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Hybrid Iterative Methods for Solving Nonlinear Equations in Banach Spaces

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Abstract. The present article contributes to the solution of equations which carry the symmetry property of the problem or not. Iterative methods with in- verses generate sequences converging faster to a solution of an equation than methods without inverses. However, the implementation of these methods has drawbacks, since the analytical form of these inverse may be unavailable or computationally very expensive. This problem is addressed in this paper by replacing the inverse with a finite sum of linear opera- tors. A convergence analysis is developed for the hybrid methods. The numerical examples demonstrate that the number of iterates is essentially the same between the hybrid and the original method. This technique is also extended to solve generalized equations.

1. Introduction

The letters X, Y denote Banach spaces; $\Omega \subset X$ is a convex and open subset of X, and $F_1 : \Omega \longrightarrow Y$ stands for a continuous operator. Numerous applications from diverse areas of computational science and Engineering can be converted by using mathematical modelling [3, 8, 14, 17, 19–21, 23, 26, 28, 33, 35] to finding a solution $s^* \in \Omega$ of the nonlinear in the general equation

$$F_1(x) = 0. \tag{1.1}$$

The closed form of the solution s^* is attainable only in special cases. This forces researchers and practitioners to solve the equation (1.1) iteratively. Single-step methods of high convergence order

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look like the Newton-type defined for each $n = 0, 1, 2, \ldots$ by

$$x_0 \in \Omega, \quad x_{n+1} = x_n - L_n^{-1} F_1(x_n), \tag{1.2}$$

where $L_n \in \mathcal{L}(X, Y)$ which is the space of continuous operators mapping X into Y, and $L_n^{-1} \in \mathcal{L}(Y, X)$ for each $n = 0, 1, 2, \ldots$. By L_n, we denote $L(x_n)$. Some choices for the operator L_n can be

$$\begin{aligned} L_n &= F_1'(x_n) \text{ (Newton's method)}, \\ L_n &= [x_n - F_1(x_n), x_n + F_1(x_n); F] \text{ (Steffensen's method)}, \\ L_n &= I \text{ (The Picard method), the identity operator.} \end{aligned}$$

Here F_1', $[.,.; F_1]$ denote Fréchet-derivative and divided differences of order one for the operator F_1, respectively [22, 25].

Many other choices are possible. It turns out that the inverse of the operator L_n is costly or impossible to find in general. This concern with the implementation of these methods constitutes the motivation for this paper. Our idea is to replace the inverse with a finite sum of linear operators converging to it. The reasoning is explained as follows. Let $p \in N$ be fixed.

Suppose there exists $\Gamma \in \mathcal{L}(X, Y)$ such that $\Gamma^{-1} \in \mathcal{L}(Y, X)$ and for $A = A(x) = \Gamma^{-1}(\Gamma - L(x))$ the operator $I - A(x)$ is also invertible, i.e. $(I - A(x))^{-1} \in \mathcal{L}(Y, X)$. In this case, the Newton-type method can read as

$$x_0 \in D, \ x_{n+1} = x_n - (I - A)^{-1} \Gamma^{-1} F_1(x_n). \tag{1.3}$$

Note that we have

$$(I - A)^{-1} \Gamma^{-1} = [\Gamma(I - A)]^{-1} = L_n^{-1}. \tag{1.4}$$

However, even if the linear operator Γ^{-1} is known it is still required to find the inverse of $(I - A)$, which is not a fixed operator (in general). But what if we replace this operator with $M = M_p(x) = I + A + \ldots + A^p$. Then, method (1.3) can be written as

$$x_0 \in \Omega, \ x_{n+1} = x_n - M \Gamma^{-1} F_1(x_n). \tag{1.5}$$

It is clear that (1.5) is a useful alternative for (1.3) because of (1.4). By letting $p \longrightarrow +\infty$, we get $\lim_{p \to +\infty} M_p = L_n^{-1}$ if the limit exists. The condition $\|A\| < 1$ for each $x \in \Omega$ assures the existence of such a limit. If the linear operator M is invertible and the sequence $\{x_n\}$ given by (1.5) converges to some s^*, then by (1.5) we get

$$M^{-1}(x_n - x_{n+1}) = \Gamma^{-1} F_1(x_n)$$

leading to

$$0 = \lim_{n \to +\infty} M^{-1}(x_n - x_{n+1}) = \lim_{n \to +\infty} \Gamma^{-1} F_1(x_n),$$

i.e. $F_1(s^*) = 0$. Thus, the point s^* solves the equation (1.1). The same reasoning leads for $A_1 = (\Gamma - L(x))\Gamma^{-1}$ and $M_1 = I + A_1 + ... + A_1^p$ to the method

$$x_0 \in \Omega \quad, x_{n+1} = x_n - M_1\Gamma^{-1}F_1(x_n). \tag{1.6}$$

It is clear that the study of the convergence of the method (1.6) is analogous to (1.5). That is why we study only method (1.5) in Section 2. We deal with two kinds of convergence: the semi-local and the local. The first utilizes knowledge in a neighborhood of x_0 and develops estimates related to $\|x_{n+1} - x_n\|$ and $\|s^* - x_n\|$, and the convergence conditions assure that $\lim_{n\to+\infty} x_n = s^*$. In the second kind, knowledge about a neighborhood of s^* is used to provide the same estimates as in the semi-local kind and again $\lim_{n\to+\infty} x_n = s^*$. It is worth noting that the iterates generated by (1.3),(1.5) and (1.6) are not the same in general. But we use the same notation for simplicity. The convergence for both kinds relies on generalized continuity conditions controlling the operators involved [5, 6, 9, 18]. In particular, our semi-local convergence analysis depends on the usage of majorizing sequences [29, 30, 34]. Notice that the method (1.5) can also be written for $D_n = \Gamma M^{-1}$ as

$$x_0 \in \Omega, \;\; F_1(x_n) + D_n(x_{n+1} - x_n) = 0. \tag{1.7}$$

In section 3 we also use the developed methodology for solving nonlinear equations to solve generalized equations. That is find $x \in X$ such that

