxK = \{k_1 * x * k_2 : k_1, k_2 \in K\} = \bigcup_{k_1, k_2 \in K} \text{supp}(\delta_{k_1} * \delta_x * \delta_{k_2}).$$

For $f \in C(G)$, we set $f(x * y) = \int_G f(z) d(\delta_x * \delta_y)(z)$ and $f^\diamond(x) = f(x^\diamond)$.

A function $f \in C(G)$ is said to be K-bi-invariant if

$$\forall k_1, k_2 \in K, \forall x \in G, f(k_1 * x * k_2) = f(x).$$

Denote by $\mathcal{K}(G)$ the set of continuous functions on G with compact support and by $\mathcal{K}^\natural(G)$ the subset of $\mathcal{K}(G)$ consisting of K-bi-invariant functions. Now, assume that the hypergroup G is provided with a left Haar measure and that K is equipped with a normalized Haar measure. For $f \in \mathcal{K}(G)$, put

$$f^\natural(x) = \int_K \int_K f(k_1 * x * k_2) dk_1 dk_2.$$

For a measure $\mu \in M(G)$, set $\mu^\natural(f) = \mu(f^\natural)$, $f \in \mathcal{K}(G)$. The measure μ is called K-bi-invariant if $\mu^\natural = \mu$. Denote by $M_c^\natural(G)$ the set of complex Radon measures with compact support that are also K-bi-invariant. For $\mu, \nu \in M(G)$, we define $\mu * \nu$ by

$$\mu * \nu(f) = \iint_G f(x * y) d\mu(x) d\nu(y), f \in C(G).$$

Also, for $f, g \in \mathcal{K}(G)$, the convolution product of f and g is the function $f * g$ defined by

$$(f * g)(x) = \int_G f(y) g(y^\diamond * x) dy = \int_G f(x * y) g(y^\diamond) dy.$$

Provided with this convolution product, $\mathcal{K}(G)$ is an algebra and $\mathcal{K}^\natural(G)$ is a subalgebra of $\mathcal{K}(G)$.

Definition 2.3. *Let G be a hypergroup and let K be a compact subhypergroup of G. The pair (G, K) is called a Gelfand pair if the space $(M_c^\natural(G), *)$ is commutative.*

We may refer to this Gelfand pair as a *hypergroup Gelfand pair*. If (G, K) is a hypergroup Gelfand pair and if G has a Haar measure then G is unimodular [6].

In the rest of the paper, (G, K) is assumed to be a hypergroup Gelfand pair. We denote by $\widehat{G^\natural}$ the set of bounded continuous functions $\phi : G \longrightarrow \mathbb{C}$ such that

(1) ϕ is K-bi-invariant,

(2) $\phi(e) = 1$,

(3) $\forall x, y \in G$, $\int_K \phi(x * k * y) dk = \phi(x)\phi(y)$,

(4) $\forall x \in G$, $\phi(x^\diamond) = \overline{\phi(x)}$, where $\overline{\phi(x)}$ is the complex conjugate of $\phi(x)$.

The set $\widehat{G^\natural}$ is called the dual set of the hypergroup G [5]. When equipped with the topology of uniform convergence on compact sets, the space $\widehat{G^\natural}$ is a locally compact Hausdorff space.

Definition 2.4 ([5]). *Let (G, K) be a hypergroup Gelfand pair. Let $f \in \mathcal{K}^\natural(G)$. The Fourier transform of f is the map $\widehat{f} : \widehat{G^\natural} \longrightarrow \mathbb{C}$ defined by*

$$\widehat{f}(\phi) = \int_G \phi(x^\diamond) f(x) dx.$$

By a classical argument, the inverse Fourier transform is given by

$$f(x) = \int_{\widehat{G^\natural}} \phi(x)\widehat{f}(\phi) d\pi(\phi)$$

where the existence of the measure π is ensured by the following theorem (Theorem 2.5).

Theorem 2.5 ([5]). *Let (G, K) be a hypergroup Gelfand pair. There exists a unique nonnegative measure π on $\widehat{G^\natural}$ such that*

$$\int_G |f(x)|^2 dx = \int_{\widehat{G^\natural}} |\widehat{f}(\phi)|^2 d\pi(\phi), \ \forall f \in L^1(G) \cap L^2(G).$$

Hereafter are the analogue of the Hausdorff-Young inequality and its inverse inequality.

