Eur. J. Math. Anal. 4 (2024) 22
doi: 10.28924/ada/ma.4.22

Boundedness of Some Commutators in Total Fofana Spaces

Pokou Nagacy

Laboratoire des Sciences et Technologies de l'Environnement, UFR Environnement, Université Jean Lorougnon GUEDE, BP 150 Daloa, Côte d'Ivoire

pokounagacy@yahoo.com

Abstract. In this paper, we find necessary and sufficient conditions for the boundedness of the commutator of the Hardy-Littlewood maximal operator in total Fofana spaces. We also give in these spaces the boundedness of some sublinear operators and their commutators.

1. Introduction and main results

Let d be a fixed positive integer and $\mathbb{R}^d$ the d-dimensional Euclidean space. The estimation of commutators plays an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance [3, 18, 20]).

We denote by $L^0(\mathbb{R}^d)$ the complex vector space of equivalent classes (modulo equality Lebesgue almost everywhere) of Lebesgue measurable complex-valued functions on $\mathbb{R}^d$. For $f \in L^0(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, dy, \tag{1.1}$$

where $|B(x,r)|$ is the Lebesgue measure of the ball

$$B(x,r) = \{y \in \mathbb{R}^d \,:\, |x-y| < r\}.$$

The maximal commutator M_b generated by the maximal operator M and a locally integrable function b is defined by

$$M_b(f)(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)||f(y)| dy.$$

Furthermore the commutator generated by the operator M and a suitable function b is defined by

$$[b, M]f(x) = b(x)M(f)(x) - M(bf)(x).$$

Recall that the operators M_b and $[b, M]$ essentially differ from each other since M_b is positive and sublinear and $[b, M]$ is neither positive nor sublinear. The operators M, $[b, M]$ and M_b play

Received: 28 Jul 2024.

Key words and phrases. Total Fofana spaces; maximal operator; commutator; sublinear operators; BMO spaces.

an important role in real and harmonic analysis and applications (see for example [1, 2, 10, 15, 16] and the references therein). The boundedness of these operators on Lebesgue spaces has been extended to several other spaces. For example, on Morrey spaces ([4], [11]), modified Morrey spaces ([14]), total Morrey spaces ([13]), Fofana spaces ([6]) to name but a few.

Recently, P. Nagacy and B. A. Kpata [17] introduced the total Fofana spaces and established in these spaces, under certain conditions, the boundedness of the Hardy-Littlewood maximal operators, the Riesz potential and fractional maximal operators.

The main purpose of the present paper is to establish the boundedness of the commutators of the Hardy-Littlewood maximal operator and some sublinear operators in total Fofana spaces. Before stating our main results, let us start with some notations and basic definitions. Let $1 \leq q,\ p \leq \infty$ and $r > 0$. For $f \in L^0(\mathbb{R}^d)$ we define

$$ {}_r\left\|f\right\|_{q,p} := \left\|\left[\int_{\mathbb{R}^d} |f\chi_{B(y,r)}|^q(x)dx\right]^{\frac{1}{q}}\right\|_p $$

with the $L^p(\mathbb{R}^d)$-norm taken with respect to the variable y. We adopt the usual convention $\frac{1}{\infty} = 0$.

In 1988, Fofana [7] introduced the functions spaces $(L^q, L^p)^\alpha(\mathbb{R}^d)$, $1 \leq q \leq \alpha \leq p \leq \infty$, which consists of the set of all functions $f \in L^0(\mathbb{R}^d)$ satisfying $\|f\|_{q,p,\alpha} < \infty$, where

$$ \|f\|_{q,p,\alpha} = \sup_{r>0} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}\ {}_r\|f\|_{q,p}\,. $$

It is proved in [7] the following properties:

• for $1 \leq q < \alpha$ fixed and p going from α to ∞, the spaces $(L^q, L^p)^\alpha(\mathbb{R}^d)$ form a chain of distinct Banach spaces beginning with the Lebesgue space $L^\alpha(\mathbb{R}^d)$ and ending by the classical Morrey space $L^{q,d(1-\frac{q}{\alpha})}(\mathbb{R}^d) = (L^q, L^\infty)^\alpha(\mathbb{R}^d)$;

• there exists a constant $C > 0$ such that

$$ \|f\|_{q,p,\alpha} \leq C\,\|f\|_\alpha\,. \tag{1.2} $$

For an in-depth study of Fofana spaces, please consult the following references (see [6, 8, 9]).