$$F_1(x) + F_2(x) \ni 0. \tag{1.8}$$

Here $F_2 : X \rightrightarrows Y$ is a set-valued operator mapping X into Y with closed graph [1–4, 13, 17–19, 21–23, 26, 28, 32, 35] and operator F is as previously defined. A plethora of applications from mathematical programming, variational inequalities, optimal control, or constrained systems are written in the form (1.8). There is extensive literature on iterative methods solving the generalized equation (1.8) [1–4, 13, 17–19, 21–23, 26, 28, 32, 35]. Notice that the method used in the literature to solve (1.8) is defined by

$$F_1(x_n) + \bar{D}_n(x_{n+1} - x_n) + F_2(x_{n+1}) \ni 0, \tag{1.9}$$

where $\bar{D}_n$ is a linear operator. It can be chosen as $\bar{D}_n = L_n$, $\bar{D}_n = F_1'(x_n)$ or $\bar{D}_n \in \partial F_1(x_n)$ or other choices [9, 11, 23, 24]. These methods have the same problems as the ones for solving nonlinear equations. That is why it is justified to consider the analog of (1.7) defined by

$$F_1(x_n) + D_n(x_{n+1} - x_n) + F_2(x_{n+1}) \ni 0 \tag{1.10}$$

The semi-local and local convergence of the method (1.10) is developed in section 3 in an analogous way to section 2 for the method (1.5) or (1.7). In numerical section 4, the examples demonstrate that the number of iterations of the hybrid methods to arrive at a predetermined error tolerance is essentially the same as with the original methods. Moreover, the convergence order is also the same.

In order to achieve all this we redevelop some standard terminology to make the paper as self-contained as possible. More details can be found in [8, 11, 19]. Let $S(z, \rho)$ and its closure $S[z, \rho]$ denote open and closed balls, respectively of center $z \in X$ and radius $\rho > 0$. Let C be a set in X. Define the distance for $x \in X$ to C by $dist(x, C) = \inf_{x \in C} \|x - y\|$. The generalized set-valued operator G relates with its graph given by $gph(G) = \{(x, y) \in X \times Y, y \in F_2(x)\}$, and its domain $dom(G) = \{x \in X | F_2(x) \neq 0\}$. The inverse of G is given as $G^{-1}(y) = \{x \in X, y \in F_2(x)\}$. Note that a set-valued operator $H : X \rightrightarrows Y$ is said to be metrically regular at x_0 for y_0 if $y_0 \in H(x_0)$ and there exists neighbourhoods V_1 of x_0 and V_2 of y_0 and $\beta > 0$ such that $gph(H \cap (V_1 \times V_2))$ is closed and for each $(x, y) \in V_1 \times V_2$

$$dist(x, H^{-1}(y)) \leq \beta dist(y, H(x)). \tag{1.11}$$

The regularity modulus of H at x_0 for y_0 is the infimum over all $\beta > 0$ and is denoted by $reg(H; x_0/y_0)$. Additionally if the operator $\Delta : V_2 \to y \to H^{-1}(y) \cap V_1$ is not multivalued on V_2, then we say that H is strongly metrically regular. In this case, Δ is Lipchitz continuous on V_2. Finally, Section 5 contains concluding remarks and directions for research.

2. Convergence for the Method (1.5)

We start with the study of the semi-local analysis in this section. Some auxiliary results and definitions are useful.

Lemma 2.1. *(Banach Lemma on Invertible Operators)([14, 22, 30, 34]) If P is a bounded linear operator in X, P^{-1} exists if and only if there is a bounded linear operator P_1 in X such that P_1^{-1} exists and*

$$\|I - P_1 P\| < 1.$$

If P^{-1} exists, then

$$P^{-1} = \sum_{n=0}^{\infty} (I - P_1 P)^n P_1$$

and

$$\|P^{-1}\| \leq \frac{\|P_1\|}{1 - \|I - P_1 P\|}.$$

Further, we use majorizing sequences to prove the semi-local convergence. Recall the definition of majorizing sequence.

Definition 2.1. *([14, 22, 30, 34]) Let $\{x_n\}$ be a sequence in a normed space X. Then a nonnegative scalar sequence $\{u_n\}$ for which*

$$\|x_{n+1} - x_n\| \le u_{n+1} - u_n \ \forall n \ge 0 \tag{2.1}$$

holds, is a majorizing sequence for $\{x_n\}$. Note that any majorizing sequence is necessarily nondecreasing. Moreover, if the sequence $\{u_n\}$ converges, then $\{x_n\}$ converges too, and for $u^* = \lim_{n \longrightarrow \infty} u_n$

$$\|s^* - x_n\| \le u^* - u_n.$$

Hence, the study of the convergence of the sequence $\{x_n\}$ reduces to that of $\{u_n\}$.

The analysis requires some conditions. Let $E = [0, +\infty)$.
Suppose

(H_1) There exists a function $\phi : E \times E \times E \to [0, +\infty)$ continuous as well nondecreasing in all three variables and invertible operators $M(.)$ and Γ such that for some $x_0 \in \Omega$, and each $x, y \in \Omega$ the following Mysovskii-like condition holds

$$\begin{aligned} & \|M(x)\Gamma^{-1}(F_1(y) - F_1(x) - \Gamma M^{-1}(x))\| \\ \le \quad & \phi(\|x - x_0\|, \|y - x_0\|, \|y - x\|)\|y - x\| \end{aligned}$$

Define the real real sequence $\{\alpha_n\}$ for $\alpha_0 = 0, \alpha_1 \ge \eta := \|M(x_0)\Gamma^{-1}F_1(x_0)\|$ and each $n = 0, 1, 2, \ldots$ by

$$\alpha_{n+1} = \alpha_n + \phi(\alpha_{n-1}, \alpha_n, \alpha_n - \alpha_{n-1})(\alpha_n - \alpha_{n-1}), \ n = 1, 2, \ldots \tag{2.2}$$

Notice that the constant η is well defined since the operator Γ is invertible. Moreover, the sequence $\{\alpha_n\}$ defined by the formula (2.2) is proven to be majorizing for the method (1.5) in Theorem 2.3. But let us present convergence conditions for it.