Theorem 2.6. *[7] Let p, q be such that $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, the following inequalities hold.*

(1) $\|\hat{f}\|_{p'} \le \|f\|_p$, *for all* $f \in L^p(G)$.
(2) $\|f\|_{p'} \le \|\widehat{f}\|_p$, *for all* $f \in L^{p'}(G)$.

3. Sobolev spaces and embedding results

Definition 3.1. *[2] Let (G, K) be a hypergroup Gelfand pair. Let $\gamma : \widehat{G^\natural} \longrightarrow \mathbb{R}_+$ be a positive measurable function and let $s \in (0, +\infty)$. The set*

$$H_\gamma^{s,\natural}(G) = \left\{ f \in L^{2,\natural}(G) : \int_{\widehat{G^\natural}} (1 + \gamma(\phi)^2)^s |\hat{f}(\phi)|^2 d\pi(\phi) < \infty \right\}$$

provided with the norm

$$\|f\|_{H_\gamma^{s,\natural}} = \left(\int_{\widehat{G^\natural}} (1 + \gamma(\phi)^2)^s |\hat{f}(\phi)|^2 d\pi(\phi) \right)^{\frac{1}{2}}$$

will be called a Sobolev space.

In the sequel, the symbol $\hookrightarrow$ denotes the continuous embedding.

Theorem 3.2. *Let (G, K) be a hypergroup Gelfand pair. Let $\alpha > s > 0$ and let $p = \frac{2\alpha}{\alpha + s}$. Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. If $(1 + \gamma^2)^{-1} \in L^\alpha(\widehat{G^\natural})$, then $H_\gamma^{s,\natural}(G) \hookrightarrow L^{p',\natural}(G)$.*

Proof. The conditions about α and s imply $1 < p < 2$. Then, by the inverse Hausdorff–Young inequality in Theorem 2.6, we have $\|f\|_{p'} \leq \|\hat{f}\|_p$.

$$\begin{aligned}\|\hat{f}\|_p^p &= \int_{\widehat{G^\natural}} |\hat{f}(\phi)|^p d\pi(\phi)\\ &= \int_{\widehat{G^\natural}} |\hat{f}(\phi)|^p \cdot \frac{(1+\gamma(\phi)^2)^{\frac{sp}{2}}}{(1+\gamma(\phi)^2)^{\frac{sp}{2}}} d\pi(\phi)\\ &= \int_{\widehat{G^\natural}} |\hat{f}(\phi)|^{\frac{2p}{2}} (1+\gamma(\phi)^2)^{\frac{sp}{2}} (1+\gamma(\phi)^2)^{-\frac{sp(2-p)}{2(2-p)}} d\pi(\phi).\end{aligned}$$

Since $\dfrac{p}{2} + \dfrac{2-p}{2} = 1$, then by the Hölder's inequality, we have

$$\begin{aligned}\|\hat{f}\|_p^p &\leq \left(\int_{\widehat{G^\natural}} (1+\gamma(\phi)^2)^s |\hat{f}(\phi)|^2 d\pi(\phi)\right)^{\frac{p}{2}} \left(\int_{\widehat{G^\natural}} (1+\gamma(\phi)^2)^{-\frac{sp}{2-p}} d\pi(\phi)\right)^{\frac{2-p}{2}}\\ \|\hat{f}\|_p &\leq \left(\int_{\widehat{G^\natural}} (1+\gamma(\phi)^2)^s |\hat{f}(\phi)|^2 d\pi(\phi)\right)^{\frac{1}{2}} \left(\int_{\widehat{G^\natural}} (1+\gamma(\phi)^2)^{-\frac{sp}{2-p}} d\pi(\phi)\right)^{\frac{2-p}{2p}}\\ &\leq \|f\|_{H_\gamma^{s,\natural}} \left(\int_{\widehat{G^\natural}} (1+\gamma(\phi)^2)^{-\frac{sp}{2-p}} d\pi(\phi)\right)^{\frac{2-p}{2p}}\\ &\leq \|f\|_{H_\gamma^{s,\natural}} \|(1+\gamma^2)^{-1}\|_\alpha^{\frac{s}{2}} \text{ since } \alpha = \frac{sp}{2-p}.\end{aligned}$$

