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A Proposal of New Extended Symmetric Cosine Distribution

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Abstract. This article presents an extended symmetric version of the cosine distribution. The corresponding probability density function is constructed by a special linear combination of cosine and sine functions. These trigonometric functions are activated by two adjustable parameters with the aim of generating modulable oscillatory shapes. This gives the new distribution greater flexibility and applicability than the cosine distribution. Its main characteristics are then examined, focusing on its functional properties, the key moment measures and the generation of distributions with different support. A new skewed version of the standard normal distribution is also derived. Potential applications in various fields are discussed. Two simulated data examples are presented and analyzed, showing the superior performance of the new distribution compared to another two-parameter extended version of the cosine distribution.

1. Introduction

Symmetric distributions model variables whose values are equally likely to deviate from a central value in both directions. Well-known examples include the normal (Gaussian), logistic, Student (often symbolized by T), Cauchy and Laplace distributions, all of which are defined over the entire real line, i.e., $\mathbb{R}$. However, many practical situations require distributions defined over a bounded interval. For example, physical measurements, such as lengths, weights and concentrations cannot be negative and often have upper limits. In addition to measurements with positive values, other examples include proportions, cosine or sine values of an angle, correlations, normalized test scores or normalized risk metrics, which are naturally bounded over an interval. In such cases, depending on the exact context, it may be a good idea to retain the symmetry property while ensuring that the support of the distribution is bounded.

Several symmetric distributions over bounded intervals have been extensively studied in the literature. These include the uniform distribution with support of the form $[-\upsilon, \upsilon]$ with $\upsilon > 0$, which assigns equal probability to all points within $[-\upsilon, \upsilon]$; it is the simplest form of bounded symmetric distribution. We can also mention the truncated normal distribution, which attempts to

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apply the properties of the normal distribution, including its symmetry, to a given interval. This is done by truncating the tails beyond certain limits and renormalizing the probability density function (pdf) accordingly. See [8], [10] and [4]. Another notable example is the cosine (C) distribution, whose pdf has a symmetric bell-shaped curve similar to that of the normal distribution, but is defined over a finite interval. Let us develop it for the purposes of this article. The (standard) C distribution with support $[-1, 1]$ is defined by the following pdf:

$$f_\circ(x) = \frac{1}{2}[1 + \cos(\pi x)], \quad x \in [-1, 1], \tag{1}$$

and $f_\circ(x) = 0$ for $x \notin [-1, 1]$. It is clear that $f_\circ(-x) = f_\circ(x)$ for any $x \in [-1, 1]$, justifying the symmetry of this pdf around $x = 0$. Furthermore, since $\cos(-\pi) = -1$, the curve of this pdf starts at 0, increases smoothly for $x \in [-1, 0)$, reaches its maximum value 1 at $x = 0$ and then decreases at the same rate thanks to the symmetry. It thus resembles a "single cosine wave", which is very close to the symmetric bell-shaped curve of the pdf of the normal distribution. The C distribution is the best known trigonometric distribution with bounded support. We also note that a scaled version of this distribution exists in the literature, with support $[-\pi, \pi]$. For more details, see [18], [12], [13], [16], and [20].

The C distribution has been the subject of recent developments in distribution theory and practice. In particular, various characterisations of the C distribution were examined in [1], an asymmetric system based on it to produce a new skewed standard normal distribution was considered in [19], a natural two-parameter symmetric version of the C distribution was proposed in [2], original distributions based on the deformation of the cumulative distribution function (cdf) of the C distribution were constructed in [7], and two different two-parameter asymmetric versions of the C distribution were proposed in [5] and [6].

Despite the variety of existing symmetric distributions over bounded intervals, there is a continuous need for new candidates that offer greater flexibility and innovative modeling capabilities. In this article, we propose a new extended symmetric C (ESC) distribution. It is designed to retain the symmetric properties of the C distribution, while introducing two additional parameters to enhance its adaptability. In particular, one of the parameters activates an additional trigonometric term capable of introducing oscillatory shapes in the curves of the corresponding pdf. From a statistical point of view, this feature allows the pdf to accommodate nuanced patterns for a normalized histogram of the data, and thus may be preferable in some situations. We illustrate this claim using the maximum likelihood (ML) estimation for the two parameters and the means of two simulated data sets. We also show that it can be more accurate in the fitting exercise than another two-parameter modified C distribution introduced in [5]. Complementing this practical aspect, we examine some understandable properties of the ESC distribution that can be used beyond the purposes of the article. These include its functional properties, the key moment measures, and the generation of distributions with different supports. A new skewed normal distribution is also derived and discussed.

The rest of this article is structured as follows: Section 2 presents a general distribution result from which we derive the mathematical formulation of the ESC distribution. Section 3 derives its main properties and additional results. Section 4 discusses possible applications and illustrates the flexibility of the ESC distribution with examples. Finally, Section 5 concludes the article and suggests directions for future research.

2. The ESC distribution

2.1. **A general distribution result.** The result below is theoretical; it shows how to choose some parameters of a special linear combination of trigonometric functions in order to satisfy the conditions of a valid pdf. The function considered is inspired by the pdf of the C distribution as defined in Equation (1), with the aim of making it more flexible in a functional sense.

Theorem 2.1. *For any $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$, let us set*

$$f(x; a, b, c) = c\left[1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}\right], \quad x \in [-1, 1],$$

and $f(x; a, b, c) = 0$ for any $x \notin [-1, 1]$. If the following conditions on a, b, and c are satisfied, then $f(x; a, b, c)$ is a valid pdf for a random variable with support $[-1, 1]$:

$$c = \frac{1}{2}, \quad a \in [-1, 1], \quad b \in [0, 1].$$

Proof. To define a valid pdf, we need to check that the following three conditions are met:

- *(condI)*: $f(x; a, b, c)$ is continuous on $\mathbb{R}$, except possibly, for a finite number of values for x.
- *(condII)*: $f(x; a, b, c) \geq 0$ for any $x \in \mathbb{R}$.
- *(condIII)*: $\int_{-\infty}^{+\infty} f(x; a, b, c)dx = 1$.

The condition (condI) is immediate; $f(x; a, b, c)$ is a linear combination of continuous trigonometric functions, it is continuous on $\mathbb{R}$, except possibly, at the extremes, i.e., $x = -1$ and $x = 1$.

Let us now investigate the condition (condII). This condition is obvious for any $x \notin [-1, 1]$, since $f(x; a, b, c) = 0$. Let us concentrate on the case $x \in [-1, 1]$. Using $c > 0$, and the standard triangle inequality, i.e., $|u - v| \geq |u| - |v|$ for any $u \in \mathbb{R}$ and $v \in \mathbb{R}$, we have

$$f(x; a, b, c) \geq c\left\{1 - |a||\cos(\pi x)||1 - b[\sin(\pi x)]^2|\right\}.$$

Since $b \in [0, 1]$ and $\sin(\pi x) \in [0, 1]$, we have $|1 - b[\sin(\pi x)]^2| = 1 - b[\sin(\pi x)]^2 \in [0, 1]$. This combined with $|\cos(\pi x)| \leq 1$ and $a \in [-1, 1]$ gives

$$f(x; a, b, c) \geq c\,[1 - |a||\cos(\pi x)|] \geq c(1 - |a|) \geq 0.$$

The condition (condIII) is now examined. Since $c = 1/2$, using standard integral rules and $\sin(-\pi) = \sin(\pi) = 0$, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x; a, b, c)dx &= \int_{-1}^{1} f(x; a, b, c)dx = c\int_{-1}^{1} \left[1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}\right] dx \\
&= c\int_{-1}^{1} \left\{1 + a\cos(\pi x) - ab\cos(\pi x)[\sin(\pi x)]^2\right\} dx \\
&= c\left\{x + \frac{a}{\pi}\sin(\pi x) - \frac{ab}{3\pi}[\sin(\pi x)]^3\right\}\Big|_{x=-1}^{x=+1} \\
&= c\left\{1 + \frac{a}{\pi}\sin(\pi) - \frac{ab}{3\pi}[\sin(\pi)]^3 - (-1) - \frac{a}{\pi}\sin(-\pi) + \frac{ab}{3\pi}[\sin(-\pi)]^3\right\} \\
&= c\left\{1 + \frac{a}{\pi} \times 0 - \frac{ab}{2\pi} \times 0 + 1 - \frac{a}{\pi} \times 0 + \frac{ab}{2\pi} \times 0\right\} = 2c = 1.
\end{aligned}$$

This ends the proof. □

This theorem is the key theoretical result for deriving the ESC distribution. It is described in the next subsection.

2.2. **Definition of the ESC distribution.** Based on Theorem 2.1, we define the ESC distribution by the following pdf:

$$f_*(x; a, b) = \frac{1}{2}\left[1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}\right], \quad x \in [-1, 1],$$

and $f_*(x; a, b) = 0$ for any $x \notin [-1, 1]$, where $a \in [-1, 1]$ and $b \in [0, 1]$. We say "symmetric" because, for any $x \in \mathbb{R}$, $f_*(x; a, b)$ satisfies $f_*(-x; a, b) = f_*(x; a, b)$; it is immediate for any $x \notin [-1, 1]$ since $f_*(x; a, b) = 0$, and, for any $x \in [-1, 1]$, this follows from the facts that $\cos(-\pi x) = \cos(\pi x)$ and $[\sin(-\pi x)]^2 = [-\sin(\pi x)]^2 = [\sin(\pi x)]^2$.