(H_2) There exists a parameter $\rho \ge \eta$ such that for each $n = 0, 1, 2, \ldots$

$$\phi(\alpha_{n-1}, \alpha_n, \alpha_n - \alpha_{n-1}) < 1 \text{ and } \alpha_n \le \rho.$$

It follows by the condition (H_2) and the formula (2.2) that

$$0 \le \alpha_{n-1} \le \alpha_n \le \rho$$

and there exists $\alpha^* \in [\eta, \rho]$ such that $\lim_{n \to +\infty} \alpha_n = \alpha^*$. H is well known that this limit is the unique least upper bound of the sequence $\{\alpha_n\}$ and

(H_3) $S[x_0, \alpha^*] \subset \Omega$.

The conditions (H_1)-(H_3) combined with the terminology are utilized for the convergence of the method (1.5).

Theorem 2.1. *Suppose that the conditions (H_1)-(H_3) hold. Then, the sequence $\{x_n\}$ generated by the method (1.5) exists in the ball $S(x_0, \alpha^*)$ remains in the same ball for each $n = 0, 1, 2, ...$ and is convergent to a unique solution $s^* \in S[x_0, \alpha^*]$ of the equation $F_1(x) = 0$ such that*

$$\|s^* - x_n\| \leq \alpha^* - \alpha_n, \;\; n = 0, 1, 2 \dots . \tag{2.3}$$

Proof. The process of induction is employed to show the assertion

$$\|x_{m+1} - x_m\| \leq \alpha_{m+1} - \alpha_m \text{ for } m = 0, 1, 2, \dots . \tag{2.4}$$

The choice of η, (2.2) and the method (1.5) give that

$$\|x_1 - x_0\| = \|M(x_0)\Gamma^{-1}F_1(x_0)\| \leq \eta = \alpha_1 - \alpha_0 < \alpha^*,$$

so the assertion (2.4) holds if $m = 0$, and the iterate $x_1 \in S(x_0, \alpha^*)$. Suppose iterates $x_0, x_1, ..., x_m$ exist and (2.4) holds for all integers smaller or equal to $m - 1$. Notice that the iterate x_{m+1} exists by the method (1.5) and the invertability of the operators $M(x_m)$ and Γ. Then, we can write by the method (1.5) the Ostrowski-type representation for $F_1(x_m)$ as

$$F_1(x_m) \;\; = \;\; F_1(x_m) - F_1(x_{m-1}) - \Gamma M^{-1}(x_m - x_{m-1}). \tag{2.5}$$

Using the condition (H_1), method (1.5), (2.2) and the induction hypothesis on (2.5) we obtain in turn that

$$\begin{aligned}
\|x_{m+1} - x_m\| &= \|M\Gamma^{-1}(F_1(x_m))\| \\
&= \|M\Gamma^{-1}(F_1(x_m) - F_1(x_{m-1}) - \Gamma M^{-1}(x_m - x_{m-1}))\| \\
&\leq \phi(\|x_{m-1} - x_0\|, \|x_m - x_0\|, \|x_m - x_{m-1}\|)\|x_m - x_{m-1}\| \\
&\leq \phi(\alpha_{m-1}, \alpha_m, \alpha_m - \alpha_{m-1})(\alpha_m - \alpha_{m-1}) \\
&= \alpha_{m+1} - \alpha_m, \qquad (2.6)
\end{aligned}$$

and

$$\begin{aligned}
\|x_{m+1} - x_0\| &\leq \|x_{m+1} - x_m\| + \|x_m - x_{m-1}\| + ... + \|x_1 - x_0\| \\
&\leq \alpha_{m+1} - \alpha_m + \alpha_m - \alpha_{m-1} + ... + \alpha_1 - \alpha_0 \\
&= \alpha_{m+1} < \alpha^*,
\end{aligned}$$

which complete the induction for the assertion (2.4) and show that all iterates $\{x_{m+1}\} \subset S(x_0, \alpha^*)$. But the sequence $\{\alpha_m\}$ is complete by the condition (H_2) as convergent. It follows by the estimate (2.4) that $\{x_m\}$ is also complete. But the space X is Banach, so there exists $s^* \in S[x_0, \alpha^*]$ such that $\lim_{m \to +\infty} x_m = s^*$. Next, by letting $m \to +\infty$ in (2.6), the invertibility of the operators $M(.), \Gamma$ and the continuity of the operator F, we deduce that $\lim_{m\to+\infty} M\Gamma^{-1}F_1(s^*) = 0$. By the invertibility of M

$$\begin{aligned} 0 &= M^{-1}(0) = M^{-1}(\lim_{m\to+\infty} M\Gamma^{-1}F_1(x_m)) = \lim_{m\to+\infty} MM^{-1}\Gamma^{-1}F_1(x_m) \\ &= \lim_{m\to+\infty} \Gamma^{-1}F_1(x_m) = \Gamma^{-1}\lim_{m\to+\infty} F_1(x_m) = \Gamma^{-1}F_1(s^*). \end{aligned}$$

So, $F_1(s^*) = 0$, since $\Gamma(0) = 0$. Let $i = 0, 1, 2, \ldots$. Then, the triangle inequality and (2.4) give

$$\|x_{m+i} - x_m\| \le \alpha_{m+i} - \alpha_m. \tag{2.7}$$

Hence, by letting $i \to +\infty$ in (2.7) we prove (2.3). Finally, to show the uniqueness part, let $w \in S[x_0, \alpha^*]$ with $F_1(w) = 0$ and $w \neq s^*$. By using the conditions (H_1), (H_2) we can write in turn that

$$\begin{aligned} \|w - s^*\| &= \|(M\Gamma^{-1})(\Gamma M^{-1})(w - s^*)\| \\ &= \|M\Gamma^{-1}(F_1(w) - F_1(s^*) - \Gamma M^{-1}(w - s^*))\| \\ &\le \phi(\|s^* - x_0\|, \|w - x_0\|, \|w - s^*\|)\|w - s^*\| \\ &\le \phi(\alpha^*, \alpha^*, \|w - s^*\|)\|w - s^*\| < \|w - s^*\|, \end{aligned}$$

which gives a contradiction. Therefore, we conclude that $w = s^*$. □

Remark 2.1. *The condition Mysoskii-type [22] condition in (H_1) can be replaced as follows:*

$(H_1)'$ *With operator Γ as in condition (H_1), suppose that there exists $a \in (0, \frac{1}{2})$ such that $\|A\| < a$ and for each $x, y \in \Omega$*

$$\begin{aligned} &\|\Gamma^{-1}(F_1(y) - F_1(x) - \Gamma M^{-1}(y - x))\| \\ \le\ &\phi_1(\|x - x_0\|, \|y - x_0\|, \|y - x\|)\|y - x\|, \end{aligned}$$

where the function ϕ_1 is as the function ϕ. Then, the condition $(H_1)'$ implies (H_1) if we take $\phi = a_0\phi_1$, where $a_0 = \frac{1}{1-a}$.

