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Eur. J. Math. Anal. 5 (2025) 1
doi: 10.28924/ada/ma.5.1

Stability Results of Positive Weak Solution for a Class of Chemically Reacting Systems

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Abstract. This paper aims to study the existence and non-existence results of positive weak solution to the quasilinear elliptic system:

$$\begin{cases} -\Delta_p u = \lambda a(x)[f(u,v) - \frac{1}{u^\alpha}], & x \in \Omega, \\ -\Delta_q v = \lambda b(x)[g(u,v) - \frac{1}{v^\beta}], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where $\Delta_r w = div(|\nabla w|^{r-2}\nabla w)$ is the r-Laplacian ($r = p, q$), $r > 1$, $\alpha, \beta \in (0,1)$, Ω is a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary $\partial\Omega$ and λ is a positive parameter. Here f, g are C^1 increasing functions such that $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$; $f(\upsilon_1, \upsilon_2) > 0$, $g(\upsilon_1, \upsilon_2) > 0$ for $\upsilon_1, \upsilon_2 > 0$. With C^1 sign-changing functions $a(x)$, $b(x)$ that perhaps have negative values nearby the boundary. We establish our results via the sub-supersolution method. In addition, we study the stability and instability results of positive weak solution with different choices of f and g.

1. Introduction

This paper aims to study the existence and non-existence results of positive weak solution to the quasilinear elliptic system:

$$\begin{cases} -\Delta_p u = \lambda a(x)[f(u,v) - \frac{1}{u^\alpha}], & x \in \Omega, \\ -\Delta_q v = \lambda b(x)[g(u,v) - \frac{1}{v^\beta}], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_r w = div(|\nabla w|^{r-2}\nabla w)$ is the r-Laplacian ($r = p, q$), $r > 1$, $\alpha, \beta \in (0,1)$, Ω is a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary $\partial\Omega$ and λ is a positive parameter. Here f, g are C^1 increasing functions such that $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$; $f(\upsilon_1, \upsilon_2) > 0$, $g(\upsilon_1, \upsilon_2) > 0$ for $\upsilon_1, \upsilon_2 > 0$. With C^1 sign-changing functions $a(x)$, $b(x)$ that perhaps have negative values nearby the boundary.

Received: 17 Jul 2024.

Key words and phrases. Positive weak solution; Sub-Supersolution method; Stability.

General evolutionary problems are defined as follows:

$$\begin{cases} u_t = \eta\Delta_p u + \lambda a(x)[f(u,v) - \frac{1}{u^\alpha}], & x \in \Omega, \\ v_t = \delta\Delta_q v + \lambda b(x)[g(u,v) - \frac{1}{v^\beta}], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{2}$$

have stationary counterpart of systems of singular equations like (1), such that η and δ are positive parameters. System (2) is an inspiration from major applications in chemically reacting systems, where the activator chemical substance's density is denoted by u, while an inhibitor is denoted by v. The slow and fast diffusion of u and v, respectively, are turned into a small η and large δ (see [1]). Furthermore, systems like (1) appear in many contexts in Engineering and Biology. It presents a simple model where u, v denote the density of two diffusing biological species for describing the interaction between these two species.

Recently, similar problems have been discussed in [2–5]. The authors in [6] investigated the positive weak solution of the system:

$$\begin{cases} -\Delta u = \lambda[f(u) - \frac{1}{u^\alpha}], & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where $f \in C^2(\mathbb{R}^+)$, $f' > 0$, $f(0) \geq 0$, $\lim\limits_{\epsilon\to\infty} \frac{f(\epsilon)}{\epsilon} = \infty$ and $\Omega \subset \mathbf{R}^N (N \geq 1)$. When $N = 1$, they used the quadrature method to discuss the multiplicity and uniqueness results, while for $N > 1$ they established their existence results using the sub-supersolution method. In [7], it was discussed the existence of positive weak solution to the non-linear system:

$$\begin{cases} -\Delta_p u = \lambda a(x)[f(v) - \frac{1}{u^\alpha}], & x \in \Omega, \\ -\Delta_q v = \lambda b(x)[g(u) - \frac{1}{v^\beta}], & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{4}$$

where $\Delta_r w = div(|\nabla w|^{r-2}\nabla w)$ is the r-Laplacian ($r = p, q$), $r > 1$. Here f, g are C^1 increasing functions such that $f, g : \mathbb{R}^+ \to \mathbb{R}^+$; $f(\omega) > 0, g(\omega) > 0$ for $\omega > 0$ and $\lim\limits_{\omega\to\infty} \frac{f(Mg(\omega)^{\frac{1}{q-1}})}{\omega^{p-1}} = 0 \quad \forall\ M > 0$. With C^1 sign-changing functions $a(x)$, $b(x)$ that perhaps have negative values nearby the boundary. See [8], where system (4) studied by some authors when $p = q = 2$. Also, we studied in [9] the existence and non-existence results of positive weak solution of (1) in case $p = q = 2$, where f, g are C^1 increasing functions, $\lim\limits_{\omega\to\infty} \frac{f(\omega, Mg(\omega,\omega))}{\omega} = 0 \quad \forall\ M > 0$ and $\lim\limits_{\omega\to\infty} \frac{g(\omega,\omega)}{\omega} = 0$. With C^1 sign-changing functions $a(x), b(x)$ that perhaps have negative values nearby the boundary.