Let $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$ and $[r]_1 = \min\{1, r\},\ r > 0$. The total Fofana spaces $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ are defined by

$$ (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d) = \left\{\ f \in L^0(\mathbb{R}^d): \quad \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} < \infty\ \right\} $$

where

$$ \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} = \sup_{r>0}[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|f\|_{q,p}\,. $$

Note that

(1) the space $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is a complex vector subspace of $L^0(\mathbb{R}^d)$.

(2) from the definition of $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ spaces we deduce that the map $L^0(\mathbb{R}^d) \ni f \mapsto \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}$ defines a norm on $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.

(3) for $1 \leq q \leq \alpha, \lambda < \infty$, the space $(L^q, L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)$ is the total Morrey space $L^{q,d(1-\frac{q}{\alpha}),d(1-\frac{q}{\lambda})}(\mathbb{R}^d)$ defined in [13] and

$$L^{q,d(1-\frac{q}{\alpha}),d(1-\frac{q}{\lambda})}(\mathbb{R}^d) = L^{q,d(1-\frac{q}{\alpha})}(\mathbb{R}^d) \cap L^{q,d(1-\frac{q}{\lambda})}(\mathbb{R}^d)$$

with $\lambda \leq \alpha$.

We define the space $BMO(\mathbb{R}^d)$ as the set of all locally integrable functions b with finite norm

$$\|b\|_{BMO(\mathbb{R}^d)} = \sup_{r>0, x\in\mathbb{R}^d} |B(x,r)|^{-1} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy$$

where $b_{B(x,r)} = |B(x,r)|^{-1} \int_{B(x,r)} b(y)dy$.

For a function b defined on $\mathbb{R}^d$, we denote by

$$b^-(x) := \begin{cases} 0 & \text{if } b(x) \geq 0 \\ |b(x)| & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Obviously $b^+(x) - b^-(x) = b(x)$.

Our first result reads as follows

Theorem 1.1. *Let $1 < q \leq \lambda \leq \alpha < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Then the operator $[b, M]$ is bounded on $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ if and only if $b \in BMO(\mathbb{R}^d)$ such that $b^- \in L^\infty(\mathbb{R}^d)$.*

The last two theorems concern the sublinear operators $\mathcal{T}$ satisfying the condition

$$|\mathcal{T}(x)| \leq C \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^d} dy \quad x \notin supp\, f, \tag{1.3}$$

for any $f \in L^1(\mathbb{R}^d)$ with compact support. We point out that the condition (1.3) was first introduced by Soria and Weiss [19]. This condition is satisfied by many operators such as the Hardy-Littewood maximal operator, Calderón-Zygmund singular integral operators, Bochner-Riesz operators at the critical index, C. Fefferman's singular multiplier. It is proved in [5] that $\mathcal{T}$ is bounded on Morrey spaces. It is also bounded on classical Fofana spaces (see [6]). In the setting of total Fofana spaces, we have the following result holds true.

Theorem 1.2. *Let $1 < q \leq \lambda, \alpha < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. If $\mathcal{T}$ is sublinear operator with is bounded on L^q and satisfies the condition (1.3) then $\mathcal{T}$ is also bounded on $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.*

If $\mathcal{T}$ is a linear operator and $b \in BMO(\mathbb{R}^d)$, we define the linear commutator $[b, \mathcal{T}]$ by

$$[b, \mathcal{T}]f(x) = \mathcal{T}(bf)(x) - b(x)\mathcal{T}(f)(x) \quad x \in \mathbb{R}^d,$$

with locally integrable functions f on $\mathbb{R}^d$. It is also proved in [6] that $[b, \mathcal{T}]$ is bounded on classical Fofana spaces and bounded on Morrey spaces in [5]. The next result shows the boundedness on total Fofana spaces of $[b, \mathcal{T}]$.