This is the case, since by the definition of the operator M, we have the estimate

$$\begin{aligned} \|M\| &= \|I + A + \ldots + A^p\| \\ &\le 1 + a + \ldots + a^p \\ &= \frac{1 - a^{p+1}}{1 - a} = a_0 < \frac{1}{1 - a}. \end{aligned} \tag{2.8}$$

In this case the invertibility of the operator M is implied by the Banach Lemma 2.1, since

$$\|I - M\| \le \|A\| + \ldots + \|A\|^p \le a + \ldots + a^p = a\frac{1 - a^p}{1 - a} < \frac{a}{1 - a} < 1.$$

So, M is invertible and

$$\|M^{-1}\| \le b = \frac{1 - a}{1 - 2a}.$$

Next, we develop the local convergence. The role of the initial point x_0 is exchanged by s^* and the function ϕ by ψ.

Suppose:

(H_4) *There exists a function $\psi : E \longrightarrow [0, +\infty)$ continuous and nondecreasing such that the equation $\psi(t) - 1 = 0$ has a smallest positive solution denoted by r.*

(H_5) *There exists a solution $s^* \in \Omega$, and invertible linear operators M, Γ such that for each $x \in \Omega$*

$$\begin{aligned} & \|M\Gamma^{-1}(F_1(x) - F_1(s^*) - \Gamma M^{-1}(x - s^*))\| \\ \leq \quad & \psi(\|x - s^*\|)\|x - s^*\| \end{aligned}$$

and

(H_6) $S[s^*, r] \subset \Omega$.

Next, the constant r is shown to be a radius of convergence for the method (1.5).

Theorem 2.2. *Suppose that the conditions $(H_4) - (H_6)$ hold. Then, the sequence $\{x_n\}$ for $x_0 \in S(s^*, r) - \{s^*\}$ exists in $S(s^*, r)$, stays in $S(s^*, r)$ and converges to s^* so that*

$$\|x_{n+1} - s^*\| \leq \psi(\|x_n - s^*\|)\|x_n - s^*\| \leq \|x_n - s^*\| < r. \tag{2.9}$$

Moreover, x^* is the only solution of the equation $F(x)$ =0 in the ball $U(s^*, r)$.

Proof. The iterates $x_1, x_2, ..., x_{m+1}$ are well defined by the method (1.5), and we can write in turn that

$$\begin{aligned} x_{m+1} - s^* &= x_m - s^* - M\Gamma^{-1}F_1(x_m) \\ &= M\Gamma^{-1}(F_1(x_m) - F_1(s^*) - \Gamma M^{-1}(x_m - s^*)). \end{aligned} \tag{2.10}$$

It follows by the conditions $(H_4), (H_5)$ and (2.10) that

$$\begin{aligned} \|x_{m+1} - s^*\| &\leq \psi(\|x_m - s^*\|)\|x_m - s^*\| \\ &\leq \xi\|x_m - s^*\| \leq \xi^{m+1}\|x_0 - s^*\| < r, \end{aligned} \tag{2.11}$$

where $\xi = \psi(\|x_0 - s^*\|) \in [0, 1)$ showing the assertion (2.9) for each $m = 1, 2, ...,$ since $x_0 \in S(s^*, r) - \{s^*\}$. By letting $m \to +\infty$ in (2.11), we deduce that $\lim_{m\to+\infty} x_m$. In order to show the uniqueness part, suppose there exists a solution $w_1 \in S(s^*, r)$ such that $w_1 \neq s^*$. Then, as in the semi-local case, we can write in turn

$$\begin{aligned} \|w_1 - s^*\| &= \|(M\Gamma^{-1})(\Gamma M^{-1}(w_1 - s^*))\| \\ &= \|M\Gamma^{-1}(F_1(w_1) - F_1(s^*) - \Gamma M^{-1}(w_1 - s^*))\| \\ &\leq \psi(\|w_1 - s^*\|)\|w_1 - s^*\| < \|w_1 - s^*\| \end{aligned} \tag{2.12}$$

by the choice of r. Hence, we conclude that $w_1 = s^*$, since (2.12) contradicts the hypothesis $w_1 \neq s^*$. □

Remark 2.2. *Comments similar to the ones in Remark 2.4 can be made provided that* x_0, ψ *are exchanged by* s^*, ψ*, respectively. It is also worth noting that the condition* (H_4) *does not necessarily imply the usual condition in local convergence studies that the operator* $F_1'(s^*)$ *is invertible, i.e. that* s^* *is a simple solution of the equation* $F_1(x) = 0$*. Consequently the method (1.5) can be applied to find solutions of multiplicity greater than one.*

3. Convergence for the Method (1.10)

Let $\lambda > 0, \mu > 0, \delta \geq 0$ and $\beta > 0$ be given parameters.

Suppose:

(C_1) There exists a function

$$\varphi_1 : [0, \lambda] \times [0, \lambda] \times [0, \lambda] \longrightarrow [0, \delta]$$

which is continuous and nondecreasing. Define the scalar sequence $\{h_n\}$ for $h_0 = 0$, some $h_1 \geq 0, \beta_1 > \beta$ and each $n = 1, 2, \ldots$ by

$$h_{n+1} = h_n + \beta_1 \varphi_1(h_{n-1}, h_n, h_n - h_{n-1})(h_n - h_{n-1}). \tag{3.1}$$

The scalar sequence $\{h_n\}$ is shown to be majorizing for $\{x_n\}$ is generated by the formula in the Theorem 3.2. However, let us present a convergence criterion for it.

