Eur. J. Math. Anal. 5 (2025) 4
doi: 10.28924/ada/ma.5.4

Correspondences Among Inner Functions, Functions With Non-Negative Real Parts and Conformal Mappings

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Abstract. We study an interesting family of dynamical systems on the set of the singular inner functions (defined on the unit disk). Starting with an inner function $S_0(z)$, we obtain new singular inner functions $S_1(z), S_2(z), \ldots$. This sequence converges to a holomorphic self-map of the unit disk which we call S. The convergence is proved with the aid of a fixed-point theorem, a special case of the Earle-Hamilton Theorem. The function S itself is not a singular inner function as $zS(z)$ is a conformal map. This conformal map has the surprising property that its inverse (which is a priori defined on a proper subset of the disk) extends to the entire disk. The motivating question for this research is whether z times a singular inner function can have an omitted value in the unit disk. This question appears within a book on the Krzyz problem written by the author. This question is still open.

1. Correspondences that involve inner functions

Let us recall few correspondences that involve inner functions in $\mathbb{H}^{\nvDash}(\mathbb{U})$. We denote the unit disk in $\mathbb{C}$ by U. We will denote the (multiplicative) group of inner functions in $\mathbb{H}^{\infty}(\mathbb{U})$ by Inn. Its subgroup which contains all the singular inner functions will be denoted by SInn. We follow the notations in, [2]. Finally, the (additive) group of holomorphic functions in U which have non-negative real parts and which have finite radial limits almost everywhere on $\mathbb{T}$ which are purely imaginary will be denoted by RP. Later on we will add one condition to this definition of RP but for now this definition suffices. Here are a few elementary facts that are well known:
(1) $f \in \text{RP} \Leftrightarrow \exists\, w \in \text{Inn}$ such that $f = \frac{1+w}{1-w}$.
This is a bijection since

$$f = \frac{1+w}{1-w} \Leftrightarrow f \cdot (1-w) = 1+w \Leftrightarrow w \cdot (f+1) = f-1 \Leftrightarrow w = \frac{f-1}{f+1}.$$

(2) $g \in \text{SInn} \Leftrightarrow \exists\, f \in \text{RP}$ such that $g = \exp(-f)$.
The correspondence $\text{RP} \to \text{SInn}$, $f \to g$ is not one-to-one. The kernel is $2\pi i\mathbb{Z}$.

Received: 13 Aug 2024.
Key words and phrases. singular inner functions; conformal mapping; Krzyz problem; complex dynamical system; Earle-Hamilton fixed-point theorem; Löwner equation.

(3) $g \in \mathrm{SInn} \Leftrightarrow \exists\, w \in \mathrm{Inn}$ such that $g = \exp\left(-\frac{1+w}{1-w}\right)$.

The correspondence $\mathrm{Inn} \to \mathrm{SInn}$, $w \to g$ is not one-to-one.

$$\frac{1+w}{1-w} + 2\pi ik = \frac{1+v}{1-v} \Leftrightarrow v = \left(\frac{1+w}{1-w} + 2\pi ik - 1\right) \Big/ \left(\frac{1+w}{1-w} + 2\pi ik + 1\right) =$$

$$= \frac{\pi ik + (1-\pi ik)w}{(1+\pi ik) - \pi ikw}.$$

So if we denote by $\phi_k(z)$ the fractional linear function

$$\phi_k(z) = \frac{\pi ik + (1-\pi ik)z}{(1+\pi ik) - \pi ikz}$$

and if we denote $M: \mathrm{Inn} \to \mathrm{SInn}$, $M(w) = g$ where

$$M(w) = \exp\left(-\frac{1+w}{1-w}\right),$$

then $M^{-1}(g) = \{\phi_k(w) \mid k \in \mathbb{Z}\}$.