Our first aim of this paper is to study system (1) as an extension of system (4) with C^1 increasing functions f, g satisfying

$$\lim_{\xi\to\infty}\frac{f(\xi, M[g(\xi,\xi)]^{\frac{1}{q-1}})}{\xi^{p-1}}=0 \quad \forall\, M>0, \quad \lim_{\xi\to\infty}\frac{g(\xi,\xi)}{\xi^{q-1}}=0.$$

On the other side, many authors have an interest in studying the stability and instability of positive solution to semiposiotne [10–12], linear [13], semilinear [14–17] and fractional [19, 20] systems, they are used in several applications such as Fluid mechanics, Newtonian fluids, Population dynamics, Reaction-diffusion problems, Glaciology, etc.; see [20–22].

Shivaji and Brown in [11] discussed the stability properties of positive solution for the system:

$$\begin{cases}-\Delta u=\lambda f(u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{5}$$

they proved that every positive solution of (5) is unstable when $f(0)\le 0$ and $f''\ge 0$. See [12], where Tertikas proved the non-monotone case. Maya and Shivaji in [16] overcame the non-monotone case through re-formulating f as a combination of a linear and monotone function. Simon and Karatson gave a direct proof of the result (see [14]). In summary, if $f(0)\ge 0\,(\le 0)$ and $f''<0\,(>0)$, then every positive solution of (5) is stable (unstable). Also in [9], we studied the stability and instability properties of system (1) in case $p=q=2$, under certain conditions such that every weak solution is stable near the boundary; otherwise, it is unstable. In [23], some authors investigated the stability of non-negative weak solution for the nonlinear system:

$$\begin{cases}-\Delta_p u=\lambda f(x,u), & x\in\Omega,\\ Bu=0, & x\in\partial\Omega,\end{cases} \tag{6}$$

where $f:\Omega\times[0,\infty)\to\mathbb{R}$ be a continuous function. They discussed (6) when $f(x,u)=w(x)f(u)$, where $w(x)$ is a continuous weight function. They showed that every positive solution is unstable (stable) if $\frac{f(x,u)}{u^{p-1}}$ is strictly increasing (decreasing) function.

Our second aim of this paper is to extend these results to (1) with different choices of f, g. For further stability and instability results on elliptic systems (see [10, 17, 18, 24–26]).

Let $\lambda_{1,r}>0, r=p,q$, be the principal eigenvalue of the following eigenvalue problem to accurately state our existence results:

$$\begin{cases}-\Delta_r\varphi=\lambda|\varphi|^{r-2}\varphi, & x\in\Omega,\\ \varphi=0, & x\in\partial\Omega,\end{cases} \tag{7}$$

where $\varphi_{1,r}$ be the corresponding eigenfunction satisfying $\varphi_{1,r}(x) > 0$ in Ω with $\|\varphi_{1,r}\|_\infty = 1$. Suppose $\mu, \delta, m > 0$ be such that

$$\frac{r}{s_r}\big(1 - \frac{rs}{s+1}\big)|\nabla\varphi_{1,r}|^r \geq m, \quad x \in \bar{\Omega}_\delta, \tag{8}$$

$$\mu \leq \varphi_{1,r} \leq 1, \quad x \in \Omega - \bar{\Omega}_\delta, \tag{9}$$

for $s = \alpha, \beta$ and $s_r = (s+1)^{r-1}$, where $\bar{\Omega}_\delta := \{x \in \Omega \,|\, d(x, \partial\Omega) \leq \delta\}$. By Hopf's lemma, we find this available since $\varphi_{1,r} = 0$ while $|\nabla\varphi_{1,r}| \neq 0$ on $\partial\Omega$. Furthermore, we suppose $e_r \in W_0^{1,r}(\Omega)$ be the unique solution of the problem:

$$\begin{cases} -\Delta_r e_r = 1, & x \in \Omega, \\ e_r = 0, & x \in \partial\Omega, \end{cases} \tag{10}$$

where $\frac{\partial}{\partial n}$ is the outer normal derivative, $e_r > 0$ in Ω and $\frac{\partial e_r}{\partial n} < 0$ on $\partial\Omega$ (see [27]). To be more specific, we will split our results into two cases:

- Case(**I**): when $x \in \bar{\Omega}_\delta$; assume $a(x)$, $b(x) < 0$ with

$$\underline{a}_0, \bar{a}_0, \underline{b}_0, \bar{b}_0 > 0 : -\underline{a}_0 \leq a(x) \leq -\bar{a}_0, \ -\underline{b}_0 \leq b(x) \leq -\bar{b}_0.$$

- Case(**II**): when $x \in \Omega - \bar{\Omega}_\delta$; assume $a(x), b(x) > 0$ with

$$\underline{a}_1, \bar{a}_1, \underline{b}_1, \bar{b}_1 > 0 : \underline{a}_1 \leq a(x) \leq \bar{a}_1, \ \underline{b}_1 \leq b(x) \leq \bar{b}_1.$$

2. Existence and non-existence results

In this section, the results of the existence and non-existence are established by using the sub-supersolution method.

Definition 2.1. *A pair of non-negative functions (u, v) is called a positive weak solution of (1) such that $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ if they satisfy*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\zeta \, dx = \lambda \int_\Omega a(x)[f(u, v) - \frac{1}{u^\alpha}]\zeta \, dx,$$
$$\int_\Omega |\nabla v|^{q-2}\nabla v \cdot \nabla\zeta \, dx = \lambda \int_\Omega b(x)[g(u, v) - \frac{1}{v^\beta}]\zeta \, dx,$$

$\forall\, \zeta \in W := \{\zeta \in C_0^\infty(\Omega) \,|\, \zeta \geq 0, \, x \in \Omega\}$.