(C_2) There exists $\lambda_0 \in [0, j]$ such that for each $n = 0, 1, 2, \ldots$ $h_n \leq \lambda_0$.

It follows by this condition and (3.1) that $0 \leq h_{n-1} \leq h_n \leq \lambda_0$ and there exists $h^* \in [0, \lambda_0]$ such that $lim_{n \to \infty} = h^*$.

The limit point h^* is the unique least upper bound of the sequence $\{h_n\}$.

(C_3) There exists $x_0 \in X$ and $y_0 \in F_1(x_0) + F_2(x_0)$ such that

$\beta\delta < 1$ and $\|y_0\| \leq (1 - \beta\delta)\ min\{\frac{\lambda}{\beta}, \mu\}$.

Choose $h_1 \leq \beta_1 \|y_0\|$.

(C_4) The operator

$$x \longrightarrow Q_{D_n}(x) := F_1(x_0) + D_n(x - x_0) + F_2(x) \tag{3.2}$$

is metrically regular at x_0 for y_0 with constant β and neighborhoods $S(x_0, \lambda)$ and $S(y_0, \mu)$, respectively.

The mapping φ_1 relates to the operators on the method (1.10).

(C_5)

$$\|F_1(x) - F_1(x_n) - D_n(x - x_n)\| \leq \varphi_1(\|x - x_0\|, \|x_n - x_0\|, \|x - x_n\|)\|x - x_n\|,$$

for each $x \in S(x_0, \lambda)$.

Next, the semi-local analysis of convergence is developed using the conditions $(C_1) - (C_5)$.

Theorem 3.1. *Suppose that the conditions* $(C_1)-(C_5)$ *hold. Then, for each* $\gamma \in (\beta\delta, 1)$ *there exists a sequence* $\{x_n\}$ *generated by the method (1.10) which is well defined in* $S(x_0, \lambda)$*, remains in* $S(x_0, \lambda)$ *for each* $n = 0, 1, 2, \dots$ *and convergence to a solution* $s^* \in S[x_0, \lambda]$ *of the generated equation (1.8). Moreover, the following assertion hold*

$$\|s^* - x_n\| \leq \gamma^n \lambda \tag{3.3}$$

and

$$dist(0, F_1(x_n) + F_2(x_n)) \leq \gamma^n \|y_0\| \tag{3.4}$$

for each $n = 0, 1, 2, \dots$

Thus, the convergence rate is R-linear. Furthermore, If the operator Q_{D_n} is strongly metrically regular with constant β and neighbourhoods $S(x_0, \lambda)$ and $S(y_0, \mu)$, respectively, then the sequence $\{x_n\}$ is the only one satisfying (1.10), and staying in $S(x_0, \lambda)$.

Remark 3.1. *The proof is similar to the one in [11, Theorem 2.2]. But it uses* (1.10), (C_4) *and* (C_5), *instead of stronger (1.9),*

$(C_4)'$ $x \longrightarrow Q_{A_n}(x) := F_1(x_0) + A_n(x - x_0) + F_2(x)$,

$(C_5)'$ $\|F_1(x) - F_1(x_n) - L_n(x - x_n)\| \leq w(\|x - x_n\|)\|x - x_n\|$

for each $x \in S(x_0, \lambda)$*, where* $w : [0.\lambda] \longrightarrow [0, \delta]$ *satisfies* $\lim_{n \to +\infty} w(t) = 0$

Proof. Pick $\gamma \in (\beta\delta, 1)$ and β_1 so that $\beta < \beta_1 \leq \frac{\gamma}{\delta}$ and

$$\|y_0\| \leq (1 - \gamma) min\left\{\frac{\lambda}{\beta_1}, \mu\right\}. \tag{3.5}$$

The choice of β_1 is certainly possible since there are infinitely many numbers between β and $\frac{\gamma}{\delta}$. We shall show the existence of the sequence $\{x_n\}$ using mathematical induction for $n = 1, 2, \dots$ satisfying

(I_n) $\|x_n - x_0\| \leq \frac{1-\gamma^n}{1-\gamma}\beta_1\|y_0\| \leq (1 - \gamma^n)\lambda < \lambda$.

(II_n) $\|x_n - x_{n-1}\| \leq h_n - h_{n-1}$.

(III_n) $0 \in F_1(x_{n-1}) + D_{n-1}(x_n - x_{n-1}) + F_2(x_n)$,

where $D_{n-1} = D_{n-1}(x_0, \dots, x_{n-1})$.

By hypothesis $0 \in S(y_0, \mu)$ and $y_0 \in Q_{D_0}(x_0)$. Using the condition (C_4) for Q_{D_0} we get

$$dist(x_0, Q_{D_0}^{-1}(0)) \leq \beta \; dist(0, Q_{D_0}(x_0)) \leq \beta\|y_0\|.$$

If $y_0 = 0$, pick $x_1 = x_0$. Otherwise, It follows that

$$dist(x_0, Q_{D_0}^{-1}(0)) \leq \beta_1\|y_0\|.$$

Thus, there exists $x_1 \in Q_{D_0}^{-1}(0)$ satisfying

$$\|x_1 - x_0\| < \beta_1\|y_0\| < (1 - \gamma)\lambda.$$

So, (I_1), (II_1) and (III_1) hold.

Suppose that for some natural number m the element x_m is defined so that the induction hypothesis $(I_m), (II_m)$ and (III_m) hold. We shall show that the iterate x_{m+1} is defined andd the assertions $(I_{m+1}), (II_{m+1})$ and (III_{m+1}) hold. It follows by (I_m) that $x_m \in S(x_0, \lambda)$. Set $e_m = F_1(x_0) - F_1(x_m) - D_m(x_0 - x_m)$.