Obviously, for $a = 1$ and $b = 0$, we have $f_*(x; a, b) = f_\circ(x)$ as given in Equation (1); the ESC distribution is thus reduced to the C distribution. More generally, note that, for any $x \in [-1, 1]$, we can write

$$f_*(x; a, b) = f_\circ(x) + g(x; a, b),$$

where

$$g(x; a, b) = \frac{1}{2}\cos(\pi x)\left\{a - 1 - ab[\sin(\pi x)]^2\right\},$$

which can be thought of as a two-parameter perturbative trigonometric function of $f_\circ(x)$ in this context.

Figure 1 shows the curves of the pdf of the ESC distribution under different but complementary configurations: when a is fixed at a positive value and b varies, when a is fixed at a negative value and b varies, when b is fixed and a varies, and a summary of the different plots.

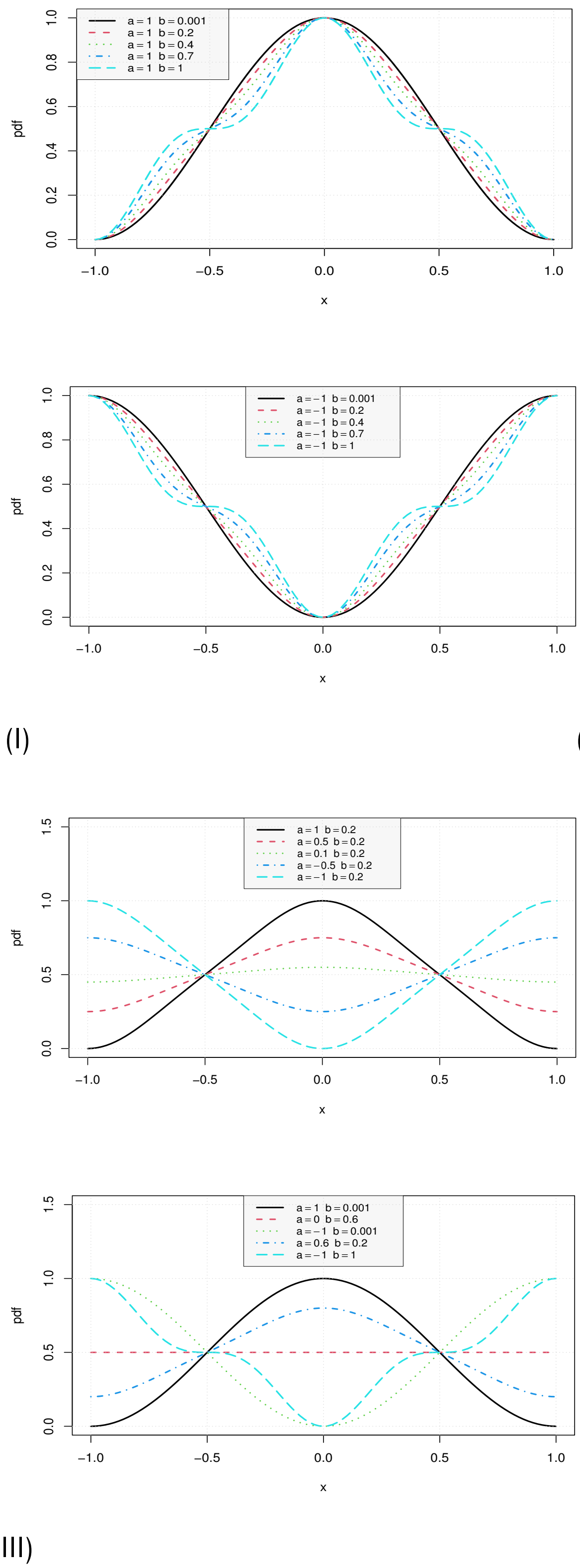

Figure 1. Curves of the pdf of the ESC distribution (I) when $a = 1$ and b varies, (II) when $a = -1$ and b varies, (III) when $b = 0.2$ and a varies, and (IV) a summary of the different plots

In this figure, we can see that all the curves are symmetrical with respect to the axis $x = 0$, with a general bell or inverted bell shape. We distinguish typical smooth curves and curves with small oscillatory shapes, mainly caused by the additional sine term in the pdf. Compared to the curves of the pdf of the C distribution, these nuanced shapes can be a plus for the fitting perspective when the histogram shows some complex symmetric features. This claim will be made more concrete in the application section, i.e., Section 5.

3. Functions and properties

The main functions of the ESC distribution, moment measures and some transformed ESC distributions are determined in this section.

3.1. The cdf.

An important function, the cdf of the ESC distribution, is shown in the result below.

Proposition 3.1. *The cdf of the ESC distribution is given as*

$$F_*(x; a, b) = \frac{1}{2}\left[1 + x + \frac{a}{\pi}\sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\}\right], \quad x \in [-1, 1],$$

$F_*(x; a, b) = 0$ *for any* $x < -1$ *and* $F_*(x; a, b) = 1$ *for any* $x > 1$.

Proof. Since the support of the ESC distribution is $[-1, 1]$, we immediately know that $F_*(x; a, b) = 0$ for any $x < -1$ and $F_*(x; a, b) = 1$ for any $x > 1$. For any $x \in [-1, 1]$, we have

$$\begin{aligned}
F_*(x; a, b) &= \int_{-\infty}^{x} f_*(t; a, b)dt = \int_{-1}^{x} f_*(t; a, b)dt = \frac{1}{2}\int_{-1}^{x}\left[1 + a\cos(\pi t)\{1 - b[\sin(\pi t)]^2\}\right]dt \\
&= \frac{1}{2}\int_{-1}^{x}\left\{1 + a\cos(\pi t) - ab\cos(\pi t)[\sin(\pi t)]^2\right\}dt \\
&= \frac{1}{2}\left\{t + \frac{a}{\pi}\sin(\pi t) - \frac{ab}{3\pi}[\sin(\pi t)]^3\right\}\Bigg|_{t=-1}^{t=x} \\
&= \frac{1}{2}\left\{x + \frac{a}{\pi}\sin(\pi x) - \frac{ab}{3\pi}[\sin(\pi x)]^3 - (-1) - \frac{a}{\pi}\sin(-\pi) + \frac{ab}{3\pi}[\sin(-\pi)]^3\right\} \\
&= \frac{1}{2}\left\{1 + x + \frac{a}{\pi}\sin(\pi x) - \frac{ab}{3\pi}[\sin(\pi x)]^3\right\} \\
&= \frac{1}{2}\left[1 + x + \frac{a}{\pi}\sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\}\right].
\end{aligned}$$

The specified expression is found. □

Clearly, we have

$$F_*(0; a, b) = \frac{1}{2}\left[1 + 0 + \frac{a}{\pi}\sin(\pi \times 0)\left\{1 - \frac{b}{3}[\sin(\pi \times 0)]^2\right\}\right] = \frac{1}{2},$$

which means that the median of the ESC distribution is 0 as expected, since the symmetry point of the corresponding pdf is $x = 0$.

Figure 2 represents the curves of the obtained cdf for different values of a and b (those considered in the subfigure (IV) of Figure 1).

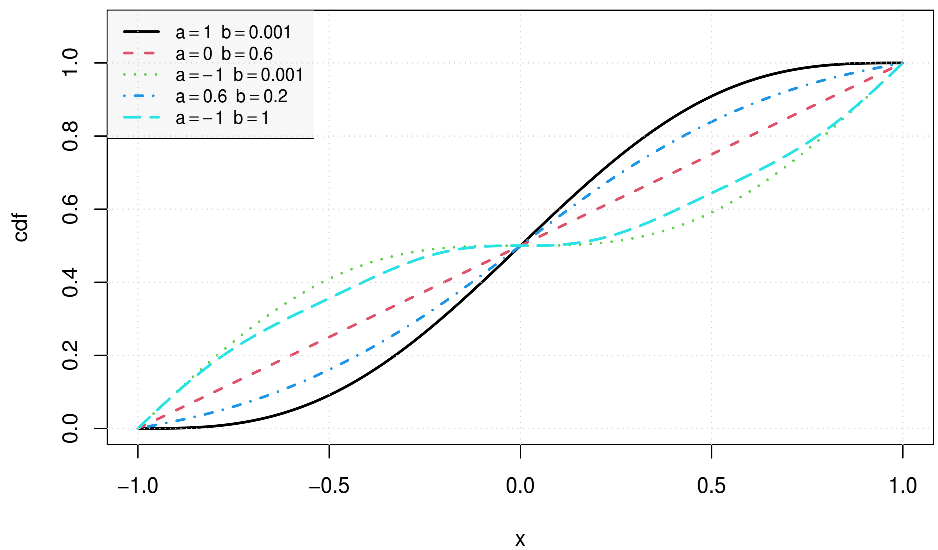

FIGURE 2. Curves of the cdf of the ESC distribution for different values of a and b

This figure confirms the flexibility of the ESC distribution, as various concave and convex increasing curves are observed. However, the "degree of distortion" is limited; in a sense, some areas of the rectangle $[-1, 1] \times [0, 1]$ cannot be reached.