Theorem 1.3. *Let $1 < q \leq \lambda, \alpha < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and $b \in BMO(\mathbb{R}^d)$. If a linear operator $\mathcal{T}$ satisfies (1.3) and $[b, \mathcal{T}]$ is bounded on L^q, then $\mathcal{T}$ is also bounded on $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.*

The remainder of this note is organized as follows: In Section 2 we recall some properties of total Fofana spaces. Section 3 is devoted to the proofs of Theorem 1.1 and section 4 deals with the proofs of Theorem 1.2 and Theorem 1.3.

The letter C will be used for positive constants not depending on the relevant variables, and theses constants may change from one occurrence to another. We propose the following abbreviation $A \lesssim B$ for the inequalities $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

2. Some properties of total Fofana spaces

The results of this section are proved in [17].
The following result examines the relationship between total Fofana spaces and Fofana spaces.

Proposition 2.1. *Let $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$. Then*

$$(L^q, L^p)^{\alpha}(\mathbb{R}^d) \cap (L^q, L^p)^{\lambda}(\mathbb{R}^d) \hookrightarrow (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$$

and for $f \in (L^q, L^p)^{\alpha}(\mathbb{R}^d) \cap (L^q, L^p)^{\lambda}(\mathbb{R}^d)$

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \leq \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.$$

Proposition 2.2. *Let $1 \leq q \leq \lambda \leq \alpha \leq p \leq \infty$. Then*

$$(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d) = (L^q, L^p)^{\alpha}(\mathbb{R}^d) \cap (L^q, L^p)^{\lambda}(\mathbb{R}^d)$$

and for $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} = \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.$$

Total Fofana spaces are generalizations of classical Fofana spaces since Proposition 2.2 asserts that $(L^q, L^p)^{\alpha,\alpha}(\mathbb{R}^d) = (L^q, L^p)^{\alpha}(\mathbb{R}^d)$.
The family of spaces $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is increasing with respect to the p power. More precisely, we have the following.

Proposition 2.3. *Let $1 \leq q \leq \alpha, \lambda \leq p_1 \leq p_2 \leq \infty$. Then:*

$$\|f\|_{(L^q,L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d)}, \qquad f \in L^0(\mathbb{R}^d)$$

and consequently, $(L^q, L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d) \subset (L^q, L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d)$.

The following result states the boundedness property of M (the Hardy-Littlewood maximal operator) on total Fofana spaces.

Theorem 2.4. *(1) Let $1 < q \leq \alpha, \lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. Then*

$$\|Mf\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

(2) Let $q = 1 < \alpha, \lambda < p < \infty$. Then

$$\|Mf\|_{(L^{1,\infty},L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^1,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^1, L^p)^{\alpha,\lambda}(\mathbb{R}^d),$$

where

$$\begin{aligned} &\|f\|_{(L^{1,\infty},L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \\ := \quad &\sup_{r>0}[r]_1^{d(\frac{1}{\alpha}-1-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+1+\frac{1}{p})}\left[\int_{\mathbb{R}^d}\left(\left\|f\chi_{B(y,r)}\right\|_{1,\infty}^*\right)^p dy\right]^{\frac{1}{p}} \end{aligned}$$

with

$$\left\|f\chi_{B(y,r)}\right\|_{1,\infty}^* = \sup_{r>0} r\left|\{x \in B(y,r) : |f(x)| > r\}\right|.$$

3. Proof of Theorem 1.1

For the proof of this theorem, we need some results.
The following result (see [1, Corollary 1.11]) will be useful in the proof of Theorem 1.1.

Lemma 3.1. *If $b \in BMO(\mathbb{R}^d)$, then there exists a positive constant C such that*

$$M_b f(x) \leq C\|b\|_{BMO(\mathbb{R}^d)} M(Mf)(x)$$

for almost every $x \in \mathbb{R}^d$ and any locally integrable functions f on $\mathbb{R}^d$.