Definition 2.2. *A pair of non-negative functions (ψ_1, ψ_2) and (z_1, z_2) are called a positive weak subsolution and supersolution of (1), respectively, such that $(\psi_1, \psi_2), (z_1, z_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ if they satisfy*

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1 \cdot \nabla\zeta \, dx \leq \lambda \int_\Omega a(x)[f(\psi_1, \psi_2) - \frac{1}{\psi_1^\alpha}]\zeta \, dx,$$
$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2 \cdot \nabla\zeta \, dx \leq \lambda \int_\Omega b(x)[g(\psi_1, \psi_2) - \frac{1}{\psi_2^\beta}]\zeta \, dx,$$

and

$$\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\zeta\,dx \ge \lambda\int_\Omega a(x)[f(z_1,z_2)-\frac{1}{z_1^\alpha}]\zeta\,dx,$$
$$\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla\zeta\,dx \ge \lambda\int_\Omega b(x)[g(z_1,z_2)-\frac{1}{z_2^\beta}]\zeta\,dx,$$

$\forall\,\zeta\in W := \{\zeta\in C_0^\infty(\Omega)\,|\,\zeta\ge 0,\, x\in\Omega\}$.

Now, we state our results as follows:

Lemma 2.1. *(see [2]): Let (ψ_1,ψ_2) and (z_1,z_2) be a subsolution and supersolution of (1), respectively, with $\psi_1\le z_1$ and $\psi_2\le z_2$. Therefore, system (1) has a solution (u,v) with $\psi_1\le u\le z_1$ and $\psi_2\le v\le z_2$.*

Our assumptions are as follows:

(S1) $f,g:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ are C^1 increasing functions such that $f(\upsilon_1,\upsilon_2)>0$, $g(\upsilon_1,\upsilon_2)>0$ for $\upsilon_1,\upsilon_2>0$ and $\lim\limits_{\upsilon_1,\upsilon_2\to\infty} f(\upsilon_1,\upsilon_2)=\lim\limits_{\upsilon_1,\upsilon_2\to\infty} g(\upsilon_1,\upsilon_2)=\infty$,

(S2) $\lim\limits_{\xi\to\infty}\dfrac{f(\xi,M[g(\xi,\xi)]^{\frac{1}{q-1}})}{\xi^{p-1}}=0\quad\forall\,M>0$ and $\lim\limits_{\xi\to\infty}\dfrac{g(\xi,\xi)}{\xi^{q-1}}=0$,

(S3) Let $\epsilon_o>0$ such that:

(i)

$$N=f\left(\frac{\mu\epsilon_o^{\frac{1}{p-1}}}{p_o},\frac{\mu\epsilon_o^{\frac{1}{q-1}}}{q_o}\right)-\left(\frac{p_o}{\mu\epsilon_o^{\frac{1}{p-1}}}\right)^\alpha>0,$$

and

$$M=g\left(\frac{\mu\epsilon_o^{\frac{1}{p-1}}}{p_o},\frac{\mu\epsilon_o^{\frac{1}{q-1}}}{q_o}\right)-\left(\frac{q_o}{\mu\epsilon_o^{\frac{1}{q-1}}}\right)^\beta>0,$$

(ii) $\frac{f(\epsilon_o^{\frac{1}{p-1}},\epsilon_o^{\frac{1}{q-1}})}{m}\le min\left\{\frac{p_o^\alpha\alpha_p}{\lambda_{1,p}\,p\epsilon_o^{\frac{\alpha}{p-1}}},\frac{N\alpha_p\underline{a}_1}{\lambda_{1,p}\,p\underline{a}_0},\frac{q_o^\beta\beta_q\underline{b}_0}{\lambda_{1,q}\,\epsilon_o^{\frac{\beta}{q-1}}\,p\underline{a}_0},\frac{M\beta_q\underline{b}_1}{\lambda_{1,q}\,p\underline{a}_0}\right\}$,

(iii) $\frac{g(\epsilon_o^{\frac{1}{p-1}},\epsilon_o^{\frac{1}{q-1}})}{m}\le min\left\{\frac{q_o^\beta\beta_q}{\lambda_{1,q}\,q\epsilon_o^{\frac{\beta}{q-1}}},\frac{N\alpha_p\underline{a}_1}{\lambda_{1,p}\,q\underline{b}_0},\frac{p_o^\alpha\alpha_p\underline{a}_0}{\lambda_{1,p}\,\epsilon_o^{\frac{\alpha}{p-1}}\,q\underline{b}_0},\frac{M\beta_q\underline{b}_1}{\lambda_{1,q}\,q\underline{b}_0}\right\}$,

with $p_o=\frac{p}{p-1}$, $q_o=\frac{q}{q-1}$, $\alpha_p=(\alpha+1)^{p-1}$ and $\beta_q=(\beta+1)^{q-1}$.

(S4) There exist $f_0,g_0>0$ where $f(\upsilon_1,\upsilon_2)\le f_0\upsilon_1^{\gamma_1}\upsilon_2^{\kappa_1}$ and $g(\upsilon_1,\upsilon_2)\le g_0\upsilon_1^{\kappa_2}\upsilon_2^{\gamma_2}$ such that $\gamma_1,\gamma_2,\kappa_1,\kappa_2$ are positive parameters, $\gamma_1,\gamma_2\in(0,1)$ and $\kappa_2+\gamma_2<min\{1,\frac{1}{\kappa_1}\}$.