Finally, $\|f\|_{p'} \leq \|\hat{f}\|_p \leq \|f\|_{H_\gamma^{s,\natural}} \|(1+\gamma^2)^{-1}\|_\alpha^{\frac{s}{2}}$. Thus, $H_\gamma^{s,\natural}(G) \hookrightarrow L^{p',\natural}(G)$. □

Lemma 3.3. *Let (G,K) be a hypergroup Gelfand pair. If $\phi \in \widehat{G^\natural}$, then $\forall g \in \mathcal{K}^\natural(G)$, $g * \phi = \widehat{g}(\phi)\phi$.*

Proof. Let $f, g \in \mathcal{K}^\natural(G)$. Consider $\Phi(g) = \widehat{g}(\phi)$. Set $A = \displaystyle\int_G f(x)\Phi(g)\phi(x^\diamond)dx$. We have

$$\begin{aligned}A &= \Phi(f)\Phi(g) = \Phi(f*g) \text{ (the convolution Theorem)}\\ &= \int_G f*g(x)\phi(x^\diamond)dx\\ &= \int_G \phi(x^\diamond)\left(\int_G f(x*y)g(y^\diamond)dy\right)dx\\ &= \int_G g(y^\diamond)\left(\int_G f(x*y)\phi(x^\diamond)dx\right)dy \text{ (the Fubini's Theorem)}\\ &= \int_G g(y^\diamond)\left(\int_G f(x)\phi(y*x^\diamond)dx\right)dy\\ &= \int_G f(x)\left(\int_G g(y^\diamond)\phi(y*x^\diamond)dy\right)dx \text{ (again the Fubini's Theorem)}\\ &= \int_G f(x)\left(\int_G g(y)\phi(y^\diamond*x^\diamond)dy\right)dx \text{ (change of variable } y \to y^\diamond)\\ &= \int_G f(x)(g*\phi)(x^\diamond)dx.\end{aligned}$$

Since $\int_G f(x)\Phi(g)\phi(x^\diamond)dx = \int_G f(x)(g*\phi)(x^\diamond)dx$ for all $f \in \mathcal{K}^\natural(G)$, then $\Phi(g)\phi(x^\diamond) = (g*\phi)(x^\diamond)$. Therefore, $g*\phi = \Phi(g)\phi = \widehat{g}(\phi)\phi$. □

Theorem 3.4. *Let (G,K) be a hypergroup Gelfand pair. Let $f \in H_\gamma^{s,\natural}(G)$. If $y \in G$, then*

$$\int_G |f(x*y^\diamond) - f(x)|^2 dx \le \left(\sup_{\phi\in\widehat{G^\natural}} \frac{|\phi(y)-1|^2}{(1+\gamma(\phi)^2)^s}\right)\cdot \|f\|^2_{H_\gamma^{s,\natural}}.$$

Proof. Fix $y \in G$. Let $f \in \mathcal{K}^\natural(G)$. Set $f_y(x) = f(x*y^\diamond)$, $x \in G$. We have,

$$\begin{aligned}\widehat{f_y}(\phi) &= \int_G \phi(x^\diamond)f(x*y^\diamond)dx\\ &= \int_G \phi(y^\diamond * x^\diamond)f(x)dx\,(\text{change of variable } x \to x*y^\diamond)\\ &= \int_G \phi(y^\diamond * x)f(x^\diamond)dx\,(\text{change of variable } x \to x^\diamond)\\ &= (\phi * f)(y^\diamond) = (f*\phi)(y^\diamond)\\ &= \widehat{f}(\phi)\phi(y^\diamond)(\text{Lemma } 3.3).\end{aligned}$$

$$\begin{aligned}\int_G |f_y(x) - f(x)|^2dx &= \int_{\widehat{G^\natural}} |\widehat{f_y}(\phi) - \widehat{f}(\phi)|^2 d\pi(\phi)(\text{Theorem } 2.5)\\ &= \int_{\widehat{G^\natural}} |\widehat{f}(\phi)\phi(y^\diamond) - \widehat{f}(\phi)|^2 d\pi(\phi)\\ &= \int_{\widehat{G^\natural}} |\widehat{f}(\phi)(\phi(y^\diamond)-1)|^2 d\pi(\phi)\\ &= \int_{\widehat{G^\natural}} |\widehat{f}(\phi)|^2|\phi(y^\diamond)-1|^2 d\pi(\phi)\\ &= \int_{\widehat{G^\natural}} |\widehat{f}(\phi)|^2|\phi(y)-1|^2 \frac{(1+\gamma(\phi)^2)^s}{(1+\gamma(\phi)^2)^s} d\pi(\phi)\\ &\le \left(\sup_{\phi\in\widehat{G^\natural}} \frac{|\phi(y)-1|^2}{(1+\gamma(\phi)^2)^s}\right)\cdot\|f\|^2_{H_\gamma^{s,\natural}}.\end{aligned}$$