3.2. **Quantile function.** The quantile function (qf) of the ESC distribution is defined as the inverse function of $F_*(x; a, b)$, i.e., $F_*^{-1}(x; a, b)$. Let us denote it by $Q_*(p; a, b)$, for $p \in [0, 1]$. Due to the trigonometric complexity of $F_*(x; a, b)$, it has no closed form expression. However, we can determine it numerically, for fixed values of a and b. Table 1 illustrates this claim, with also different values of p.

TABLE 1. Some values of $Q_*(p; a, b)$ for different values of a, b, and p.

$p \rightarrow$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$a = 1$, $b = 0.001$	-1.00	-0.48	-0.33	-0.21	-0.10	0.00	0.10	0.21	0.33	0.48	1.00
$a = 0$, $b = 0.6$	-1.00	-0.80	-0.60	-0.40	-0.20	0.00	0.20	0.40	0.60	0.80	1.00
$a = -1$, $b = 0.001$	-1.00	-0.90	-0.79	-0.67	-0.52	0.00	0.52	0.67	0.79	0.90	1.00
$a = 0.6$, $b = 0.2$	-1.00	-0.64	-0.43	-0.26	-0.13	0.00	0.13	0.26	0.43	0.64	1.00
$a = -1$, $b = 1$	-1.00	-0.90	-0.78	-0.61	-0.41	0.00	0.41	0.61	0.78	0.90	1.00

For example, for $a = 0.6$, $b = 0.2$ and $p = 0.6$ we have $Q_*(p; a, b) = 0.13$. For $p = 0.5$, we refind that the median of the ESC distribution is 0. This table shows that a computational quantile analysis of the ESC distribution is possible, including the determination of various quantile measures (see [11] and [14]).

3.3. **Hazard rate function.** Complementary to the cdf, we can determine the survival function of the ESC distribution. It is obtained as $S_*(x; a, b) = 1 - F_*(x; a, b)$, so that

$$S_*(x; a, b) = \frac{1}{2}\left[1 - x - \frac{a}{\pi}\sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\}\right], \quad x \in [-1, 1],$$

$S_*(x; a, b) = 1$ for any $x < -1$ and $S_*(x; a, b) = 0$ for any $x > 1$.

The hazard rate function (hrf) of the ESC distribution is obtained by the following ratio formula: $h_*(x; a, b) = f_*(x; a, b)/S_*(x; a, b)$, which can be expressed as

$$h_*(x; a, b) = \frac{1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}}{1 - x - (a/\pi)\sin(\pi x)\{1 - (b/3)[\sin(\pi x)]^2\}}, \quad x \in [-1, 1],$$

and $h_*(x; a, b) = 0$ for any $x \notin [-1, 1]$. The shape behavior of this function is informative about the flexibility of the ESC distribution. With this in mind, Figure 3 shows the curves of this hrf for different values of a and b.

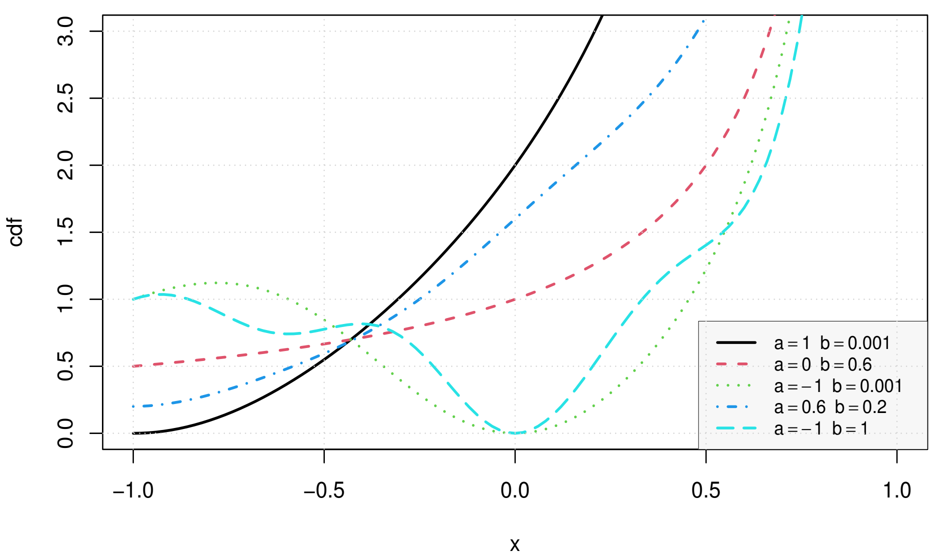

FIGURE 3. Curves of the hrf of the ESC distribution for different values of a and b

In this figure, we distinguish smooth increasing curves as well as oscillating curves, which are characteristics of the trigonometric type distributions. This shows that the ESC distribution is able to model both simple and complex statistical situations involving data in $[-1, 1]$.

3.4. **Moment analysis.** Moment analysis is particularly interesting for distributions with bounded supports, as most of the derived moment measures and functions are automatically well defined.

3.4.1. *Main moment results.* We start this analysis with the mean and variance associated with the ESC distribution in the result below.

Proposition 3.2. *Let X be a random variable with the ESC distribution. Then we have*

$$\mathbb{E}(X) = 0$$

and

$$\mathbb{V}(X) = \frac{1}{3} + \frac{2a}{\pi^2}\left(\frac{2b}{9} - 1\right),$$

where $\mathbb{E}$ and $\mathbb{V}$ denote the mean and variance operators, respectively.

Proof. The symmetry of the ESC distribution (around 0) implies that X and $-X$ follow the ESC distribution (with the same parameters). We thus have $\mathbb{E}(X) = \mathbb{E}(-X) = -\mathbb{E}(X)$, so that $\mathbb{E}(X) = 0$. On the other hand, we have $\mathbb{V}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \mathbb{E}(X^2)$, where

$$\begin{aligned}
\mathbb{E}(X^2) &= \int_{-\infty}^{+\infty} x^2 f_*(x; a, b)dx = \int_{-1}^{1} x^2 f_*(x; a, b)dx \\
&= \frac{1}{2}\int_{-1}^{1} x^2 \left[1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}\right] dx \\
&= \frac{1}{2}\int_{-1}^{1} x^2 dx + \frac{a}{2}\int_{-1}^{1} x^2 \cos(\pi x)\{1 - b[\sin(\pi x)]^2\}dx \\
&= \frac{1}{3} + \frac{a}{2}\int_{-1}^{1} x^2 \cos(\pi x)\{1 - b[\sin(\pi x)]^2\}dx.
\end{aligned}$$

For the remaining integral term, using two integrations by parts in a row and $[\sin(\pi x)]^2 = 1 - [\cos(\pi x)]^2$, we get

$$\begin{aligned}
&\int_{-1}^{1} x^2 \cos(\pi x)\{1 - b[\sin(\pi x)]^2\}dx \\
&= \left[x^2 \frac{1}{\pi}\sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\}\right]\Bigg|_{x=-1}^{x=1} - \frac{2}{\pi}\int_{-1}^{1} x\sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\} dx \\
&= 0 - \frac{2}{\pi}\int_{-1}^{1} x\sin(\pi x)\left\{1 - \frac{b}{3} + \frac{b}{3}[\cos(\pi x)]^2\right\} dx \\
&= -\frac{2}{\pi}\left\{ \left[-x\frac{1}{\pi}\cos(\pi x)\left\{1 - \frac{b}{3} + \frac{b}{9}[\cos(\pi x)]^2\right\}\right]\Bigg|_{x=-1}^{x=1} \right. \\
&\left. + \frac{1}{\pi}\int_{-1}^{1} \cos(\pi x)\left\{1 - \frac{b}{3} + \frac{b}{9}[\cos(\pi x)]^2\right\} dx\right\} \\
&= -\frac{2}{\pi}\left\{\frac{2}{\pi}\left(1 - \frac{2b}{9}\right) + \left(1 - \frac{b}{3}\right)\frac{1}{\pi}\int_{-1}^{1} \cos(\pi x)dx + \frac{b}{9\pi}\int_{-1}^{1} [\cos(\pi x)]^3 dx\right\} \\
&= -\frac{2}{\pi}\left\{\frac{2}{\pi}\left(1 - \frac{2b}{9}\right) + 0 + \frac{b}{9\pi}\int_{-1}^{1} \cos(\pi x)\{1 - [\sin(\pi x)]^2\}dx\right\}
\end{aligned}$$

$$= -\frac{2}{\pi}\left\{\frac{2}{\pi}\left(1-\frac{2b}{9}\right)+\frac{b}{9\pi}\left[\frac{1}{\pi}\sin(\pi x)\left\{1-\frac{1}{3}[\sin(\pi x)]^2\right\}\right]\Big|_{x=-1}^{x=1}\right\}$$

$$= -\frac{2}{\pi}\left\{\frac{2}{\pi}\left(1-\frac{2b}{9}\right)+0\right\} = \frac{8b}{9\pi^2}-\frac{4}{\pi^2} = \frac{4}{\pi^2}\left(\frac{2b}{9}-1\right).$$