2. An example of our construction

It will be convenient to first demonstrate the construction on a particular case where concrete computations are possible. This construction was motivated by a problem that appeared in the book, [4]: **let $S(z)$ be a singular inner function ($S \in \mathrm{SInn}$). Is it true that the inner function $z \cdot S(z)$ is a surjection $U \to U$?**

Theorem 2.1. *Let $\{S_n(z)\}_{n=0}^{\infty}$ be a sequence of singular inner functions defined recursively by: $S_0 \in \mathrm{SInn}$ (an arbitrary initial point),*

$$S_{n+1}(z) = \exp\left(-\frac{1+z\cdot S_n(z)}{1-z\cdot S_n(z)}\right) \quad \text{for} \quad n \in \mathbb{Z}_{\geq \nvdash}.$$

Then $\lim_{n\to\infty} S_n = S$ uniformly on compact subsets of U. $S(z)$ is in $\mathbb{H}^\infty(\mathbb{U})$ and it satisfies the fixed-point equation

$$S = \exp\left(-\frac{1+z\cdot S}{1-z\cdot S}\right).$$

Also the mapping $z \cdot S(z) \in \mathbb{H}^\infty(\mathbb{U})$ is injective $U \to \mathrm{Im}(z \cdot S(z)) \subset U$ but it can not be an inner function.

Proof.

Since $S_0 \in \mathrm{SInn}$ and since an inductive argument shows that if $S_n \in \mathrm{SInn}$ then

$$S_{n+1} = \exp\left(-\frac{1+z\cdot S_n}{1-z\cdot S_n}\right) \in \mathrm{SInn} \ \text{for} \ n \in \mathbb{Z}_{\geq \nvdash},$$

it follows that the sequence $\{S_n(z)\}_{n=0}^{\infty}$ is a sequence of singular inner functions. The family of functions in the sequence is a normal family. Even more, for a fixed-point $z \in U$ the function of $t \in U$ given by

$$\exp\left(-\frac{1+z\cdot t}{1-z\cdot t}\right),$$

is a contraction and so by the fixed-point theorem of S. Banach iterations of this contraction converge to a unique fixed-point $S(z)$. So $\lim_{n\to\infty} S_n = S$ uniformly on compact subsets of U, and $S(z)$ satisfies the fixed-point equation

$$S(z) = \exp\left(-\frac{1 + z \cdot S(z)}{1 - z \cdot S(z)}\right).$$

Clearly $S(z)$ is a non-vanishing function in $\mathbb{H}^\infty(\mathbb{U})$. Next, let us consider the following holomorphic function of w, defined on the once punctured plane as follows:

$$f : \mathbb{C} - \{1\} \to \mathbb{C}, \quad f(w) = w \exp\left(\frac{1+w}{1-w}\right).$$

Then by the fixed-point equation satisfied by $S(z)$ we get $f(z \cdot S(z)) = z$. So f is a left inverse of $z \cdot S(z)$ and hence $z \cdot S(z) : U \to \mathrm{Im}(z \cdot S(z))$ is an injection. More concretely, if we denote $g(z) = z \cdot S(z)$ then the assumption $g(z_1) = g(z_2)$ implies that $z_1 = f(g(z_1)) = f(g(z_2)) = z_2$. Since $S(z)$ can not be a constant function (by the fixed-point equation), $z \cdot S(z)$ can not be an inner function (see [3], remarked by Raymond Mortini). □

3. A generalization

Definition 3.1. We will denote by RP, the family of all the $F \in H(U)$, that satisfy the following four conditions:
(i) $\Re F(z) \geq 0$, $\forall z \in U$.
(ii) $\Re F(e^{i\theta}) = 0$ almost everywhere on $\mathbb{T}$ with respect to the Lebesgue measure on $\mathbb{T}$.
(iii) The function of $t \in U$ given by $\exp(-F(z \cdot t))$ is a contraction (with respect to the Euclidean metric) where $z \in U$ is fixed.
(iv) F is a non-constant function.