Proposition 3.2. *Let $1 < q \leq \alpha, \lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and $b \in BMO(\mathbb{R}^d)$. Then M_b is bounded on $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.*

Proof. Let $1 < q \leq \alpha, \lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$, $b \in BMO(\mathbb{R}^d)$ and $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.
By taking the $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$-norm of both sides of the estimate in Lemma 3.1, we obtain

$$\|M_b f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|b\|_{BMO(\mathbb{R}^d)}\|M(Mf)\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

According to the first point of Theorem 2.4, we have

$$\|M(Mf)\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|Mf\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

We deduce that

$$\|M_b f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|b\|_{BMO(\mathbb{R}^d)}\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \tag{3.1}$$

□

Lemma 3.3. *Let $1 \leq q \leq \lambda \leq \alpha < \infty$ and $r > 0$. Then*

$$r^{-\frac{d}{q}}[r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}\max\{r^{\frac{d}{\alpha}}, r^{\frac{d}{\lambda}}\} \leq 2.$$

Proof. Let $1 \leq q \leq \lambda \leq \alpha < \infty$ and $r > 0$.
Put $C(r) = r^{\frac{-d}{q}} [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q})} \max\{r^{\frac{d}{\alpha}}, r^{\frac{d}{\lambda}}\}$.

$$\begin{aligned} C(r) &\leq \begin{cases} r^{-\frac{d}{\alpha}}(r^{\frac{d}{\alpha}} + r^{\frac{d}{\lambda}}), & 0 < r \leq 1 \\ r^{-\frac{d}{\lambda}}(r^{\frac{d}{\alpha}} + r^{\frac{d}{\lambda}}), & r > 1 \end{cases} \\ &\leq \begin{cases} 1 + r^{d(-\frac{1}{\alpha}+\frac{1}{\lambda})}, & 0 < r \leq 1 \\ 1 + r^{d(\frac{1}{\alpha}-\frac{1}{\lambda})}, & r > 1. \end{cases} \end{aligned}$$

Thus, $C(r) \leq 2$ for all $r > 0$. □

Proof of Theorem 1.1. Let $1 < q \leq \lambda \leq \alpha < p < \infty$ such that
$\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$.
(1) Assume that $b \in BMO(\mathbb{R}^d)$ such that $b^- \in L^\infty(\mathbb{R}^d)$ and $f \in (L^q, L^p)^{\alpha,\lambda}(\mu)$.
Proceeding as in the proof of Theorem 4 in [13], we have

$$\begin{aligned} \|[b, M]f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} &\leq \left\|M_b f + 2b^- Mf\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \\ &\leq \|M_b f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \\ &\quad + 2\left\|b^-\right\|_\infty \|Mf\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \end{aligned}$$

From (3.1) and the first point of Theorem 2.4, we deduce that

$$\|[b, M]f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \left(\|b\|_{BMO(\mathbb{R}^d)} + \left\|b^-\right\|_\infty\right) \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

(2) Conversely, assume that $[b, M]$ is bounded on $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.
Let $t > 0$ and $x \in \mathbb{R}^d$. Put $B = B(x, t)$. Denote by $M_B f$ the local maximal function of f defined by:

$$M_B f(x) = \sup_{B' \ni x: B' \subset B} |B'|^{-1} \int_{B'} |f(y)|\, dy.$$

Since $\chi_B \in L^\alpha(\mathbb{R}^d) \cap L^\lambda(\mathbb{R}^d)$, it follows from (1.2) that
$\chi_B \in (L^q, L^p)^\alpha(\mathbb{R}^d) \cap (L^q, L^p)^\lambda(\mathbb{R}^d)$. From Proposition 2.2, we deduce that $\chi_B \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.
Therefore, there exists a constant $C > 0$ such that

$$\|[b, M]\chi_B\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \leq C \|\chi_B\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

We also have

$$\begin{aligned} |M_B(b) - b\chi_B| &= |M(b\chi_B)\chi_B - bM(\chi_B)\chi_B| \\ &\leq |M(b\chi_B) - bM(\chi_B)| = |[b, M]\chi_B|. \end{aligned}$$