By (C_5) for $x = x_0, (I_m), (II_m)$ and the properties of the function φ_1 we can write

$$\begin{aligned}
\|e_m - y_0\| &\leq \|y_0\| + \|F_1(x_0) - F_1(x_m) - D_m(x_0 - x_m)\| \\
&\leq \|y_0\| + \varphi_1(\|x_0 - x_0\|, \|x_m - x_0\|, \|x_m - x_0\|)\|x_0 - x_m\| \\
&\leq \|y_0\| + \varphi_1(0, h_m, h_m)\|x_0 - x_m\| \\
&\leq \|y_0\| + \delta\|x_0 - x_m\| \leq \|y_0\| + \frac{1-\gamma^m}{1-\gamma}\beta\|y_0\| \\
&\leq (1 + \frac{1-\gamma^m}{1-\gamma}\gamma)\|y_0\| = \frac{1-\gamma^{m+1}}{1-\gamma}\|y_0\| < \mu.
\end{aligned}$$

If $e_m \in Q_{D_m}(x_m)$, set $x_{m+1} = x_m$. Otherwise by the condition (C_4) we have

$$\begin{aligned}
dist(x_m, Q_{D_m}^{-1}(e_m)) &\leq \beta\ dist(e_m, Q_{D_m}(x_n)) \\
&\leq \beta_1\ dist(e_m, Q_{D_m}(x_m)).
\end{aligned}$$

Next, there exists an element $x_{m+1} \in Q_{D_m}^{-1}(e_m)$ satisfying

$$\|x_{m+1} - x_m\| \leq \beta_1\ dist(e_m, Q_{D_m}(x_m)).$$

By (III_m) it follows

$$\begin{aligned}
Q_{D_m}(x_m) &= F_1(x_0) + D_m(x_m - x_0) + F_2(x_m) \\
&\ni F_1(x_0) + D_m(x_m - x_0) - F_1(x_{m-1} - D_{m-1}(x_m - x_{m-1}).
\end{aligned}$$

In view of the condition (C_4) with $x = x_m$, (3.5) and (II_m), we obtain in turn that

$$\begin{aligned}
\|x_{m+1} - x_m\| &\leq \beta_1\big\|e_m - \big[F_1(x_0) - F_1(x_{m-1}) + D_m(x_m - x_0) \\
&\quad - D_{m-1}(x_m - x_{m-1})\big]\big\| \\
&= \beta_1\|F_1(x_m) - F_1(x_{m-1}) - D_{m-1}(x_m - x_{m-1})\| \\
&\leq \beta_1\varphi_1(\|x_{m-1} - x_0\|, \|x_m - x_0\|, \|x_m - x_{m-1}\| \\
&\leq \beta_1\varphi_1(h_{m-1}, h_m, h_m - h_{m-1}) = h_{m+1} - h_m,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\|x_{m+1} - x_m\| &\leq \beta_1\delta\|x_m - x_{m-1}\| = \gamma\|x_m - x_{m-1}\| \leq \gamma^m\|x_1 - x_0\| \\
&\leq \gamma^m\beta_1\|y_0\|.
\end{aligned}$$

Thus, the condition (III_m) holds if $m+1$ replaces m. Moreover, by the selection of x_{m+1}, we get

$$e_m \in Q_{D_m}(x_{m+1}) = F_1(x_0) + D_m(x_{m+1-} - x_0) + F_2(x_{m+1}).$$

So, the assertion (III_n) holds if $m+1$ replaces m. Moreover, by (I_m) we get in turn that

$$\begin{aligned} \|x_{m+1} - x_0\| &\leq \|x_{m+1} - x_m\| + \|x_m - x_0\| \\ &\leq \gamma^m \beta_1 \|y_0\| + \frac{1-\gamma^m}{1-\gamma}\beta_1\|y_0\| \\ &= \frac{1-\gamma^{m+1}}{1-\gamma}\beta_1\|y_0\|, \end{aligned}$$

which terminates the induction for (I_m). Then, by (C_2) the sequence is complete as convergent. It follows by (II_m) that the sequence $\{x_m\}$ is also complete in a Banach space X and as such it converges to some s^*. By (I_m) we have $s^* \in S(x_0, \lambda)$. We must show that the limit point s^* solves (1.8).

Set $y_m := F_1(x_m) - F_1(x_{m-1}) - D_{m-1}(x_m - x_{m-1})$. It follows by (III_m) that $y_m \in F_1(x_m) + F_2(x_m)$. By using (II_m) and the condition (C_4) for $x = x_m$, we get in turn that

$$\begin{aligned} \|y_m\| &= \|F_1(x_m) - F_1(x_{m-1}) - D_{m-1}(x_m - x_{m-1})\| \\ &\leq \psi_1(\|x_{m-1} - x_0\|, \|x_m - x_0\|, \|x_m - x_{m-1}\|)\|x_m - x_{m-1}\| \\ &\leq \psi_1(h_{m-1}, h_m, h_m - h_{m-1})(\alpha_m - \alpha_{m-1}) = \alpha_{m+1} - \alpha_m \longrightarrow 0 \end{aligned}$$

as $m \to +\infty$.

Consequently, we deduce that $(x_m, y_m) \to (s^*, 0)$ as $m \to +\infty$.

But F is continuous whereas G has a closed graph. Thus, we conclude $0 \in F_1(s^*) + F_2(s^*)$.

Finally, If Q_{D_m} is a strongly metrically regular operator. It follows that the iterate x_{m+1} is unique and is obtained from x_m. □

Next, we develop the local convergence analysis of the method (1.10).

Suppose :

(C_6) There exists a parameter $\beta > 1$ and a function $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous and nondecreasing such that the equation $\beta_1\varphi_2(t) - 1 = 0$ has a smallest positive solution. Denote such a solution by r_0.

(C_7) There exists a solution $s^* \in X$ of the generalized equation (1.8) and parameters $\beta > 0$ and $\delta \geq 0$ such that $\beta\delta < 1$.

(C_8) The operator $x \longrightarrow T_{D_n}(x) := F_1(s^*) + D_n(x - s^*) + F_2(x)$ is a metrically regular at s^* for 0 with constant β and neighbourhoods $S(s^*, \lambda)$ and $(0, \mu)$ for some $\mu > 0$, respectively.

The function φ_2 relates to the operators on the method (1.10).

(C_9)

$$\|F_1(s^*) - F_1(x_n) - D_n(s^* - x_n)\| \leq \varphi_2(\|s^* - x_n\|)\|s^* - x_n\| \tag{3.7}$$

for $x_0 \in S(s^*, \lambda)$. Next, the local convergence analysis of the method (1.10) is presented using the conditions $(C_6) - (C_4)$.