Since $\mathcal{K}^\natural(G)$ is dense in $H_\gamma^{s,\natural}(G)$, the result holds for all $f \in H_\gamma^{s,\natural}(G)$. □

Theorem 3.5. *Let (G,K) be a hypergroup Gelfand pair. If $f \in H_\gamma^{s,\natural}(G)$, then there exists $\eta \in \mathcal{K}^\natural(G)$ such that*

$$\|f*\eta - f\|_2 \le \sup_{y\in supp(\eta)}\ \sup_{\phi\in\widehat{G^\natural}} \frac{|\phi(y)-1|}{(1+\gamma(\phi)^2)^{\frac{s}{2}}}\cdot\|f\|_{H_\gamma^{s,\natural}}.$$

Proof. Since G is a locally compact compact Hausdorff space, then it is a Tychonoff space. Therefore, there exists $\eta \in \mathcal{K}^\natural(G)$ such that $\eta(e) \neq 0$, $\eta \ge 0$ and $\int_G \eta(x)dx = 1$. Then, we have

$$\|f*\eta - f\|_2 = \left(\int_G |f*\eta(x) - f(x)|^2dx\right)^{\frac{1}{2}}$$

$$= \left(\int_G \left| \int_G f(x * y^\diamond)\eta(y)dy - f(x) \right|^2 dx \right)^{\frac{1}{2}}$$
$$= \left(\int_G \left| \int_G f(x * y^\diamond)\eta(y)dy - f(x) \int_G \eta(y)dy \right|^2 dx \right)^{\frac{1}{2}}$$
$$= \left(\int_G \left| \int_G (f(x * y^\diamond) - f(x))\eta(y)dy \right|^2 dx \right)^{\frac{1}{2}}$$
$$\leq \int_G \left(\int_G |f(x * y^\diamond) - f(x)|^2 |\eta(y)|^2 dx \right)^{\frac{1}{2}} dy$$
$$\leq \int_G |\eta(y)| \left(\int_G |f(x * y^\diamond) - f(x)|^2 dx \right)^{\frac{1}{2}} dy$$
$$\leq \sup_{y \in supp(\eta)} \sup_{\phi \in \widehat{G^\natural}} \frac{|\phi(y) - 1|}{(1 + \gamma(\phi)^2)^{\frac{s}{2}}} \cdot \|f\|_{H_\gamma^{s,\natural}} \text{ (use Theorem 3.4).}$$

□

In the rest of the paper, we assume that G is compact.

Theorem 3.6. *Let G be a compact hypergroup. Let (G, K) be a hypergroup Gelfand pair. Let $p, q \in (1, \infty)$. If a sequence $(f_n) \subset L^{p,\natural}(G)$ converges weakly to a function f, then for every $\eta \in \mathcal{K}^\natural(G)$ the sequence $(f_n * \eta)$ converges strongly to $f * \eta$ in $L^{q,\natural}(G)$.*

Proof. Since the sequence (f_n) converges weakly to f, then by [4, Proposition 3.5] there exists a positive real M such that

$$\|f_n\|_p \leq M \text{ and } \|f\|_p \leq M.$$

We have

$$|f_n * \eta(x)| = \left| \int_G f_n(y)\eta(y^\diamond * x)dy \right|$$
$$\leq \int_G |f_n(y)\eta(y^\diamond * x)| \, dy$$
$$\leq \|f_n\|_p \left(\int_G |\eta(y^\diamond * x)|^{p'} dy \right)^{\frac{1}{p'}}$$
$$\leq M\|\eta\|_{p'}$$

where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. Since G is compact, the constant function $x \longmapsto M\|\eta\|_{p'}$ is integrable. Therefore, by the Dominated Convergence Theorem, we have

$$f_n * \eta(x) = \int_G f_n(y)\eta(y^\diamond * x)dy \xrightarrow{n \to \infty} \int_G f(y)\eta(y^\diamond * x)dy = f * \eta(x).$$