To be more specific we consider $\lambda_o(\epsilon_o)$ and $\lambda^o(\epsilon_o)$ by the following

$$\lambda^o=min\left\{\frac{m\epsilon_o}{p\underline{a}_0f(\epsilon_o^{\frac{1}{p-1}},\epsilon_o^{\frac{1}{q-1}})},\frac{m\epsilon_o}{q\underline{b}_0g(\epsilon_o^{\frac{1}{p-1}},\epsilon_o^{\frac{1}{q-1}})}\right\},$$

and

$$\lambda_o=max\left\{\frac{\epsilon_o^{\frac{\alpha+p-1}{p-1}}\lambda_{1,p}}{p_o^\alpha\alpha_p\underline{a}_0},\frac{\epsilon_o^{\frac{\beta+q-1}{q-1}}\lambda_{1,q}}{q_o^\beta\beta_q\underline{b}_0},\frac{\epsilon_o\,\lambda_{1,p}}{N\alpha_p\underline{a}_1},\frac{\epsilon_o\,\lambda_{1,q}}{M\beta_q\underline{b}_1}\right\}.$$

Example 2.1. *Assume* $f(\upsilon_1,\upsilon_2)=[\upsilon_2^{k_1}+(\upsilon_1\upsilon_2)^{l_1}-1]$, $g(\upsilon_1,\upsilon_2)=[\upsilon_1^{k_2}+(\upsilon_1\upsilon_2)^{\frac{l_2}{2}}-1]$ *where* k_1,k_2,l_1,l_2 *are positive parameters. Thus,* f,g *clearly satisfy (S1) and (S2) if* $max\{k_2,l_2\}\frac{k_1}{q-1}<p-1$, $max\{k_2,l_2\}<q-1$ *and* $(max\{k_2,l_2\}\frac{1}{q-1}+1)l_1<p-1$ *such that*

$$\lim_{\xi\to\infty}\frac{f(\xi,M[g(\xi,\xi)]^{\frac{1}{q-1}})}{\xi^{p-1}}=0\quad\forall M>0,\qquad \lim_{\xi\to\infty}\frac{g(\xi,\xi)}{\xi^{q-1}}=0,$$

and $\lim_{\xi\to\infty}g(\xi,\xi)=\infty$. *We can take* $\epsilon_o>0$ *small enough that* f,g *satisfy (S3).*

Remark 2.1. *Note that (S3) implies* $\lambda_o<\lambda^o$.

Here, we can establish our existence results.

Theorem 2.1. *Suppose (S1)-(S3) hold, hence (1) has a positive weak solution for every* $\lambda\in[\lambda_o(\epsilon_o),\lambda^o(\epsilon_o)]$.

Proof. We shall verify that

$$(\psi_1,\psi_2)=\left(\frac{\epsilon_o^{\frac{1}{p-1}}\varphi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o},\frac{\epsilon_o^{\frac{1}{q-1}}\varphi_{1,q}^{\frac{q_o}{\beta+1}}}{q_o}\right)$$

is a positive weak subsolution of (1). Then

$$\nabla\psi_1=\frac{\epsilon_o^{\frac{1}{p-1}}\nabla\varphi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o}=\frac{\epsilon_o^{\frac{1}{p-1}}}{1+\alpha}\varphi_{1,p}^{\frac{p_o-1-\alpha}{\alpha+1}}\nabla\varphi_{1,p},$$

and

$$\begin{aligned}
&\int_\Omega|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\zeta\,dx\\
&=\frac{\epsilon_o}{\alpha_p}\int_\Omega\varphi_{1,p}^{(1-\frac{\alpha p}{\alpha+1})}|\nabla\varphi_{1,p}|^{p-2}\nabla\varphi_{1,p}\cdot\nabla\zeta dx\\
&=\frac{\epsilon_o}{\alpha_p}\int_\Omega|\nabla\varphi_{1,p}|^{p-2}\nabla\varphi_{1,p}\Big[\nabla\big(\varphi_{1,p}^{(1-\frac{\alpha p}{\alpha+1})}\cdot\zeta\big)-\big(1-\frac{\alpha p}{\alpha+1}\big)\varphi_{1,p}^{\frac{-\alpha p}{\alpha+1}}\nabla\varphi_{1,p}\cdot\zeta\Big]dx\\
&=\frac{\epsilon_o}{\alpha_p}\int_\Omega\Big[|\nabla\varphi_{1,p}|^{p-2}\nabla\varphi_{1,p}\nabla\big(\varphi_{1,p}^{(1-\frac{\alpha p}{\alpha+1})}\cdot\zeta\big)-\big(1-\frac{\alpha p}{\alpha+1}\big)\varphi_{1,p}^{\frac{-\alpha p}{\alpha+1}}|\nabla\varphi_{1,p}|^p\cdot\zeta\Big]dx\\
&=\frac{\epsilon_o}{\alpha_p}\int_\Omega\Big[\lambda_{1,p}\varphi_{1,p}^{\frac{p}{\alpha+1}}-\big(1-\frac{\alpha p}{\alpha+1}\big)\varphi_{1,p}^{\frac{-\alpha p}{\alpha+1}}|\nabla\varphi_{1,p}|^p\Big]\zeta\,dx,
\end{aligned}$$

then,

$$\int_\Omega|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\zeta\,dx=\frac{\epsilon_o}{\alpha_p}\int_\Omega\Big[\lambda_{1,p}\varphi_{1,p}^{\frac{p}{\alpha+1}}-\big(1-\frac{\alpha p}{\alpha+1}\big)\varphi_{1,p}^{\frac{-\alpha p}{\alpha+1}}|\nabla\varphi_{1,p}|^p\Big]\zeta\,dx. \tag{11}$$

Similarly,

$$\int_\Omega|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla\zeta\,dx=\frac{\epsilon_o}{\beta_q}\int_\Omega\Big[\lambda_{1,q}\varphi_{1,q}^{\frac{q}{\beta+1}}-\big(1-\frac{\beta q}{\beta+1}\big)\varphi_{1,q}^{\frac{-\beta q}{\beta+1}}|\nabla\varphi_{1,q}|^q\Big]\zeta\,dx.$$