Hence, we have

$$\mathbb{V}(X) = \frac{1}{3}+\frac{a}{2}\times\frac{4}{\pi^2}\left(\frac{2b}{9}-1\right) = \frac{1}{3}+\frac{2a}{\pi^2}\left(\frac{2b}{9}-1\right).$$

This concludes the proof. □

Now let us complete this result. Let X be a random variable with the ESC distribution. Then, for any positive integer n, since the ESC distribution is symmetric (around 0), we have

$$\mathbb{E}(X^{2n+1}) = 0.$$

More generally, for any odd function $p(x)$, we have $\mathbb{E}[p(X)] = 0$. In particular, we rediscover $\mathbb{E}(X) = 0$, and we can mention that $\mathbb{E}(X^3) = 0$. On the other hand, using several integral techniques and trigonometric formulas, we find that

$$\mathbb{E}(X^4) = \frac{1}{5}+\frac{4a}{\pi^4}\left[\frac{2(3\pi^2-20)b}{27}-\pi^2+6\right].$$

Since $\mathbb{E}(X) = 0$, the skewness coefficient of X is equal to

$$\gamma(X) = \frac{1}{[\mathbb{E}(X^2)]^{3/2}}\mathbb{E}(X^3) = 0.$$

The ESC distribution therefore has a neutral skewness. Furthermore, the kurtosis coefficient of X can be calculated as

$$\beta(X) = \frac{1}{[\mathbb{E}(X^2)]^2}\mathbb{E}(X^4) = \frac{81\pi^4+60a\left[(6\pi^2-40)b-27(\pi^2-6)\right]}{5\left[2a(2b-9)+3\pi^2\right]^2}.$$

Since the above moment measures are not easy to handle, especially for $\beta(X)$, we propose a numerical work. Table 2 gives some values of the moment measures for different values of a and b.

Table 2. Some values of moment measures of a random variable X with the ESC distribution for different values of a and b.

	$\mathbb{E}(X)$	$\mathbb{E}(X^2)$	$\mathbb{E}(X^3)$	$\mathbb{E}(X^4)$	$\gamma(X)$	$\beta(X)$
$a = 1$, $b = 0.001$	0.000	0.131	0.000	0.041	0.000	2.406
$a = 0$, $b = 0.6$	0.000	0.333	0.000	0.200	0.000	1.800
$a = -1$, $b = 0.001$	0.000	0.536	0.000	0.359	0.000	1.249
$a = 0.6$, $b = 0.2$	0.000	0.217	0.000	0.108	0.000	2.294
$a = -1$, $b = 1$	0.000	0.491	0.000	0.330	0.000	1.368

For the values considered, we have $\beta(X) < 3$, indicating that the ESC distribution is mainly platykurtic. Thus, for the same variance, the distribution is relatively "flattened".

We now determine the incomplete mean associated with the ESC distribution.

Proposition 3.3. *Let X be a random variable with the ESC distribution. Then, for any $x \in [-1, 1]$, we have*

$$\mathbb{E}(X1_{\{X\leq x\}}) = \frac{1}{4}(x^2-1) + \frac{a}{2\pi^2}\left[\pi x\sin(\pi x)\left\{1-\frac{b}{3}[\sin(\pi x)]^2\right\}\right.$$
$$\left. + \cos(\pi x)\left\{1-\frac{b}{3}+\frac{b}{9}[\cos(\pi x)]^2\right\} + 1 - \frac{2b}{9}\right],$$

where $1_{\{.\}}$ denotes the indicator operator.

Proof. For any $x \in [-1, 1]$, we have

$$\begin{aligned}\mathbb{E}(X1_{\{X\leq x\}}) &= \int_{-\infty}^{x} tf_*(t;a,b)dt = \int_{-1}^{x} tf_*(t;a,b)dt\\ &= \frac{1}{2}\int_{-1}^{x} t\left[1+a\cos(\pi t)\{1-b[\sin(\pi t)]^2\}\right]dt\\ &= \frac{1}{2}\int_{-1}^{x} tdt + \frac{a}{2}\int_{-1}^{x} t\cos(\pi t)\{1-b[\sin(\pi t)]^2\}dt\\ &= \frac{1}{4}(x^2-1) + \frac{a}{2}\int_{-1}^{x} t\cos(\pi t)\{1-b[\sin(\pi t)]^2\}dt.\end{aligned}$$

For the remaining integral term, using an integration by parts and $[\sin(\pi x)]^2 = 1-[\cos(\pi x)]^2$, we get

$$\begin{aligned}&\int_{-1}^{x} t\cos(\pi t)\{1-b[\sin(\pi t)]^2\}dt\\ &= \left[t\frac{1}{\pi}\sin(\pi t)\left\{1-\frac{b}{3}[\sin(\pi t)]^2\right\}\right]\Bigg|_{t=-1}^{t=x} - \frac{1}{\pi}\int_{-1}^{x}\sin(\pi t)\left\{1-\frac{b}{3}[\sin(\pi t)]^2\right\}dt\\ &= \frac{1}{\pi}x\sin(\pi x)\left\{1-\frac{b}{3}[\sin(\pi x)]^2\right\} - \frac{1}{\pi}\int_{-1}^{x}\sin(\pi t)\left\{1-\frac{b}{3}+\frac{b}{3}[\cos(\pi t)]^2\right\}dt\\ &= \frac{1}{\pi}x\sin(\pi x)\left\{1-\frac{b}{3}[\sin(\pi x)]^2\right\} - \frac{1}{\pi}\left[-\frac{1}{\pi}\cos(\pi t)\left\{1-\frac{b}{3}+\frac{b}{9}[\cos(\pi t)]^2\right\}\right]\Bigg|_{t=-1}^{t=x}\\ &= \frac{1}{\pi}x\sin(\pi x)\left\{1-\frac{b}{3}[\sin(\pi x)]^2\right\} + \frac{1}{\pi^2}\cos(\pi x)\left\{1-\frac{b}{3}+\frac{b}{9}[\cos(\pi x)]^2\right\}\\ &+ \frac{1}{\pi^2}\left(1-\frac{2b}{9}\right).\end{aligned}$$

Hence, we have

$$\mathbb{E}(X1_{\{X\leq x\}}) = \frac{1}{4}(x^2-1) + \frac{a}{2\pi^2}\left[\pi x\sin(\pi x)\left\{1-\frac{b}{3}[\sin(\pi x)]^2\right\}\right.$$
$$\left. + \cos(\pi x)\left\{1-\frac{b}{3}+\frac{b}{9}[\cos(\pi x)]^2\right\} + 1 - \frac{2b}{9}\right].$$

This ends the proof. □

Clearly, if we take $x = 1$, we refind that $\mathbb{E}(X1_{\{X\leq 1\}}) = \mathbb{E}(X) = 0$. More generally, the expression of this incomplete mean can be useful in assessing the expected value of X within a certain range, which is crucial in areas such as reliability. It also helps in the calculation and interpretation of other related functions, such as the mean residual life function. The incomplete mean also plays a theoretical role in some characterization results (see [17]).

We now determine the mean deviation associated with the ESC distribution.

Proposition 3.4. *Let X be a random variable with the ESC distribution. Then we have*

$$\mathbb{E}(|X|) = \frac{1}{2} + \frac{2a}{\pi^2}\left(\frac{2b}{9} - 1\right).$$

Proof. Since the pdf $f_*(x; a, b)$ is symmetric (around 0), we have

$$\begin{aligned}
\mathbb{E}(|X|) &= \int_{-\infty}^{+\infty} |x| f_*(x; a, b) dx = \int_{-1}^{1} |x| f_*(x; a, b) dx \\
&= 2\int_0^1 x f_*(x; a, b) dx = \int_0^1 x\left[1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}\right] dx \\
&= \int_0^1 x dx + a\int_0^1 x\cos(\pi x)\{1 - b[\sin(\pi x)]^2\} dx \\
&= \frac{1}{2} + a\int_0^1 x\cos(\pi x)\{1 - b[\sin(\pi x)]^2\} dx.
\end{aligned}$$

For the remaining integral term, using an integration by parts and $[\sin(\pi x)]^2 = 1 - [\cos(\pi x)]^2$, we get

$$\begin{aligned}
&\int_0^1 x\cos(\pi x)\{1 - b[\sin(\pi x)]^2\} dx \\
&= \left[x\frac{1}{\pi}\sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\}\right]\Bigg|_{x=0}^{x=1} - \frac{1}{\pi}\int_0^1 \sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\} dx \\
&= 0 - \frac{1}{\pi}\int_0^1 \sin(\pi x)\left\{1 - \frac{b}{3} + \frac{b}{3}[\cos(\pi x)]^2\right\} dx \\
&= -\frac{1}{\pi}\left[-\frac{1}{\pi}\cos(\pi x)\left\{1 - \frac{b}{3} + \frac{b}{9}[\cos(\pi x)]^2\right\}\right]\Bigg|_{x=0}^{x=1} \\
&= \frac{1}{\pi^2}\left(-1 + \frac{2b}{9} - 1 + \frac{2b}{9}\right) = \frac{2}{\pi^2}\left(\frac{2b}{9} - 1\right).
\end{aligned}$$

We therefore have

$$\mathbb{E}(|X|) = \frac{1}{2} + \frac{2a}{\pi^2}\left(\frac{2b}{9} - 1\right).$$

This concludes the proof. □

As the ESC distribution is symmetric, the associated mean deviation of X can be considered as a measure of dispersion of the same unit as the modeled variable. It is comparable to the standard deviation of X, which is defined as the square of the corresponding variance. We notice also the following formula:

$$\mathbb{V}(X) - \mathbb{E}(|X|) = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6},$$

which has the property of being independent of a and b.