Then,

$$\begin{aligned}|f_n * \eta(x) - f * \eta(x)| &= \left|\int_G f_n(y)\eta(y^\diamond * x)dy - \int_G f(y)\eta(y^\diamond * x)dy\right| \\ &= \left|\int_G (f_n(y) - f(y))\eta(y^\diamond * x)dy\right| \\ &\leq \|f_n - f\|_p \left(\int_G |\eta(y^\diamond * x)|^{p'}dy\right)^{\frac{1}{p'}} \\ &\leq 2M\left(\int_G |\eta(z)|^{p'}d(\delta_{y^\diamond} * \delta_x)(z)\right)^{\frac{1}{p'}} \\ &\leq 2M\|\eta\|_{p'}.\end{aligned}$$

Again, by the Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty} \|f_n * \eta - f * \eta\|_q = 0.$$

□

Hereafter is the analogue of the Rellich-Kondrachov theorem for hypergroup Gelfand pairs.

Theorem 3.7. *Assume that G is compact. Let (G, K) be a hypergroup Gelfand pair. Let $\alpha > s > 0$. Let $p := \frac{2\alpha}{\alpha+s}$ and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. If $(1+\gamma^2)^{-1} \in L^\alpha(\widehat{G^\natural})$ and*

$$\lim_{y\to e}\left(\sup_{\phi\in\widehat{G^\natural}} \frac{|\phi(y) - 1|}{(1+\gamma(\phi)^2)^{\frac{s}{2}}}\right) = 0,$$

then $H^{s,\natural}_\gamma(G)$ embeds compactly in $L^{q,\natural}(G)$ for every $q \in [1, p']$.

Proof. In Theorem 3.2, we obtained that $H^{s,\natural}_\gamma(G) \hookrightarrow L^{p',\natural}(G)$; since G is compact and $p' > q$, then $L^{p',\natural}(G) \hookrightarrow L^{q,\natural}(G)$. Therefore, we have that $H^{s,\natural}_\gamma(G) \hookrightarrow L^{q,\natural}(G)$. Now, let (f_n) be a bounded sequence in $H^{s,\natural}_\gamma(G)$. Then, (f_n) is a bounded sequence in $L^{p',\natural}(G)$. There exists $M > 0$ such that

$$\forall n \in \mathbb{N}, \|f_n\|_{p'} \leqslant M.$$

For $g \in L^{p,\natural}(G)$, we have

$$|\langle f_n, g\rangle| \leqslant \|f_n\|_{p'}\|g\|_p \leqslant M\|g\|_p.$$

Therefore, (f_n) is weakly bounded. It admits a subsequence (h_n) which converges weakly to $h \in L^{p',\natural}(G)$. Take $\varepsilon > 0$ and $\eta \in \mathcal{K}^\natural(G)$ such that $\|h * \eta - h\|_2 < \varepsilon$.

By Theorem 3.5 and Theorem 3.6, we have

$$\begin{aligned}\|h_n - h\|_2 &\leq \|h_n - h_n * \eta\|_2 + \|h_n * \eta - h * \eta\|_2 + \|h * \eta - h\|_2 \\ &\leq \sup_{y\in supp(\eta)}\left(\sup_{\phi\in\widehat{G^\natural}} \frac{|\phi(y) - 1|}{(1+\gamma(\phi)^2)^{\frac{s}{2}}}\right)\cdot\|h_n\|_{H^{s,\natural}_\gamma} + \|h_n * \eta - h * \eta\|_2 + \varepsilon \\ &\leq 2\varepsilon + \|h_n * \eta - h * \eta\|_2.\end{aligned}$$

As the above inequality is realized for an arbitrary ε, then

$$\|h_n - h\|_2 \leq \|h_n * \eta - h * \eta\|_2.$$

Therefore,

$$\lim_{n\to\infty} \|h_n - h\|_2 = \lim_{n\to\infty} \|h_n * \eta - h * \eta\|_2 = 0.$$

Thus, (h_n) converges to h in $L^{2,\natural}(G)$. Since G is compact, we apply the Vitali's convergence theorem to conclude that (h_n) converges to h in $L^{q,\natural}(G)$. $\square$

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