Case(**I**): when $x \in \bar{\Omega}_\delta$. Put $s = \alpha, r = p$ in (8), we have

$$\frac{-p}{\alpha_p}(1 - \frac{\alpha p}{\alpha + 1})|\nabla \varphi_{1,p}|^p \le -m.$$

Hence,

$$\frac{-\epsilon_o \varphi_{1,p}^{\frac{-\alpha p}{\alpha+1}}}{\alpha_p}(1 - \frac{\alpha p}{\alpha + 1})|\nabla \varphi_{1,p}|^p \le \frac{-m\epsilon_o}{p},$$

and since $\lambda \le \lambda^o$, then $\lambda \le \dfrac{m\epsilon_o}{p\underline{a}_0 f(\epsilon_o^{\frac{1}{p-1}}, \epsilon_o^{\frac{1}{q-1}})}$. Hence,

$$\frac{-m\epsilon_o}{p} \le -\lambda \underline{a}_0 f(\epsilon_o^{\frac{1}{p-1}}, \epsilon_o^{\frac{1}{q-1}}) \le -\lambda \underline{a}_0 f\left(\frac{\epsilon_o^{\frac{1}{p-1}} \varphi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o}, \frac{\epsilon_o^{\frac{1}{q-1}} \varphi_{1,q}^{\frac{q_o}{\beta+1}}}{q_o}\right),$$

so,

$$\frac{-\epsilon_o \varphi_{1,p}^{\frac{-\alpha p}{\alpha+1}}}{\alpha_p}(1 - \frac{\alpha p}{\alpha + 1})|\nabla \varphi_{1,p}|^p \le -\lambda \underline{a}_0 f\left(\frac{\epsilon_o^{\frac{1}{p-1}} \varphi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o}, \frac{\epsilon_o^{\frac{1}{q-1}} \varphi_{1,q}^{\frac{q_o}{\beta+1}}}{q_o}\right), \tag{12}$$

and since $\lambda \ge \lambda_o$, then $\lambda \ge \dfrac{\epsilon_o^{\frac{\alpha+p-1}{p-1}} \lambda_{1,p}}{p_o^\alpha \alpha_p \underline{a}_0}$. Hence,

$$\frac{\epsilon_o \lambda_{1,p} \varphi_{1,p}^{\frac{p}{\alpha+1}}}{\alpha_p} \le \frac{\epsilon_o \lambda_{1,p}}{\alpha_p} \le \frac{\lambda \underline{a}_0}{(\frac{\epsilon_o^{\frac{1}{p-1}}}{p_o})^\alpha} \le \frac{\lambda \underline{a}_0}{\left(\frac{\epsilon_o^{\frac{1}{p-1}} \varphi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o}\right)^\alpha}. \tag{13}$$

Using (12) and (13) in (11), we see that

$$\begin{aligned}
\int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla \zeta \, dx &\le \int_\Omega \frac{\lambda \underline{a}_0 \zeta}{\left(\frac{\epsilon_o^{\frac{1}{p-1}} \varphi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o}\right)^\alpha} dx - \int_\Omega \lambda \underline{a}_0 f\left(\frac{\epsilon_o^{\frac{1}{p-1}} \varphi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o}, \frac{\epsilon_o^{\frac{1}{q-1}} \varphi_{1,q}^{\frac{q_o}{\beta+1}}}{q_o}\right) \zeta dx \\
&= -\lambda \int_\Omega \underline{a}_0 [f(\psi_1, \psi_2) - \frac{1}{\psi_1^\alpha}] \zeta \, dx \\
&\le \lambda \int_\Omega a(x) [f(\psi_1, \psi_2) - \frac{1}{\psi_1^\alpha}] \zeta \, dx.
\end{aligned}$$

Case(**II**): when $x \in \Omega - \bar{\Omega}_\delta$; $\mu \le \varphi_{1,p} \le 1$. Since $\lambda \ge \lambda_o$, then

$$\frac{\epsilon_o \lambda_{1,p}}{N \alpha_p \underline{a}_1} \le \lambda.$$

Hence,

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\zeta\, dx = \frac{\epsilon_o}{\alpha_p}\int_\Omega \Big[\lambda_{1,p}\,\phi_{1,p}^{\frac{p}{\alpha+1}} - \big(1-\frac{\alpha p}{\alpha+1}\big)\phi_{1,p}^{\frac{-\alpha p}{\alpha+1}}|\nabla\phi_{1,p}|^p\Big]\zeta dx$$
$$\leq \frac{\epsilon_o}{\alpha_p}\int_\Omega \lambda_{1,p}\,\phi_{1,p}^{\frac{p}{\alpha+1}}\zeta\, dx$$
$$\leq \lambda\int_\Omega \underline{a}_1 N\zeta\, dx$$
$$= \lambda\int_\Omega \underline{a}_1\Big[f\Big(\frac{\mu\epsilon_o^{\frac{1}{p-1}}}{p_o}, \frac{\mu\epsilon_o^{\frac{1}{q-1}}}{q_o}\Big) - \Big(\frac{p_o}{\mu\epsilon_o^{\frac{1}{p-1}}}\Big)^\alpha\Big]\zeta\, dx$$
$$\leq \lambda\int_\Omega \underline{a}_1\Big[f\Big(\frac{\epsilon_o^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o}, \frac{\epsilon_o^{\frac{1}{q-1}}\phi_{1,q}^{\frac{q_o}{\beta+1}}}{q_o}\Big) - \frac{1}{\Big(\frac{\epsilon_o^{\frac{1}{p-1}}\phi_{1,p}^{\frac{p_o}{\alpha+1}}}{p_o}\Big)^\alpha}\Big]\zeta\, dx$$
$$= \lambda\int_\Omega \underline{a}_1[f(\psi_1,\psi_2) - \frac{1}{\psi_1^\alpha}]\zeta\, dx$$
$$\leq \lambda\int_\Omega a(x)[f(\psi_1,\psi_2) - \frac{1}{\psi_1^\alpha}]\zeta\, dx.$$

Similarly, we can get also

$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla\zeta\, dx \leq \lambda\int_\Omega b(x)[g(\psi_1,\psi_2) - \frac{1}{\psi_2^\beta}]\zeta\, dx.$$

Thus, (ψ_1,ψ_2) be a positive weak subsolution of (1).