The moment generating function of the ESC distribution can be determined with mathematical efforts. This is developed in the result below.

Proposition 3.5. *Let X be a random variable with the ESC distribution. Then the moment generating function of X is given as $\varphi(t; a, b) = \mathbb{E}\left[e^{tX}\right]$, and can be expressed as*

$$\varphi(t; a, b) = \sinh(t)\left\{\frac{1}{t} - \frac{at[\pi^2(9-2b)+t^2]}{(t^2+\pi^2)(t^2+9\pi^2)}\right\}, \quad t \in \mathbb{R},$$

where $\sinh(t) = (e^t - e^{-t})/2$.

Proof. We have

$$\begin{aligned}\varphi(t; a, b) &= \mathbb{E}\left[e^{tX}\right] = \int_{-\infty}^{+\infty} e^{tx} f_*(x; a, b)dx = \int_{-1}^{1} e^{tx} f_*(x; a, b)dx \\ &= \frac{1}{2}\int_{-1}^{1} e^{tx}\left[1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}\right] dx \\ &= \frac{1}{2}\int_{-1}^{1} e^{tx}dx + \frac{a}{2}\int_{-1}^{1}\cos(\pi x)e^{tx}dx - \frac{ab}{2}\int_{-1}^{1}\cos(\pi x)[\sin(\pi x)]^2 e^{tx}dx.\end{aligned}$$

The last two integral terms require special treatment. It follows from [9, Formula number 2.663.3] that

$$\int_{-1}^{1}\cos(\pi x)e^{tx}dx = -\frac{2t\sinh(t)}{t^2+\pi^2}.$$

On the other hand, after a work on the formula in [9, Formula number 2.664.2], we obtain

$$\int_{-1}^{1}\cos(\pi x)[\sin(\pi x)]^2 e^{tx}dx = -\frac{4\pi^2 t\sinh(t)}{(t^2+9\pi^2)(t^2+\pi^2)}.$$

We therefore have

$$\begin{aligned}\varphi(t; a, b) &= \frac{1}{2}\left[\frac{1}{t}e^{tx}\right]\Bigg|_{x=-1}^{x=1} - \frac{at\sinh(t)}{t^2+\pi^2} + \frac{2ab\pi^2 t\sinh(t)}{(t^2+9\pi^2)(t^2+\pi^2)} \\ &= \frac{\sinh(t)}{t} - \frac{at\sinh(t)}{t^2+\pi^2} + \frac{2ab\pi^2 t\sinh(t)}{(t^2+9\pi^2)(t^2+\pi^2)} \\ &= \sinh(t)\left\{\frac{1}{t} - \frac{at[\pi^2(9-2b)+t^2]}{(t^2+\pi^2)(t^2+9\pi^2)}\right\}.\end{aligned}$$

The stated formula is established, concluding the proof. □

We rediscover the moment generating function associated with the C distribution by taking $a = 1$ and $b = 0$, i.e,

$$\varphi_\circ(t) = \frac{\pi^2 \sinh(t)}{t(t^2 + \pi^2)}, \quad t \in \mathbb{R}.$$

See [1] (with the following configuration: $\mu = 0$ and $\sigma = 1$). The characteristic function associated with the ESC distribution follows from the same mathematical development. By introducing i, the imaginary unit solution of the equation $x^2 + 1 = 0$, since $\sinh(it) = i\sin(t)$, we have

$$\begin{aligned}\phi(t; a, b) = {}''\varphi(it; a, b)'' &= \sinh(it)\left\{\frac{1}{it} - \frac{ait[\pi^2(9-2b) + (it)^2]}{((it)^2+\pi^2)((it)^2+9\pi^2)}\right\} \\ &= \sin(t)\left\{\frac{1}{t} - \frac{at(2\pi^2 b + t^2 - 9\pi^2)}{(t^2-9\pi^2)(t^2-\pi^2)}\right\}, \quad t \in \mathbb{R}.\end{aligned}$$

This function also fully defines the ESC distribution. In fact, we have the following formula linking the pdf and the characteristic function:

$$f_*(x; a, b) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(t; a, b) e^{-itx} dt, \quad x \in [-1, 1],$$

which can be written as, for any $x \in [-1, 1]$,

$$\frac{1}{2}\left[1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sin(t)\left\{\frac{1}{t} - \frac{at(2\pi^2 b + t^2 - 9\pi^2)}{(t^2-9\pi^2)(t^2-\pi^2)}\right\} e^{-itx} dt$$

or, equivalently,

$$1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \sin(t)\left\{\frac{1}{t} - \frac{at(2\pi^2 b + t^2 - 9\pi^2)}{(t^2-9\pi^2)(t^2-\pi^2)}\right\} e^{-itx} dt.$$

This two-parameter formula can be of some mathematical interest, especially in harmonic analysis dealing with the Fourier transform of various trigonometric functions.

We can also note that, since X is symmetric (around 0), we have $\varphi(t; a, b) = \varphi(-t; a, b)$, a property we can also check using the expression we found. As a direct consequence, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}[\cosh(tX)] &= \frac{1}{2}[\varphi(t; a, b) + \varphi(-t; a, b)] = \varphi(t; a, b) \\ &= \sinh(t)\left\{\frac{1}{t} - \frac{at[\pi^2(9-2b) + t^2]}{(t^2+\pi^2)(t^2+9\pi^2)}\right\},\end{aligned}$$

recalling that $\cosh(x) = (e^x + e^{-x})/2$.

We end this part by investigating the moments of the power of the sine transformed random variable with the ESC distribution. These moments have the property of being very simple, dependent on an adjustable positive integer, and independent of a and b.

Proposition 3.6. *Let n be a positive integer, and X be a random variable with the ESC distribution. Then we have*

$$\mathbb{E}\left\{[\sin(\pi X)]^{2n}\right\} = \frac{(2n)!}{2^{2n}(n!)^2}$$

and

$$\mathbb{E}\left\{[\sin(\pi X)]^{2n+1}\right\} = 0.$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left\{[\sin(\pi X)]^{2n}\right\} &= \int_{-\infty}^{+\infty} [\sin(\pi x)]^{2n} f_*(x; a, b) dx = \int_{-1}^{1} [\sin(\pi x)]^{2n} f_*(x; a, b) dx \\
&= \frac{1}{2} \int_{-1}^{1} [\sin(\pi x)]^{2n} \left[1 + a\cos(\pi x)\{1 - b[\sin(\pi x)]^2\}\right] dx \\
&= \frac{1}{2} \int_{-1}^{1} [\sin(\pi x)]^{2n} dx + \frac{a}{2} \int_{-1}^{1} \cos(\pi x)[\sin(\pi x)]^{2n}\{1 - b[\sin(\pi x)]^2\} dx.
\end{aligned}$$

After a work on the formula in [9, Formula number 2.513.1], we find that

$$\int_{-1}^{1} [\sin(\pi x)]^{2n} dx = \frac{(2n)!}{2^{2n-1}(n!)^2}. \tag{2}$$

We therefore have

$$\begin{aligned}
\mathbb{E}\left\{[\sin(\pi X)]^{2n}\right\} &= \frac{(2n)!}{2^{2n}(n!)^2} + \frac{a}{2}\left[\frac{1}{\pi}[\sin(\pi x)]^{2n+1}\left\{\frac{1}{2n+1} - \frac{b}{2n+3}[\sin(\pi x)]^2\right\}\right]\Bigg|_{x=-1}^{x=1} \\
&= \frac{(2n)!}{2^{2n}(n!)^2} + 0 = \frac{(2n)!}{2^{2n}(n!)^2}.
\end{aligned}$$

On the other hand, since $p(x) = [\sin(\pi x)]^{2n+1}$ is an odd function, we have already discussed that $\mathbb{E}\left\{[\sin(\pi X)]^{2n+1}\right\} = \mathbb{E}[p(X)] = 0$. The desired results are obtained. □

3.4.2. *Secondary moment results.* During our investigations of the ESC distribution, we found other moment results of potential interest. We present them below, considering a random variable X with the ESC distribution. The mathematical details are omitted for reasons of space.

- A special moment formula is as follows:

$$\mathbb{E}\left[\frac{1}{\sqrt{X+1}}\right] = \frac{1}{12\sqrt{2}}\left[3a(b-4)C(2) - ab\sqrt{3}C[2\sqrt{3}] + 24\right],$$

  where $C(x)$ denotes the Fresnel C integral defined by $C(x) = \int_0^x \cos(\pi t^2/2) dt$ (see [9, Section 8.25]). This integral is implemented in most mathematical software, such as R with the package entitled pracma.