Applying Hölder's inequality, Proposition 2.3 and Proposition 2.2, we get

$$\begin{aligned} & |B|^{-1}\int_B |b(z)-M_B(b)(z)|dz \\ \leq & \left(|B|^{-1}\int_B |b(z)-M_B(b)(z)|^q dz\right)^{\frac{1}{q}} \\ \leq & |B|^{-\frac{1}{q}}\left(\int_B |[b,M]\chi_B(z)|^q dz\right)^{\frac{1}{q}} \\ \lesssim & t^{-\frac{d}{q}}[t]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})}[1/t]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}\|[b,M]\chi_B\|_{(L^q,L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)} \\ \lesssim & t^{-\frac{d}{q}}[t]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})}[1/t]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}\|[b,M]\chi_B\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \\ \lesssim & t^{-\frac{d}{q}}[t]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})}[1/t]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}\|\chi_B\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \\ \lesssim & t^{-\frac{d}{q}}[t]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})}[1/t]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}\max\{\|\chi_B\|_{q,p,\alpha},\|\chi_B\|_{q,p,\lambda}\}. \end{aligned}$$

It follows from (1.2) that

$$|B|^{-1}\int_B |b(z)-M_B(b)(z)|dz \lesssim t^{-\frac{d}{q}}[t]_1^{d(-\frac{1}{\alpha}+\frac{1}{q})}[1/t]_1^{d(\frac{1}{\lambda}-\frac{1}{q})}\max\{t^{\frac{d}{\alpha}},t^{\frac{d}{\lambda}}\}.$$

So, by Lemma 3.3, we obtain

$$|B|^{-1}\int_B |b(z)-M_B(b)(z)|dz \lesssim 2.$$

Denote by

$$E := \{y\in B : b(y)\leq b_B\},\quad F := \{y\in B : b(y)>b_B\}.$$

Since

$$\int_E |b(z)-b_B|dz = \int_F |b(z)-b_B|dz,$$

in view of the inequality $b(x)\leq b_B\leq M_B(b)$, for $x\in E$, we get

$$\begin{aligned} |B|^{-1}\int_B |b(z)-b_B|dz &= 2|B|^{-1}\int_E |b(z)-b_B|dz \\ &\leq 2|B|^{-1}\int_E |b(z)-M_B(b)(z)|dz \\ &\leq 2|B|^{-1}\int_B |b(z)-M_B(b)(z)|dz \lesssim 4. \end{aligned}$$

By taking in the left hand side the supremum over all $t>0$ and $x\in\mathbb{R}^d$, we obtain $\|b\|_{BMO(\mathbb{R}^d)}<\infty$.

In order to show that $b^-\in L^\infty(\mathbb{R}^d)$, note that $M_B(b)\geq |b|$. Hence

$$0\leq b^- = |b|-b^+ \leq M_B(b)-b^+ \leq M_B(b)-b^+ + b^- = M_B(b)-b.$$

Thus

$$(b^-)_B \lesssim 2,$$

and by the Lebesgue Differentiation Theorem we get

$$b^-(x) \lesssim 2$$

for almost every $x \in \mathbb{R}^d$. □

4. Proof of Theorem 1.2 and Theorem 1.3

We recall that the proofs of Theorem 1.2 and Theorem 1.3 are simply an adaptation of those given in [5] (see also [6]).

Proof of Theorem 1.2. Let $1 < q \leq \lambda, \alpha < p < \infty$ such that $\frac{1}{q}+\frac{2}{p} < \frac{1}{\alpha}+\frac{1}{\lambda}$ and $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$. Fix $y \in \mathbb{R}^d$ and $r > 0$ we have

$$f = f\chi_{B(y,2r)} + \sum_{i=1}^{\infty} f\chi_{B(y,2^{i+1}r)\backslash B(y,2^i r)}.$$