Remark 3.2. $\forall F, G \in \mathrm{RP}$, $\forall a, b \in \mathbb{R}_{\geq 0}$, such that $0 < a + b \leq 1$, we have $a \cdot F + b \cdot G \in \mathrm{RP}$. Also if $F(z) = \frac{1+w(z)}{1-w(z)}$, where, as always $w(z) \in \mathrm{Inn}$, then $F'(z) = \frac{2w'(z)}{(1-w(z))^2}$. By

$$\frac{d}{dt}\exp(-F(z\cdot t)) = -zF'(z \cdot t)\exp(-F(z \cdot t)),$$

it follows by (iii)

$$\left| z\frac{2w'(z\cdot t)}{(1-w(z\cdot t))^2}\exp(-F(z\cdot t))\right| \leq c < 1.$$

In particular we obtain that the generating inner function $w(z)$ of $F(z)$ satisfies:

$$|z|\frac{|w'(z\cdot t)|}{|1-w(z\cdot t)|^2}\exp\left(-\frac{1-|w(z\cdot t)|^2}{|1-w(z\cdot t)|^2}\right) \leq \frac{c}{2} < \frac{1}{2}.$$

Thus we conclude that $\forall z \in U$ and $\forall w \in \mathrm{Inn}$, such that $\frac{1+w}{1-w} \in \mathrm{RP}$ we have:

$$|z|\frac{|w'(z)|}{|1-w(z)|^2}\exp\left(-\frac{1-|w(z)|^2}{|1-w(z)|^2}\right) < \frac{1}{2}.$$

<u>The construction.</u> Let $F \in \mathrm{RP}$. We will use $F(z)$ to define a sequence $\{S_n(z)\}_{n=0}^{\infty}$ of singular inner functions. The definition will use the following recursion: $S_0 \in \mathrm{SInn}$ (an arbitrary initial point).

$$S_{n+1}(z) = \exp\left(-F(z \cdot S_n(z))\right) \text{ for } n \in \mathbb{Z}_{\geq \nvdash}.$$

Theorem 3.3. *The limit* $\lim_{n\to\infty} S_n(z) = S(z)$ *exists and is uniform on compact subsets of* U. $S \in H(U)$ *satisfies* $|S(z)| \leq 1\,\forall\, z \in U$, *and satisfies the following fixed-point equation,* $S = \exp(-F(z \cdot S))$.
The function $z \cdot S(z) \in \mathbb{H}^\infty(\mathbb{U})$ *is a conformal mapping* $z \cdot S(z) : U \to \mathrm{Im}(z \cdot S) \subseteq U$ *but it is not an inner function.*

Proof.
Since $S_0 \in \mathrm{SInn}$ and since an inductive argument shows that if $S_n \in \mathrm{SInn}$, then $S_{n+1} = \exp(-F(z \cdot S_n)) \in \mathrm{SInn}$ for $n \in \mathbb{Z}_{\geq \nvdash}$, it follows that all the members of the sequence $\{S_n\}_{n=0}^{\infty}$ belong to SInn. The reason for the validity of the inductive argument is that $|z \cdot S_n| = |z||S_n| \leq |z|$ for all $z \in U$, using the induction hypothesis $S_n \in \mathrm{SInn}$. Thus $z \cdot S_n \in \mathbb{B}_{\mathbb{H}^\infty}$, the unit ball of $\mathbb{H}^\infty$. Also $|e^{i\theta} \cdot S_n(e^{i\theta})| = 1$ almost everywhere on $\mathbb{T}$ with respect to the Lebesgue measure on $\mathbb{T}$. Also this follows by the induction hypothesis on S_n. Hence $\Re F(z \cdot S_n(z)) \geq 0\ \forall\, z \in U$ and $\Re\left(e^{i\theta} \cdot S_n(e^{i\theta})\right) = 0$ almost everywhere on $\mathbb{T}$ (recall that $F \in \mathrm{RP}$ satisfies by the definition $\Re F(e^{i\theta}) = 0$ almost everywhere on $\mathbb{T}$). Hence

$$|\exp\left(-F(z \cdot S_n(z))\right)| = \exp\left(-\Re F(z \cdot S_n(z))\right) \leq 1\ \ \forall\, z \in U$$

and also

$$\left|\exp\left(-F(e^{i\theta} \cdot S_n(e^{i\theta}))\right)\right| = 1 \text{ almost everywhere on } \mathbb{T}.$$