Theorem 3.2. *Suppose that the conditions $(C_6) - (C_9)$ hold. Then, for each $\gamma \in (\beta\delta, 1)$ the re exists a sequence $\{x_n\}$ generated by the method (1.10) which is well defined in $S(s^*, \lambda)$, remains in $S(s^*, \lambda)$ for each $n = 0, 1, 2, ..$ and converges to s^* so that*

$$\|s^* - x_n\| \leq d\|s^* - x_{n-1}\| \leq d^n\|s^* - x_0\| < \lambda,$$

where $d = \beta_1\varphi_2(\|s^* - x_0)\| \in [0, 1)$. Additionally, If the operator T_{D_n} is strongly metrically regular with constant β and neighbourhoods $S(s^*, \lambda)$ and $S(0, \mu)$, respectively, then the sequence $\{x_n\}$ is the only one satisfying (1.8), and satisfying in (s^*, λ).

Proof. Simply follow the proof of Theorem 3.1 for $\gamma \in (\beta\delta, 1)x_0 = s^*$ and $\beta < \beta_1 \leq \frac{\gamma}{\delta}$ to obtain as in (3.6) but using (C_8) and (3.7) instead of (C_4) and (3.6), respectively to obtain

$$\begin{aligned} \|s^* - x_m\| &\leq \beta_1\varphi_2(\|s^* - x_{m-1}\|)\|s^* - x_{m-1}\| \\ &\leq d\|s^* - x_{m-1}\| \leq ... \leq d^m\|s^* - x_0\| < \lambda. \end{aligned}$$

Therefore, we conclude that $\lim_{m \longrightarrow +\infty} x_m = s^*$ and the iterate $x_m \in S(s^*, \lambda)$.

Finally, if the operator T_{D_m} is strongly metrically regular, it follows that the iterate x_m is unique in $S(s^*, \lambda)$ by the way the iterate x_m is derived from x_{m-1}. □

4. Numerical Examples

The examples use $L_n = F_1'(x_n)$, $\Gamma = I$ which is independent of x_0 and s^*.

Example 4.1. *The solution sought for the nonlinear system*

$$\begin{aligned} f_1 &= x - 0.1\sin x - 0.3\cos y + 0.4 \\ f_2 &= y - 0.2\cos x + 0.1\sin y + 0.3 \end{aligned}$$

Let $F_1 = (f_1, f_2)$. Then, the system becomes

$$F_1(s) = 0 \;\; for \;\; s = (x, y)^T.$$

Then

$$F_1'((x, y)) = \begin{bmatrix} 1 - 0.1\cos(x) & 0.3\sin(y) \\ 0.2\sin(x) & 0.1\cos(y) + 1 \end{bmatrix}.$$

Method (1.2)

$$x_{p+1} = x_p - F_1'(x_p)^{-1}F_1(x_p).$$

Method (1.5), $p = 1$, $\Gamma = I$,

$$M_1(x) = I + (I - F_1'(x)),$$
$$Q_1(x) = x - (I + (I - F_1'(x)))F_1(x), \tag{4.1}$$
$$x_{n+1} = Q_1(x_n).$$

Method (1.5), $p = 2$, $\Gamma = I$,

$$M_2(x) = I + (I - F_1'(x)) + (I - F_1'(x))^2,$$
$$Q_2(x) = x - M_2(x)F_1(x), \tag{4.2}$$
$$x_{n+1} = Q_2(x_n).$$

Method (1.5), $p = 3$, $\Gamma = I$,

$$M_3(x) = I + (I - F_1'(x)) + (I - F_1'(x))^2 + (I - F_1'(x))^3,$$
$$Q_3(x) = x - M_3(x)F_1(x), \tag{4.3}$$
$$x_{n+1} = Q_3(x_n).$$

Method (1.5), $p = 4$, $\Gamma = I$,

$$M_4(x) = I + (I - F_1'(x)) + (I - F_1'(x))^2 + (I - F_1'(x))^3 + (I - F_1'(x))^4,$$
$$Q_4(x) = x - M_4(x)F_1(x), \tag{4.4}$$
$$x_{n+1} = Q_4(x_n).$$

Method (1.5), $p = 5$, $\Gamma = I$,

$$M_5(x) = I + (I - F_1'(x)) + (I - F_1'(x))^2 + (I - F'(x))^3 + (I - F_1'(x))^4 + (I - F_1'(x))^5,$$
$$Q_5(x) = x - M_5(x)F_1(x), \tag{4.5}$$
$$x_{n+1} = Q_5(x_n).$$

Method (1.5), $p = 1, 5$, $\Gamma = F_1'(x_0)$,

$$x_{n+1} = x_n - M\Gamma^{-1}F_1(x_n),$$
$$A = B^{-1}(B - F_1'(x)), \tag{4.6}$$
$$M = I + \sum_{i=1}^{p} A^i.$$

Thus, the comparison shows that the behavior of the method (1.5) is essentially the same as Newton's method (1.2). However, the iterates of the method (1.5) are cheaper to obtain than Newton's. As observed in Table 1 - Table 4, the number of iterations required for the proposed methods with k ranging from 3 to 5 closely aligns with those of Newton's method.

Method	*Iterations*	*Method*	*Iterations*
(1.2) Newton	4	(1.2) Newton	4
(4.1), $p = 1$	6	(4.6), $p = 1$	8
(4.2), $p = 2$	5	(4.6), $p = 2$	6
(4.3), $p = 3$	4	(4.6), $p = 3$	5
(4.4), $p = 4$	4	(4.6), $p = 4$	5
(4.5), $p = 5$	4	(4.6), $p = 5$	4

TABLE 1. The number of iterations to reach error tolerance $\varepsilon = 10^{-9}$ with initial guess $x_0 = (1, 1)$ and $\|I - F_1'(x_0)\| = 0.3129 < 1$.