On the other side, we will construct a positive weak supersolution of (1). Suppose

$$(z_1, z_2) = \big(c\, e_p(x), [\lambda\mu_b g(c\mu_p, c\mu_p)]^{\frac{1}{q-1}} e_q(x)\big)$$

where $\mu_a = \|a(x)\|_\infty$, $\mu_b = \|b(x)\|_\infty$ and $\mu_r = \|e_r(x)\|_\infty$ for $r = p, q$. Now by (S2), we can take c large enough such that

$$c^{p-1} \geq \lambda\mu_a f\big(c\mu_p, [\lambda\mu_b g(c\mu_p, c\mu_p)]^{\frac{1}{q-1}}\mu_q\big),$$

then,

$$\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\zeta\, dx = c^{p-1}\int_\Omega |\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\zeta\, dx$$
$$= c^{p-1}\int_\Omega \zeta\, dx$$
$$\geq \int_\Omega \lambda\mu_a f\big(c\mu_p, [\lambda\mu_b g(c\mu_p, c\mu_p)]^{\frac{1}{q-1}}\mu_q\big)\cdot\zeta\, dx$$
$$\geq \lambda\int_\Omega a(x) f\big(c\, e_p(x), [\lambda\mu_b g(c\mu_p, c\mu_p)]^{\frac{1}{q-1}} e_q(x)\big)\cdot\zeta\, dx$$
$$= \lambda\int_\Omega a(x) f(z_1, z_2)\cdot\zeta\, dx$$

$$\geq \lambda \int_\Omega a(x)[f(z_1, z_2) - \frac{1}{z_1^\alpha}]\zeta \, dx.$$

Also, by (S2) we can take $c\mu_p \geq \mu_q[\lambda \mu_b g(c\mu_p, c\mu_p)]^{\frac{1}{q-1}}$. Then

$$\begin{aligned}\int_\Omega |\nabla z_2|^{q-2}\nabla z_2 \cdot \nabla \zeta \, dx &= \lambda \int_\Omega \mu_b \, g(c\mu_p, c\mu_p)|\nabla e_q|^{q-2}\nabla e_q \cdot \nabla \zeta \, dx \\ &= \lambda \int_\Omega \mu_b \, g(c\mu_p, c\mu_p) \cdot \zeta \, dx \\ &\geq \lambda \int_\Omega b(x) g\big(c \, e_p(x), [\lambda \mu_b g(c\mu_p, c\mu_p)]^{\frac{1}{q-1}} e_q(x)\big) \cdot \zeta \, dx \\ &\geq \lambda \int_\Omega b(x) g(z_1, z_2) \cdot \zeta \, dx \\ &\geq \lambda \int_\Omega b(x)[g(z_1, z_2) - \frac{1}{z_2^\beta}]\zeta \, dx.\end{aligned}$$

Thus, (z_1, z_2) be a positive weak supersolution of (1) for c large with $\psi_1 \leq z_1$ and $\psi_2 \leq z_2$. Thus, there exists a positive weak solution (u, v) of (1) such that $\psi_1 \leq u \leq z_1$ and $\psi_2 \leq v \leq z_2$. □

Theorem 2.2. *Let (S4) holds with $(p-1-\gamma_1)(q-1-\gamma_2) = \kappa_1\kappa_2$ and $p\kappa_1 = q(p-1-\gamma_1)$. Hence (1) has no positive weak solution if $\lambda \in (\lambda_{max}, \lambda_{min})$, where $\lambda_{max} = max\{\frac{\lambda_{1,p}}{-2t}, \frac{\lambda_{1,q}}{-2t}\}$ and $\lambda_{min} = min\{\frac{\lambda_{1,p}}{2s}, \frac{\lambda_{1,q}}{2s}\}$ with $t = min\{f_0\bar{a}_0, g_0\bar{b}_0\}$ and $s = max\{f_0\bar{a}_1, g_0\bar{b}_1\}$.*

Proof. Let (u, v) be a positive weak solution of (1). Proof's idea is that a contradiction will be obtained in the end. Multiplying the 1^{st} and 2^{nd} equation of (1) by u, v, respectively. Applying Young's inequality, so

$$\int_\Omega |\nabla u|^p dx \leq \lambda \int_\Omega f_0 a(x)(\frac{u^p}{\mu_1} + \frac{v^q}{\mu_2})dx, \tag{14}$$

with $\mu_1 = \frac{p}{1+\gamma_1} > 1$ and $\mu_2 = \frac{p}{p-1-\gamma_1} > 1$. Similarly, we have

$$\int_\Omega |\nabla v|^q dx \leq \lambda \int_\Omega g_0 b(x)(\frac{u^p}{\vartheta_1} + \frac{v^q}{\vartheta_2})dx, \tag{15}$$

with $\vartheta_1 = \frac{q}{q-1-\gamma_2} > 1$ and $\vartheta_2 = \frac{q}{1+\gamma_2} > 1$. Note that

$$\lambda_{1,p} \int_\Omega u^p dx \leq \int_\Omega |\nabla u|^p dx, \quad \lambda_{1,q} \int_\Omega v^q dx \leq \int_\Omega |\nabla v|^q dx. \tag{16}$$