- Another special moment formula is as follows:

$$\mathbb{E}\left[e^{-|X|}\right] = \frac{e[a\pi^2(9-2b) + 1 + a + 10\pi^2 + 9\pi^4] - 1 + a + \pi^2(9a - 10 - 2ab) - 9\pi^4}{e(1+\pi^2)(1+9\pi^2)}.$$

  More succinct moment formulas are given below, some of which may be useful for various theoretical or practical applications involving the ESC distribution.

- We have

$$\mathbb{E}\left\{\frac{1}{1 + [\cos(\pi X)]^2}\right\} = \frac{1}{\sqrt{2}}.$$

- We have
$$\mathbb{E}\left\{\frac{1}{1+[\sin(\pi X)]^2}\right\}=\frac{1}{\sqrt{2}}.$$
The two results for the moments above are therefore the same.
- We have
$$\mathbb{E}\left\{\frac{1}{1+[\tan(\pi X)]^2}\right\}=\frac{1}{2}.$$
- We have
$$\mathbb{E}\left\{\cos[\pi|X|]\right\}=\frac{a}{8}(4-b).$$
- We have
$$\mathbb{E}\left\{\sin[\pi|X|]\right\}=\frac{2}{\pi}.$$
- We have
$$\mathbb{E}\left[|\cos(\pi X)|\right]=\frac{2}{\pi}.$$
- We have
$$\mathbb{E}\left[|\sin(\pi X)|\right]=\frac{2}{\pi}.$$
The three results for the moments above are therefore the same.
- We have
$$\mathbb{E}\left[\sqrt{1+\cos(\pi X)}\right]=-\frac{2\sqrt{2}}{105\pi}[a(8b-35)-105].$$
- We have
$$\mathbb{E}\left\{\frac{\cos(\pi X)}{1+[\sin(\pi X)]^2}\right\}=a\left\{\sqrt{2}-1+b\left[\sqrt{2}-\frac{3}{2}\right]\right\}.$$

3.5. **Some derived distributions.**

3.5.1. *Transformed ESC distributions.* Since the ESC distribution is new in the literature, it may be interesting to derive transformed ESC distributions with different properties, such as different supports. For this purpose, we consider a random variable X with the ESC distribution and investigate the following transformed random variable:

$$Y=m(X),$$

where $m(x)$ is a well-defined function on $[-1,1]$ or $(-1,1)$.

- For $m(x)=\pi x$, we get $Y=\pi X$, which is of support $[-\pi,\pi]$. Based on Proposition 3.1, Y has the following cdf:
$$\begin{aligned}F_{\wedge}(x;a,b)&=F_{*}\left(\frac{x}{\pi};a,b\right)\\&=\frac{1}{2\pi}\left[\pi+x+a\sin(x)\left\{1-\frac{b}{3}[\sin(x)]^2\right\}\right],\quad x\in[-\pi,\pi],\end{aligned}$$
$F_{\wedge}(x;a,b)=0$ for any $x<-\pi$ and $F_{\wedge}(x;a,b)=1$ for any $x>\pi$ (still with $a\in[-1,1]$ and $b\in[0,1]$).

- For $m(x) = |x|$, we get $Y = |X|$, which is of support $[0, 1]$. Based on Proposition 3.1, Y has the following cdf:

$$\begin{aligned} F_{\diamond}(x; a, b) &= 2F_{*}(x; a, b) - 1 \\ &= x + \frac{a}{\pi}\sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\}, \quad x \in [0, 1], \end{aligned}$$

$F_{\diamond}(x; a, b) = 0$ for any $x < 0$ and $F_{\diamond}(x; a, b) = 1$ for any $x > 1$.

- For $m(x) = (x + 1)/2$, we get $Y = (X + 1)/2$, which is of support $[0, 1]$. Based on Proposition 3.1, Y has the following cdf:

$$\begin{aligned} F_{\star}(x; a, b) &= F_{*}(2x - 1; a, b) \\ &= x - \frac{a}{2\pi}\sin(\pi x)\left\{1 - \frac{b}{3}[\sin(\pi x)]^2\right\}, \quad x \in [0, 1], \end{aligned}$$

$F_{\star}(x; a, b) = 0$ for any $x < 0$ and $F_{\star}(x; a, b) = 1$ for any $x > 1$. This and the previous cdf can be seen as modified versions of the cdf of the uniform distribution over $[0, 1]$.

- For $m(x) = \tan[(\pi/2)x]$, we get $Y = \tan[(\pi/2)X]$, which is of support $\mathbb{R}$. Based on Proposition 3.1, Y has the following cdf:

$$\begin{aligned} F_{\odot}(x; a, b) &= F_{*}\left[\frac{2}{\pi}\arctan(x); a, b\right] \\ &= \frac{1}{2}\left\{1 + \frac{2}{\pi}\arctan(x) + \frac{a}{\pi}\sin[2\arctan(x)]\left[1 - \frac{b}{3}\{\sin[2\arctan(x)]\}^2\right]\right\} \\ &= \frac{1}{2}\left\{1 + \frac{2}{\pi}\arctan(x) + \frac{2ax}{\pi(1+x^2)}\left[1 - \frac{4bx^2}{3(1+x^2)^2}\right]\right\}, \quad x \in \mathbb{R}. \end{aligned}$$

This cdf can be seen as a modified cdf of the standard Cauchy distribution.

- For $m(x) = \sin[(\pi/2)x]$, we get $Y = \sin[(\pi/2)X]$, which is of support $[-1, 1]$. Based on Proposition 3.1, Y has the following cdf:

$$\begin{aligned} F_{\vee}(x; a, b) &= F_{*}\left[\frac{2}{\pi}\arcsin(x); a, b\right] \\ &= \frac{1}{2}\left\{1 + \frac{2}{\pi}\arcsin(x) + \frac{a}{\pi}\sin[2\arcsin(x)]\left[1 - \frac{b}{3}\{\sin[2\arcsin(x)]\}^2\right]\right\} \\ &= \frac{1}{2}\left\{1 + \frac{2}{\pi}\arcsin(x) + \frac{2a}{\pi}x\sqrt{1-x^2}\left[1 - \frac{4b}{3}x^2(1-x^2)\right]\right\}, \quad x \in [-1, 1], \end{aligned}$$

$F_{\vee}(x; a, b) = 0$ for any $x < -1$ and $F_{\vee}(x; a, b) = 1$ for any $x > 1$.

- For $m(x) = (1-x)/(1+x)$, we get $Y = (1-X)/(1+X)$, which is of support $(0, +\infty)$. Based on Proposition 3.1, Y has the following cdf:

$$\begin{aligned} F_{\triangleleft}(x; a, b) &= 1 - F_*\left(\frac{1-x}{1+x}; a, b\right) \\ &= \frac{1}{2}\left\{1 - \frac{1-x}{1+x} - \frac{a}{\pi}\sin\left[\frac{\pi(1-x)}{1+x}\right]\left[1 - \frac{b}{3}\left\{\sin\left[\frac{\pi(1-x)}{1+x}\right]\right\}^2\right]\right\} \\ &= \frac{x}{x+1} - \frac{a}{2\pi}\sin\left(\frac{2\pi x}{1+x}\right)\left\{1 - \frac{b}{3}\left[\sin\left(\frac{2\pi x}{1+x}\right)\right]^2\right\}, \quad x > 0 \end{aligned}$$

and $F_{\triangleleft}(x; a, b) = 0$ for any $x \leq 0$. This cdf can be seen as a modified cdf of the standard Lomax distribution, i.e., defined by $q(x) = x/(x+1) = 1 - (x+1)^{-1}$.

These are only examples of new distributions derived from the ESC distribution. Others may be considered depending on the objectives.

3.5.2. *A new skewed normal distribution.* In [19], the C distribution is used to create a trigonometric skewed version of the standard normal distribution by applying the Azzalini scheme introduced in [3]. We can follow the same idea but using the ESC distribution, also for more flexibility thanks to the two additional parameters. To be precise, let us consider the pdf of the standard normal distribution defined by

$$g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad x \in \mathbb{R}.$$

We then construct a new trigonometric skewed version of this pdf by considering the following function:

$$f_{\triangle}(x; a, b, \lambda) = 2g(x)F_*(\lambda x; a, b), \quad x \in \mathbb{R},$$

where $\lambda > 0$ in an additional skew parameter, and $F_*(x; a, b)$ is still the cdf of the ESC distribution given in Proposition 3.1. Thus defined, $f_{\triangle}(x; a, b, \lambda)$ is a valid pdf. It can be expressed as

$$f_{\triangle}(x; a, b, \lambda) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\left[1 + \lambda x + \frac{a}{\pi}\sin(\lambda\pi x)\left\{1 - \frac{b}{3}[\sin(\lambda\pi x)]^2\right\}\right], \quad x \in \left(-\frac{1}{\lambda}, \frac{1}{\lambda}\right),$$

completed by

$$f_{\triangle}(x; a, b, \lambda) = \sqrt{\frac{2}{\pi}}e^{-x^2/2}, \quad x \geq \frac{1}{\lambda}$$

and $f_{\triangle}(x; a, b, \lambda) = 0$ for any $x \leq -1/\lambda$.