By the sublinearity of $\mathcal{T}$ and the condition (1.3) we obtain

$$|\mathcal{T}f| \lesssim |\mathcal{T}(f\chi_{B(y,2r)})| + \sum_{i=1}^{\infty} |B(y,2^{i+1}r)|^{-1} \int_{B(y,2^{i+1}r)} |f(x)|dx$$

and therefore, an application of Hölder inequality leads to

$$|\mathcal{T}f| \lesssim |\mathcal{T}(f\chi_{B(y,2r)})| + \sum_{i=1}^{\infty} |B(y,2^{i+1}r)|^{-\frac{1}{q}} \|f\chi_{B(y,2^{i+1}r)}\|_q.$$

Taking the L^q-norm of both sides on the ball $B(y,r)$ and using the boundedness of $\mathcal{T}$ on L^q, we get

$$\|(\mathcal{T}f)\chi_{B(y,r)}\|_q \lesssim \|f\chi_{B(y,2r)}\|_q + \sum_{i=1}^{\infty} (2^i)^{-\frac{d}{q}} \|f\chi_{B(y,2^{i+1}r)}\|_q.$$

Taking the L^p-norm of both sides with respect to y, it comes that

$${}_r\|\mathcal{T}f\|_{q,p} \lesssim {}_{2r}\|f\|_{q,p} + \sum_{i=1}^{\infty} (2^i)^{-\frac{d}{q}} {}_{2^{i+1}r}\|f\|_{q,p}.$$

On the one hand, we have,

$$\begin{aligned}
{}_{2r}\|f\|_{q,p} &= \frac{[2r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}} {}_{2r}\|f\|_{q,p} \\
&\leq [2r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/2r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \\
&\leq (2[r]_1)^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} \left(\frac{1}{2}[1/r]_1\right)^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.
\end{aligned}$$

Hence,

$${}_{2r}\|f\|_{q,p} \lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \tag{4.1}$$

On the other hand,

$$\begin{aligned}
&\sum_{i=1}^{\infty}(2^i)^{-\frac{d}{q}}\ {}_{2^{i+1}r}\|f\|_{q,p}\\
=\ &\sum_{i=1}^{\infty}(2^i)^{-\frac{d}{q}}\frac{[2^{i+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{i+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2^{i+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{i+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}\ {}_{2^{i+1}r}\|f\|_{q,p}\\
\leq\ &\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\sum_{i=1}^{\infty}(2^i)^{-\frac{d}{q}}[2^{i+1}r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/2^{i+1}r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\\
\lesssim\ &[r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\sum_{i=1}^{\infty}(2^i)^{d(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}.
\end{aligned}$$

Since $-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p}<0$, $\sum_{i=1}^{\infty}(2^i)^{d(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}<\infty$. Therefore,

$$\begin{aligned}
&\sum_{i=1}^{\infty}(2^i)^{-\frac{d}{q}}\ {}_{2^{i+1}r}\|f\|_{q,p}\\
\lesssim\ &[r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.
\end{aligned}\tag{4.2}$$

From (4.1) and (4.2), we deduce that

$${}_r\|\mathcal{T}f\|_{q,p}\lesssim[r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

It follows that

$$[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\|\mathcal{T}f\|_{q,p}\lesssim\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.\tag{4.3}$$

We obtain the desired result by taking the supremum over all $r>0$ in the left hand side of (4.3). □

Proof of Theorem 1.3. Let $1<q\leq\lambda,\alpha<p<\infty$ such that $\frac{1}{q}+\frac{2}{p}<\frac{1}{\alpha}+\frac{1}{\lambda}$ and $b\in BMO(\mathbb{R}^d)$. Let f be any element of $f\in(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$. We recall that $(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is a subspace of the Morrey space $L^{q,d(1-\frac{q}{\alpha})}(\mathbb{R}^d)$. Proceeding as in the proof of [5, Theorem 2.2], we have that for all $y\in\mathbb{R}^d$ and $r>0$,

$$\begin{aligned}
&\|[b,\mathcal{T}]f\chi_{B(y,r)}\|_q\\
\lesssim\ &\|f\chi_{B(y,2r)}\|_q\\
&+\sum_{i=1}^{\infty}(2^ir)^{-d}\left[\int_{B(y,r)}\left(\int_{B(y,2^{i+1}r)}|b(x)-b(z)||f(x)|dx\right)^q dz\right]^{\frac{1}{p}}.
\end{aligned}$$