Combining (14)-(16), we obtain

$$\lambda_{1,p} \int_\Omega u^p dx + \lambda_{1,q} \int_\Omega v^q dx \leq \lambda\Big[\int_\Omega \Big(\frac{f_0 a(x)}{\mu_1} + \frac{g_0 b(x)}{\vartheta_1}\Big) u^p dx + \int_\Omega \Big(\frac{f_0 a(x)}{\mu_2} + \frac{g_0 b(x)}{\vartheta_2}\Big) v^q dx\Big]. \tag{17}$$

Case(**I**): when $x \in \bar{\Omega}_\delta$; $a(x) \le -\bar{a}_0$, $b(x) \le -\bar{b}_0$, hence

$$(\lambda_{1,p} + 2\lambda t) \int_\Omega u^p dx + (\lambda_{1,q} + 2\lambda t) \int_\Omega v^q dx \le 0, \tag{18}$$

where $t = min\{f_0\bar{a}_0, g_0\bar{b}_0\}$, that is a contradiction when $\lambda > \lambda_{max}$.

Case(**II**): when $x \in \Omega - \bar{\Omega}_\delta$; $a(x) \le \bar{a}_1$, $b(x) \le \bar{b}_1$, hence

$$(\lambda_{1,p} - 2\lambda s) \int_\Omega u^p dx + (\lambda_{1,q} - 2\lambda s) \int_\Omega v^q dx \le 0, \tag{19}$$

where $s = max\{f_0\bar{a}_1, g_0\bar{b}_1\}$, that is a contradiction when $\lambda < \lambda_{min}$. □

3. Stability and instability results

Now, we study the stability and instability results of positive weak solution for (1) with different choices of f and g (see [28, 29]).

Suppose (u, v) be any positive weak solution of (1), hence the linearized system associated with (1) is defined as follows:

$$\begin{cases} -(p-1)div(|\nabla u|^{p-2}\nabla\varphi) - \lambda a(x)\big[\big(f_u + \frac{\alpha}{u^{\alpha+1}}\big)\varphi + f_v\psi\big] = \mu\varphi, & x \in \Omega, \\ -(q-1)div(|\nabla v|^{q-2}\nabla\psi) - \lambda b(x)\big[g_u\varphi + \big(g_v + \frac{\beta}{v^{\beta+1}}\big)\psi\big] = \mu\psi, & x \in \Omega, \\ \varphi = 0 = \psi, & x \in \partial\Omega, \end{cases} \tag{20}$$

where subscripts refer to the partial derivative of f or g (see [30]). Let μ_1 be the first eigenvalue and (φ_1, ψ_1) be the corresponding eigenfunction of (20) such that $\varphi_1, \psi_1 > 0$ in Ω.

Definition 3.1. *We say (u, v) is a stable solution of (1) if all eigenvalues of (20) are strictly positive, which can be implied if the first eigenvalue $\mu_1 > 0$. Otherwise (u, v) is unstable.*

Our assumptions are as follows:

(T1): For $u, v > 0$, the functions f_v, g_u are positive.

(T2): For every $v > 0$, the function $\big(f(u, v) - u^{-\alpha}\big)/u^{p-1}$ is strictly increasing at u.

(T3): For every $u > 0$, the function $\big(g(u, v) - v^{-\beta}\big)/v^{q-1}$ is strictly increasing at v.

Theorem 3.1. *Suppose that (T1)-(T3) are satisfied, hence every positive weak solution of (1) is stable in $\bar{\Omega}_\delta$ and unstable in $\Omega - \bar{\Omega}_\delta$.*

Proof. Let (u_o, v_o) be any positive weak solution of (1). Multiplying the 1^{st} and 2^{nd} equation of (1) by $(p-1)\varphi_1$, $(q-1)\psi_1$, respectively and integrating over Ω, so

$$-(p-1)\int_\Omega \varphi_1(x)div(|\nabla u_o|^{p-2}\nabla u_o)dx = (p-1)\lambda \int_\Omega \varphi_1(x)a(x)\big[f(u_o, v_o) - \frac{1}{{u_o}^\alpha}\big]dx, \tag{21}$$

and

$$-(q-1)\int_{\Omega}\psi_1(x)div(|\nabla v_o|^{q-2}\nabla v_o)dx=(q-1)\lambda\int_{\Omega}\psi_1(x)b(x)\big[g(u_o,v_o)-\frac{1}{{v_o}^{\beta}}\big]dx. \quad (22)$$

Similarly, multiplying the 1^{st} and 2^{nd} equation of (20) by $-u_o$, $-v_o$, respectively and integrating over Ω, so

$$\begin{aligned}-\mu_1\int_{\Omega}u_o\varphi_1(x)dx=&(p-1)\int_{\Omega}u_o\,div(|\nabla u_o|^{p-2}\nabla\varphi_1)dx\\&+\lambda\int_{\Omega}\varphi_1(x)a(x)\big[u_of_u+\frac{\alpha}{{u_o}^{\alpha}}\big]dx\\&+\lambda\int_{\Omega}\psi_1(x)a(x)f_vu_o\,dx,\end{aligned} \quad (23)$$

and

$$\begin{aligned}-\mu_1\int_{\Omega}v_o\psi_1(x)dx=&(q-1)\int_{\Omega}v_o\,div(|\nabla v_o|^{q-2}\nabla\psi_1)dx\\&+\lambda\int_{\Omega}\varphi_1(x)b(x)\big[v_og_v+\frac{\beta}{{v_o}^{\beta}}\big]dx\\&+\lambda\int_{\Omega}\psi_1(x)b(x)g_uv_o\,dx.\end{aligned} \quad (24)$$