Figure 4 shows the curves of this pdf for different values of a, b and λ.

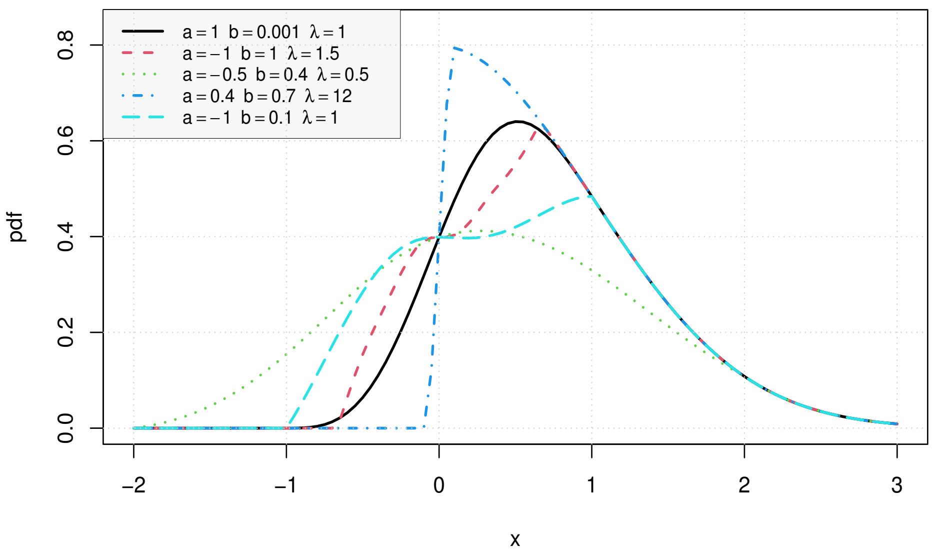

Figure 4. Curves of the pdf of the derived skewed normal distribution for different values of a, b and λ

This figure illustrates the complexity of the deformations applied to the pdf of the standard normal distribution based on our trigonometric skew scheme. In particular, we see some abrupt changes and pronounced peaks in the mode, which are quite rare properties in the family of skewed normal distributions.

The derived skewed normal distribution may be of interest for the analysis of versatile skewed data. It is more adaptable than the one developed in [19], mainly due to the presence of the additional parameter b. We leave this practical aspect to future work.

Of course, the same methodology can be applied to other basic symmetric distributions around 0 than the standard normal. For example, we can consider the pdf of the logistic, Student, Cauchy and Laplace distributions for $g(x)$, which opens up new research directions.

4. Applications

This section looks at some statistical aspects of the ESC distribution, mainly when it is going to be used to analyze data.

4.1. **Estimation method.** In practice, a variable with values in $[-1, 1]$ can potentially be modeled by a random variable X with the ESC distribution. To get a precise idea of this probabilistic assumption, we assume that the parameters a and b as unknown, and try to estimate them efficiently based on the available information of X that make up the data. A wide range of estimation methods can be used to do this. One of the most common and efficient is the ML estimation method. Let us formalize this in the context of the ESC distribution below.

Suppose we have n data, observations of X, classically denoted as $x_1, \ldots, x_n$. Then the ML estimates of a and b are given as

$$(\hat{a}, \hat{b}) = \text{argmax}_{(a,b)\in[-1,1]\times[0,1]} \ell(a, b),$$

where

$$\begin{aligned}\ell(a, b) &= \sum_{i=1}^{n} \ln [f_*(x_i; a, b)] \\ &= -n\ln(2) + \sum_{i=1}^{n} \ln \left[1 + a\cos(\pi x_i)\{1 - b[\sin(\pi x_i)]^2\}\right].\end{aligned}$$

This procedure of maximization can be carried out with any mathematical software, such as R with the function nlminb. Once $\hat{a}$ and $\hat{b}$ are obtained, the corresponding estimated pdf is given by

$$\hat{f}_*(x) = f_*(x; \hat{a}, \hat{b}) = \frac{1}{2}\left[1 + \hat{a}\cos(\pi x)\{1 - \hat{b}[\sin(\pi x)]^2\}\right], \quad x \in [-1, 1]. \tag{3}$$

Using this function, we can visually assess the fit of the ESC distribution to the data. More specifically, we have an acceptable result if the corresponding normalized histogram of the data has a shape that is well fitted by the curve of $\hat{f}_*(x)$ for $x \in [-1, 1]$.

The goodness of fit of different distributions can be compared using well-established mathematical tools, such as the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). In the context of the ESC distribution, these criteria are defined as

$$AIC = 2[k - \ell(\hat{a}, \hat{b})], \quad BIC = k\ln(n) - 2\ell(\hat{a}, \hat{b}),$$

respectively, where k is the number of unknown parameters, i.e., $k = 2$.

As the main competitor of the ESC distribution, we consider another variant of the C distribution: the AC distribution introduced in [5]. It is defined by the following two-parameter pdf:

$$f_\dagger(x; \alpha, \beta) = \frac{\beta(\pi^2 + \beta^2)}{2[\pi^2 + (1-\alpha)\beta^2]\sinh(\beta)}[1 + \alpha\cos(\pi x)]e^{\beta x}, \quad x \in [-1, 1], \tag{4}$$

and $f_\dagger(x; \alpha, \beta) = 0$ for all $x \notin [-1, 1]$, where $\alpha \in [-1, 1]$ and $\beta \in \mathbb{R}$. Unlike the ESC distribution, the AC distribution is not designed to be exclusively symmetric. This makes it a challenger of the ESC distribution when dealing with "possibly symmetric" data, especially when the data are not "perfectly symmetric". For this distribution, we can also use the ML estimation method to estimate the parameters α and β, say $\hat{\alpha}$ and $\hat{\beta}$, respectively, determine the estimated pdf as $\hat{f}_\dagger(x) = f_\dagger(x; \hat{\alpha}, \hat{\beta})$, and compute the AIC and BIC for comparison purposes.

For more general information on the ML estimation method, see [4].

In the remaining parts of this section, for illustrative purposes, we test the ML estimation method on the ESC distribution using two different examples of simulated data, i.e., data that can be obtained in credible scenarios (not real data). In particular, we compare the fit obtained with that of the AC distribution.

4.2. **Simulated example 1.** In this simulated example, we look at the normalized daily air quality index (AQI) in a particular urban area over a period of 57 days. Each measurement of the AQI is contained in the range $[-1, 1]$, where a value close to 0 indicates moderate air quality, while values close to -1 represent poor air quality and values close to 1 represent good air quality. Conceptually, the data capture the natural variability of air quality due to various environmental factors, such as traffic patterns, weather conditions and industrial activities.

The data are as follows: 0.877, -0.803, 0.623, -0.132, 0.063, 0.282, -0.725, 0.452, -0.290, 0.114, -0.029, -0.520, 0.711, -0.775, 0.906, 0.106, -0.506, 0.194, -0.744, 0.549, -0.339, 0.186, -0.263, 0.109, -0.888, 0.739, -0.591, 0.731, -0.253, 0.533, -0.037, 0.307, -0.205, 0.122, -0.106, 0.858, -0.760, 0.586, -0.723, 0.280, -0.496, 0.037, -0.236, 0.159, -0.253, 0.105, -0.901, 0.709, -0.590, 0.725, -0.298, 0.432, -0.052, 0.347, -0.194, 0.297, -0.108.

To summarize these data, the minimum is -0.926, the first quartile is -0.194, the median is 0.002, the mean is -0.005122, the third quartile is 0.199, and the maximum is 0.920.

This information indicates some symmetry in the distribution of the data. The main quantile features of the data are illustrated by a boxplot in Figure 5.

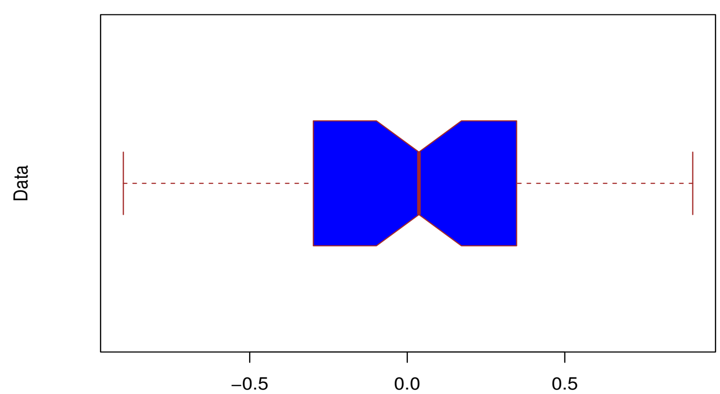

FIGURE 5. Boxplot of the data of the simulated example 1

This boxplot confirms a certain symmetry of the data, centered around 0, with no outliers. The ESC distribution may therefore be appropriate for their analysis.

Using the ML estimation method, we obtain $\hat{a} = 0.5688681$ and $\hat{b} = 1$. From these estimates, we derive the estimated pdf $\hat{f}_*(x)$. Note that the high value of $\hat{b}$ makes the estimated pdf of the

ESC distribution really different from that of the C distribution. We also find $AIC = 77.49385$ and $BIC = 81.57996$.