Method	*Iterations*	*Method*	*Iterations*
(1.2) Newton	3	(1.2) Newton	3
(4.1), $p = 1$	5	(4.6), $p = 1$	3
(4.2), $p = 2$	4	(4.6), $p = 2$	3
(4.3), $p = 3$	3	(4.6), $p = 3$	3
(4.4), $p = 4$	3	(4.6), $p = 4$	3
(4.5), $p = 5$	3	(4.6), $p = 5$	3

TABLE 2. The number of iterations to reach error tolerance $\varepsilon = 10^{-9}$ with initial guess $x_0 = (0, 0)$ and $\|I - F_1'(x_0)\| = 0.1414 < 1$.

Method	*Iterations*	*Method*	*Iterations*
(1.2) Newton	5	(1.2) Newton	5
(4.1), $p = 1$	7	(4.6), $p = 1$	9
(4.2), $p = 2$	5	(4.6), $p = 2$	7
(4.3), $p = 3$	5	(4.6), $p = 3$	6
(4.4), $p = 4$	5	(4.6), $p = 4$	6
(4.5), $p = 5$	5	(4.6), $p = 5$	5

TABLE 3. The number of iterations to reach error tolerance $\varepsilon = 10^{-9}$, where $x_0 = (-15, -15)$ and $\|I - F_1'(x_0)\| = 0.257 < 1$.

Table 5 shows the results of calculations to determine the Computational Order of Convergence (COC) and the Approximated Computational Order of Convergence (ACOC) aiming to compare the convergence order of method (1.5) with the convergence order of Newton's method (1.2)*.*

Definition 4.1. *Computational order of convergence of a sequence* $\{x_j\}_{j\geq 0}$ *is defined by*

Method	Iterations	Method	Iterations
(1.2) Newton	7	(1.2) Newton	7
(4.1), $p = 1$	8	(4.6), $p = 1$	12
(4.2), $p = 2$	7	(4.6), $p = 2$	8
(4.3), $p = 3$	7	(4.6), $p = 3$	8
(4.4), $p = 4$	7	(4.6), $p = 4$	7
(4.5), $p = 5$	7	(4.6), $p = 5$	7

TABLE 4. The number of iterations to reach error tolerance $\varepsilon = 10^{-12}$, where $x_0 = (-15, -15)$ and $\|I - F_1'(x_0)\| = 0.257 < 1$.

$$\overline{\Upsilon}_j = \frac{\ln |e_{j+1}/e_j|}{\ln |e_j/e_{j-1}|},$$

where x_{j-1}, x_j, x_{j+1} are three consecutive iterations near the root α and $e_j = x_j - \alpha$ [6].

Definition 4.2. *The approximated computational order of convergence of a sequence $\{x_j\}_{j\geq 0}$ is defined by*

$$\widehat{\Upsilon}_n = \frac{\ln |\widehat{e}_{j+1}/\widehat{e}_j|}{\ln |\widehat{e}_j/\widehat{e}_{j-1}|},$$

where $\widehat{e}_j = x_j - x_{j-1}$. x_j, x_{j-1}, x_{j-2} are three consecutive iterates [6].

Method	COC	ACOC
(1.2) Newton	1.8624	1.9697
(4.1), $p = 1$	0.863	1
(4.2), $p = 2$	0.2695	1.0438
(4.3), $p = 3$	1.9714	2.3569
(4.4), $p = 4$	1.8354	1.9453
(4.5), $p = 5$	1.8642	1.9661
(4.6), $p = 1$	0.9065	1.0118
(4.6), $p = 2$	0.5912	0.999
(4.6), $p = 3$	0.7321	0.9926
(4.6), $p = 4$	1.933	2.0151
(4.6), $p = 5$	1.8679	1.9578

TABLE 5. The computational Order of Convergence and the Approximated Computational Order of Convergence, where $x_0 = (-15, -15)$, $\varepsilon = 10^{-12}$.

Table 5 shows that the convergence of the proposed methods closely corresponds with the convergence of Newton's method, particularly for values of k ranging from 4 to 5 with the convergence order closely approximating 2.

Example 4.2. *Let $X = Y = \mathbb{R}^3$ and $\Omega = S[s^*, 1]$. The mapping F is defined on Ω for $a = (a_1, a_2, a_3)^{tr} \in \mathbb{R}^3$ as*

$$F_1(a) = (a_1, e^{a_2} - 1, \frac{e-1}{2}a_3^2 + a_3)^{tr}.$$

Then, the definition of the derivative according to Fréchet [22, 25] is given for the mapping F_1

$$F_1'(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{a_2} & 0 \\ 0 & 0 & (e-1)a_3 + 1 \end{bmatrix}.$$

The point $s^* = (0, 0, 0)^{tr}$ solves the equation $F_1(a) = 0$. Moreover, $F_1'(s^*) = I$. The conditions of the Theorem 2.2 hold for $p = 1$, if $\varphi(t) = a(t) = (e-1)t$. Then, we can have $r \in (0, 0.2909883534)$.

Example 4.3. *Let $H[0, 1]$ stand for the space of continuous functions mapping the interval $[0, 1]$ into the real numbers. Let $X = Y = H[0, 1]$ and $\Omega = S[s^*, 1]$ with $s^*(v) = 0$. The operator F_1 is defined on $H[0, 1]$ as*

$$F_1(z)(v) = z(v) - 4\int_0^1 v z(\tau)^3 d\tau.$$

Then, of the derivative according to Fréchet [1, 10, 15, 22, 30] is given below for the operator F_1

$$F_1'(z(w))(v) = w(v) - 12\int_0^1 v\tau z(\tau)^2 w(\tau) d\tau$$

for each $w \in H[0, 1]$. Therefore, the conditions of the Theorem 2.2 hold if, since for $s^* = 0$, $F_1'(x^*(v)) = I$ hold for that $p = 1$, if $\varphi(t) = a(t) = 6t$. Then, again by the definition of r, we can choose $r \in (0, 0.8\bar{3})$.

5. Concluding Remarks

The paper addresses the issue with the inverses appearing in the study of the convergence of simple-step iterative methods. It is shown that the inverse can be replaced by a finite sum of linear operators related to the operator involved. The resulting hybrid methods demonstrate the effectiveness of these methods since the number of iterations is essentially the same as well as the convergence order of the methods. However, the hybrid method is cheaper to implement. This idea can be used for multiple steps and multiple point methods with the same advantages [3, 5–7, 9, 14, 23, 24, 31]. This is the direction of future research.

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