Combining (21) to (24), we get

$$\begin{aligned}&-(p-1)\int_{\Omega}\big[\varphi_1(x)div(|\nabla u_o|^{p-2}\nabla u_o)-u_o\,div(|\nabla u_o|^{p-2}\nabla\varphi_1)\big]dx\\&-(q-1)\int_{\Omega}\big[\psi_1(x)div(|\nabla v_o|^{q-2}\nabla v_o)-v_o\,div(|\nabla v_o|^{q-2}\nabla\psi_1)\big]dx\\&+\lambda\int_{\Omega}\varphi_1(x)a(x)\big[u_of_u+\frac{\alpha}{{u_o}^{\alpha}}\big]dx+\lambda\int_{\Omega}\psi_1(x)b(x)\big[v_og_v+\frac{\beta}{{v_o}^{\beta}}\big]dx\\&-(p-1)\lambda\int_{\Omega}\varphi_1(x)a(x)\big[f(u_o,v_o)-\frac{1}{{u_o}^{\alpha}}\big]dx\\&-(q-1)\lambda\int_{\Omega}\psi_1(x)b(x)[g(u_o,v_o)-\frac{1}{{v_o}^{\beta}}]dx\\&+\lambda\int_{\Omega}a(x)\psi_1(x)f_vu_o\,dx+\lambda\int_{\Omega}b(x)\varphi_1(x)g_uv_o\,dx\\&=-\mu_1\int_{\Omega}[u_o\varphi_1(x)+v_o\psi_1(x)]dx.\end{aligned} \quad (25)$$

Using Green's first identity, then

$$\int_{\Omega}u_o\,div(|\nabla u_o|^{p-2}\nabla\varphi_1)dx=\int_{\Omega}\varphi_1(x)div(|\nabla u_o|^{p-2}\nabla u_o)dx, \quad (26)$$

and

$$\int_{\Omega}v_o\,div(|\nabla v_o|^{q-2}\nabla\psi_1)dx=\int_{\Omega}\psi_1(x)div(|\nabla v_o|^{q-2}\nabla v_o)dx. \quad (27)$$

By using (26) and (27) in (25), then

$$\begin{aligned}
&\lambda \int_\Omega \varphi_1(x)a(x)\left[u_o f_u-(p-1)f(u_o,v_o)+\frac{\alpha+p-1}{{u_o}^{\alpha}}\right]dx\\
&+\lambda \int_\Omega \psi_1(x)b(x)\left[v_o g_v-(q-1)g(u_o,v_o)+\frac{\beta+q-1}{{v_o}^{\beta}}\right]dx\\
&+\lambda \int_\Omega \psi_1(x)a(x)f_v u_o dx+\lambda \int_\Omega \varphi_1(x)b(x)g_u v_o dx\\
&=-\mu_1 \int_\Omega [u_o\varphi_1(x)+v_o\psi_1(x)]dx.
\end{aligned} \tag{28}$$

Also, since $\left(f(u_o,v_o)-u_o^{-\alpha}\right)/{u_o}^{p-1}$ is strictly increasing at u_o $\forall v_o>0$, then for $u_o,v_o>0$

$$\frac{u_o f_u-(p-1)f(u_o,v_o)+(\alpha+p-1){u_o}^{-\alpha}}{{u_o}^{p}}>0, \tag{29}$$

and since $\left(g(u_o,v_o)-{v_o}^{-\beta}\right)/{v_o}^{q-1}$ is strictly increasing at v_o $\forall u_o>0$, then for $u_o,v_o>0$

$$\frac{v_o g_v-(q-1)g(u_o,v_o)+(\beta+q-1){v_o}^{-\beta}}{{v_o}^{q}}>0. \tag{30}$$

Case(**I**): when $x\in\bar{\Omega}_\delta$; $a(x)$, $b(x)<0$. Thus substituting (29)-(30) in (28), so

$$-\mu_1 \int_\Omega [u_o\varphi_1(x)+v_o\psi_1(x)]dx<0, \tag{31}$$

then, $\mu_1>0$ and the solution is stable.
Case(**II**): when $x\in\Omega-\bar{\Omega}_\delta$; $a(x)$, $b(x)>0$. Thus substituting (29)-(30) in (28), so

$$-\mu_1 \int_\Omega [u_o\varphi_1(x)+v_o\psi_1(x)]dx>0, \tag{32}$$

then $\mu_1<0$ and the solution is unstable. □

Remark 3.1. *By replacing assumptions (T1)-(T3) with next:*
***(L1)**: For $u,v>0$, the functions f_v, g_u are negative.*
***(L2)**: For every $v>0$, the function $\left(f(u,v)-u^{-\alpha}\right)/u^{p-1}$ is strictly decreasing at u.*
***(L3)**: For every $u>0$, the function $\left(g(u,v)-v^{-\beta}\right)/v^{q-1}$ is strictly decreasing at v.*

We deduce the following:

Corollary 3.1. *Suppose that (L1)-(L3) are satisfied, hence every positive weak solution of (1) is unstable in $\bar{\Omega}_\delta$ and stable in $\Omega-\bar{\Omega}_\delta$.*

Proof. In the same way that Theorem 3.1 is proved, the proof procedure is similar. □

Remark 3.2. *As shown in the preceding theorem and corollary, the stability results of positive weak solution are dependent on the domain, in addition to the provided assumptions.*

Remark 3.3. *If $p=q=2$ in system (1), we get the results of the system which have been studied in [9].*

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