The AC distribution and the ML estimation method are used for comparison. The estimates of the parameters involved in the AC distribution are $\hat{\alpha} = 0.36297403$ and $\hat{\beta} = 0.02174637$, and we obtain $AIC = 79.11784$ and $BIC = 83.20395$. Since the AIC and BIC associated with the ESC distribution are lower than those associated with the AC distribution, the ESC distribution provides the best fit to the data. As a visual check, in Figure 6, we plot the histogram of the data and overlay the estimated pdfs of the ESC and AC distributions.

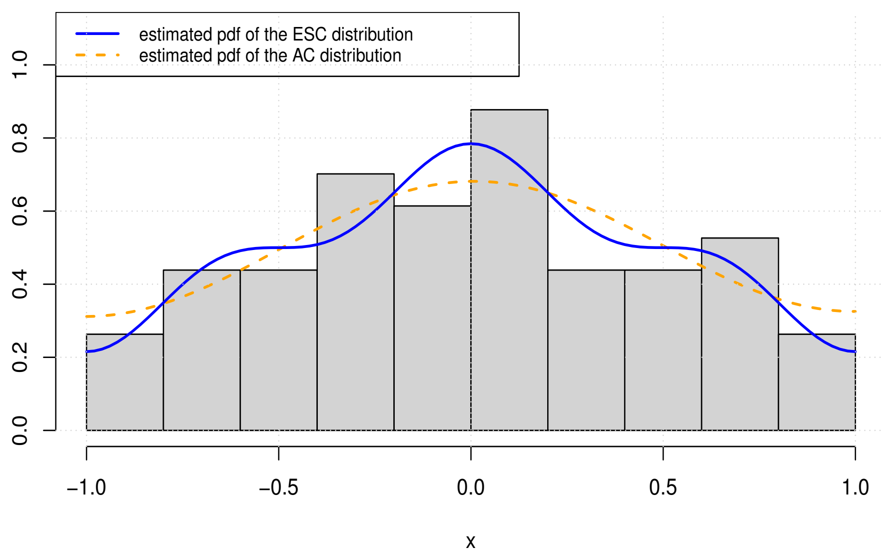

Figure 6. Histogram of the data of the simulated example 1 and estimated pdfs of the ESC and AC distributions

The normalized histogram is well fitted by the two estimated pdfs, but that of the ESC distribution seems to be closer to the overall shape thanks to oscillatory features. This follows the results interpreted from the values of the AIC and BIC.

4.3. **Simulated example 2.** This example simulates a different scenario. We consider a series of measurements taken over time from a sensor array monitoring water quality parameters in a river. There are 49 values, ranging from -1 to 1, reflecting the variation in a particular water quality parameter. Most values are centered around 0, due to the overall stability of the condition of the river, but with a complex symmetry feature, including a slight asymmetry caused by occasional fluctuations due to external factors such as run-off, pollution or weather changes.

The data are as follows: 0.096, -0.590, 0.811, -0.194, -0.309, 0.593, -0.926, 0.378, -0.635, 0.285, 0.116, -0.382, 0.439, 0.020, -0.682, 0.724, -0.771, 0.920, -0.048, -0.121, 0.199, -0.251, 0.445, -0.599, 0.536, -0.807, 0.899, -0.797, -0.044, 0.030, 0.055, -0.035, -0.001, -0.064, 0.072, -0.073, 0.131, 0.002, 0.150, -0.140, 0.217, -0.128, 0.192, -0.167, 0.186, -0.136, 0.264, -0.274, 0.163.

To summarize these data, the minimum is -0.901, the first quartile is -0.298, the median is 0.037, the mean is 0.005649, the third quartile is 0.347, and the maximum is 0.906. This information indicates some symmetry in the distribution of the data. As for the previous example, the main quantile features of the data are illustrated by a boxplot in Figure 7.

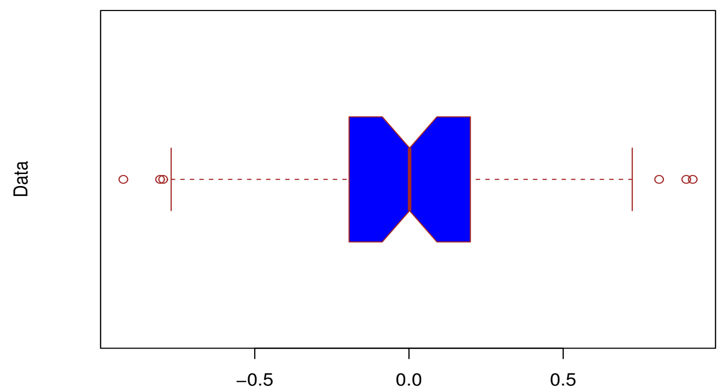

Figure 7. Boxplot of the data of the simulated example 2

There are some outliers in this boxplot, but also some symmetry. Again, the ESC distribution may be able to analyze these data.

Using the ML estimation method for the ESC distribution, we obtain $\hat{a} = 0.7993845$ and $\hat{b} = 1$. From these estimates, we derive the estimated pdf $\hat{f}_*(x)$. Note that the high value of $\hat{b}$ makes the estimated pdf of the ESC distribution really different from that of the C distribution. We also find $AIC = 57.03759$ and $BIC = 60.82123$.

For comparison, we look at the AC distribution. We also use the ML estimation method. The corresponding estimates of the parameters are $\hat{\alpha} = 0.65030333$ and $\hat{\beta} = -0.02541488$, and we get $AIC = 58.46521$ and $BIC = 62.24885$. Since the AIC and BIC associated with the ESC distribution are lower than those associated with the AC distribution, the ESC distribution provides the best fit for the data. To support this claim, in Figure 8, we show the histogram of the data and overlay the estimated pdfs of the ESC and AC distributions.

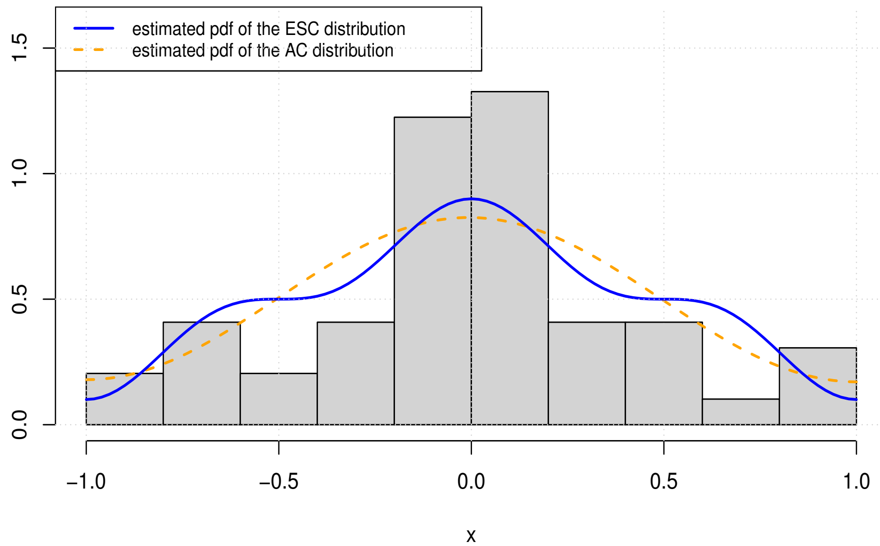

Figure 8. Histogram of the data of the simulated example 2 and estimated pdfs of the ESC and AC distributions

We see that both estimated pdf fit the general shape of the normalized histogram. The advantage of the pdf of the ESC distribution is that it better captures the bars around $x = 0$, and also the original variability of the data represented by the other bars.

It is worth noting that during our statistical investigations, other simulated data examples were produced and it results that the ESC distribution was not uniformly the best compared to the AC distribution. This is mainly due to the fact that the ESC distribution is only designed to analyze data with a symmetric distribution, whereas the AC distribution offers more possibilities to capture different skewnesses and is clearly more appropriate in some scenarios. If the normalized histogram of the data is symmetric in shape, the ESC distribution is more recommended. If there is skewness, the AC distribution is preferable.

5. Conclusion

In this article, we have developed a new symmetric extension of the famous C distribution, called the ESC distribution. In addition to being symmetric, it has the property of having the support $[-1, 1]$, while maintaining a high degree of flexibility thanks to two adjustable parameters. In particular, one of them activates an original sine term in the main functions. This results in some oscillations in the pdf and hrf, which can be useful in different data fitting scenarios. We have investigated some of the main properties of the ESC distributions. In particular, a moment analysis expresses several key measures in closed form, including the mean, variance, skewness, kurtosis, incomplete mean and mean deviation. A number of transformed ESC distributions with different supports were also defined, together with a new skewed version of the standard normal distribution. The statistical aspects of the ESC distribution were then examined using the ML estimation of the two parameters and two examples of simulated data. For these examples, the fit of the new distribution is satisfactory and outperforms that of another two-parameter extended (mainly asymmetric) version of the C distribution.

Thus, this article provides a valuable extension of the C distribution and opens some perspectives for new statistical models with trigonometric properties. The ESC distribution can also be used as it is in different regression contexts, work that deserves a complete study, which we leave to future research.

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