Therefore, using the John-Nirenberg theorem on BMO-functions (see [12, Corollary 7.1.8]), we obtain

$$\|[b,\mathcal{T}]f\chi_{B(y,r)}\|_q\lesssim\|f\chi_{B(y,2r)}\|_q+\|b\|_{BMO(\mathbb{R}^d)}\sum_{i=1}^{\infty}(2^i)^{-\frac{d}{q}}\|f\chi_{B(y,2^{i+1}r)}\|_q.$$

Using the same argument as in the proof of Theorem 1.2, we end the proof. □

References

[1] M. Agcayazi, A. Gogatishvili, K. Koca, R. Ch. Mustafayev, A note on maximal commutators and commutators of maximal functions, J. Math. Soc. Japan, 67 (2015), 581-593.

[2] J. Bastero, M. Milman, F. J. Ruiz, Commutators for the maximal and sharp functions, Proc. Amer. Math. Soc., 128 (2000), 3329-3334.

[3] S. Chanillo, A note on commutators, Indiana Univ. Math. J., 31 (1982), 7-16.

[4] F. Chiarenza, M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Math. 7 (1987), 273-279.

[5] D. Fan, S. Lu and D. Yang, Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients, Georgian Math. J. 5 (1998), 425-440.

[6] J. Feuto, Norm inequalities in some subspaces of Morrey space, Ann. Math. Blaise Pascal 21 (2014), 21-37.

[7] I. Fofana, Etude d'une classe d'espaces de fonctions contenant les espaces de Lorentz, Afr. Mat. 2 (1988), 29-50.

[8] I. Fofana, Continuité de l'intégrale fractionnaire et espace $(L^q, l^p)^\alpha$, C. R. Acad. Sci. Paris 308 (1989), 525-527.

[9] I. Fofana, Espace $(L^q, l^p)^\alpha$ et continuité de l'opérateur maximal fractionnaire de Hardy-Littlewood, Afr. Mat. 3 (2001), 23-37.

[10] J. Garciá-Cuerva, E. Harboure, C. Segovia, J. L. Torrea, Weighted norm inequalities for commutators of strongly singular integrals, Indiana Univ. Math. J., 40 (1991), 1397-1420.

[11] A. Gogatishvili, R.Ch. Mustafayev, M. Agcayazi, Weak-type estimates in Morrey spaces for maximal commuatator and commutator of maximal function, Tokyo J. Math. 41 (2018), 193-218.

[12] L. Grafakos, Modern Fourier analysis, 2nd ed., Graduate Texts in Mathematics vol. 250, Springer, New York, 2009.

[13] V.S. Guliyev, Maximal commutator and commutator of maximal function on total Morrey spaces, J. Math. Inequal. 16 (2022), 1509-1524

[14] V. S. Guliyev, J. J. Hasanov , Y. Zeren, Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces, J. Math. Inequal. 5 (2011), 491-506.

[15] G. Hu, D. Yang, Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type, J. Math. Anal. Appl. 354 (2009), 249-262.

[16] S. Janson, Mean oscillation and commutators of singular integral operators, Ark. Mat., 16 (1978), 263-270.

[17] P. Nagacy, B. A. Kpata, Norm inequalities for fractional integral and fractional maximal operators in the total Fofana spaces, Preprint.

[18] Y. Sawano, G. Di Fazio, D. I. Hakim, Morrey spaces-introduction and applications to integral operators and PDE's, Vol. I, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 409 pp. (2020).

[19] F. Soria, G. Weiss and D. I. Hakim, A remark on singular integrals and power weights, Indiana Univ. Math. J. 43 (1994), 187-204.

[20] E.M. Stein, Harmonic analysis: real variable methods, Orthogonality, and Oscillatory Integrals Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, New Jersey (1993).