We just proved that $S_{n+1}(z) = \exp\left(-F(z \cdot S_n(z))\right) \in \mathrm{SInn}$ for $n \in \mathbb{Z}_{\geq \nvdash}$. Hence the family of functions in the sequence $\{S_n\}_{n=0}^{\infty}$ is a normal family. Moreover, by condition (iii) in Definition 3.1, for a fixed $z \in U$ iterations of the function of $t \in U$ given by $\exp(-F(z \cdot t))$ converge (by Banach fixed-point theorem) to a unique fixed-point $t_0 = S(z)$. So $\lim_{n\to\infty} S_n(z) = S(z)$ uniformly on compact subsets of U, and $S(z)$ satisfies the fixed-point equation $S(z) = \exp(-F(z \cdot S(z)))$. Clearly, the $\mathbb{H}^\infty(\mathbb{U})$ function is a non-vanishing function that belongs to the unit ball $\mathbb{B}_{\mathbb{H}^\infty(\mathbb{U})}$. Next, let us consider the following holomorphic function of w, defined on U as follows: $f : U \to \mathbb{C}$, $\mho(\lessapprox) = \lessapprox \exp(\mathbb{F}(\lessapprox))$. Then by the fixed-point equation satisfied by $S(z)$ we get: $f(z \cdot S(z)) = z$. The reason is that

$$f(z \cdot S(z)) = z \cdot S(z) \exp(F(z \cdot S(z))) = z \cdot S(z) \cdot S(z)^{-1} = z.$$

Thus f is a left is a left inverse of $z \cdot S(z)$ and hence the mapping: $z \cdot S(z) : U \to \mathrm{Im}(z \cdot S(z))$ is an injection. More concretely, if we denote $g(z) = z \cdot S(z)$ then the assumption $g(z_1) = g(z_2)$ implies that $z_1 = f(g(z_1)) = f(g(z_2)) = z_2$. The function $S(z)$ can not be a constant function, for if $S(z) = e^{i\theta_0}$, then

$$e^{i\theta_0} = \exp\left(-F(e^{i\theta} \cdot S(e^{i\theta_0}))\right) = \exp\left(-F(e^{i(\theta+\theta_0)})\right).$$

But any function such as F in RP is non-constant by the definition. Since the only injective inner functions are Blaschke factors, [3], $z \cdot S(z)$ can not be an inner function (for that would imply that $S(z)$ is a unimodular constant). □

4. A parametrization of a family of conformal mappings by the functions in RP

We note that we may replace $F(z)$ by any of the functions in the sequence $F(z) + 2\pi i\mathbb{Z}$. All of these are members of RP that will generate $S(z)$ just as $F(z)$ does. However, if $G(z) - F(z) \notin 2\pi i\mathbb{Z}$ and $G(z)$ (like $F(z)$) belongs to RP, then $\exp(-F)$ and $\exp(-G)$ are different holomorphic functions. Can they share the same fixed-point $S(z)$? That is, can the following be true?

$$S(z) = \exp\left(-F(z \cdot S(z))\right) = \exp\left(-G(z \cdot S(z))\right) \quad z \in U.$$

By the Permanence Principle for holomorphic functions this holds if and only if $G - F \in 2\pi i\mathbb{Z}$ (which is not the case). So the assignment: $[F] := F + 2\pi i\mathbb{Z} \to \mathbb{S}$ is an injection of the quotient space of RP, namely of $\mathrm{RP}/2\pi i\mathbb{Z}$ onto the family of functions $S_{[F]}(z)$ in the unit ball of $\mathbb{H}^\infty(\mathbb{U})$, such that $z \cdot S_{[F]}(z) : U \to \mathrm{Im}(z \cdot S_{[F]}(z)) \subseteq U$ is a conformal mapping, where $S_{[F]}$ is the usual $F(z)$ fixed-point: $S_{[F]} = \exp(-F(z \cdot S_{[F]}(z)))$. Here (in the notation of section 3) $[F]$ stands for any of the members of the equivalence class $[F]$ in $\mathrm{RP}/2\pi i\mathbb{Z}$. We got our parametrization that the title of this section refers to. $[F]$ determines the conformal mapping $z \cdot S_{[F]}$ via the fixed-point equation. So $[F]$ is the parameter of the conformal mapping $z \cdot S_{[F]}(z)$. If we define the conformal mapping by $w = f_{[F]}(z) = z \cdot S_{[F]}(z)$, then $f_{[F]} : U \to \mathrm{Im}(f_{[F]})$, is invertible, so that $z = f^{-1}_{[F]}(w)$. Using the fixed-point equation: $-\log S_{[F]} = F(z \cdot S_{[F]})$ we see that $S_{[F]}$ determines its parameter $[F]$ by:

$$F(w) = -\log\left(\frac{w}{z}\right) = -\log\left(\frac{w}{f^{-1}(w)}\right).$$

The family RP is algebraically easy to understand, unlike the family of the conformal mappings:

$$\left\{S_{[F]} \mid [F] \in \mathrm{RP}/2\pi i\mathbb{Z}\right\}.$$

For example, see our Remark 3.2: RP is closed for taking linear combinations with coefficients $a, b \in \mathbb{R}_{\geq \not{\vdash}}$, such that $0 < a+b \leq 1$, i.e. $\forall\, F, G \in \mathrm{RP}(\text{or } \mathrm{RP}/2\pi i\mathbb{Z})$, $a \cdot F + b \cdot G \in \mathrm{RP}(\text{or } \mathrm{RP}/2\pi i\mathbb{Z})$. Geometrically we are dealing with cones. Let (as usual) $S_{[F]} = \exp(-F(z \cdot S_{[F]}))$, $S_{[G]} = \exp(-G(z \cdot S_{[G]}))$, so that $z \cdot S_{[F]} : U \to \mathrm{Im}(z \cdot S_{[F]})$, $z \cdot S_{[G]} : U \to \mathrm{Im}(z \cdot S_{[G]})$ are conformal mappings.Then we make the following:

Definition 4.1. $\forall\, a, b \in \mathbb{R}_{\geq \not{\vdash}}$, $0 < a+b \leq 1$ we define the conic linear combination by the equation:

$$a \cdot (z \cdot S_{[F]}) \hat{+} b \cdot (z \cdot S_{[G]}) = z \cdot S_{[a \cdot F + b \cdot G]}.$$

This definition (and a similar one for multiplication by a real non-negative scalar) induces on the family of our conformal mappings the same conic structure as the one we easily have on RP (or $\mathrm{RP}/2\pi i\mathbb{Z}$).

Remark 4.2. It is well known that there exist natural and elementary parametrizations between RP and SInn and also between Inn and SInn. See the explanations in Section 1. Thus, our less elementary parametrization of the family of conformal mappings $z \cdot S_{[F]}(z)$ by RP$/2\pi i\mathbb{Z}$ can now be related to the tree of correspondences among SInn, Inn and RP.

We naturally inquire as to what are the conformal members that form the family of conformal mappings in the tree of correspondences. We will deal with that on the next section. As expected a main ingredient of these conformal mappings will be the boundary behaviour of the inverse conformal mappings $\mathrm{Im}(z \cdot S_{[F]}) \to U$.

5. The family of conformal mappings CONF

Our family of conformal mappings is clearly given by the following:

Definition 5.1.

$$\mathrm{CONF} = \left\{ z \cdot S_{[F]} \in H(U) \,\middle|\, [F] \in \mathrm{RP}/2\pi i\mathbb{Z},\ \mathbb{S}_{[\mathbb{F}]}(\digamma) = \exp\left(-\mathbb{F}(\digamma \cdot \mathbb{S}_{[\mathbb{F}]}(\digamma))\right),\ \forall \digamma \in \mathbb{U} \right\}.$$

We recall the following facts:
(1) $z \cdot S_{[F]} : U \to \mathrm{Im}(z \cdot S_{[F]})$ is an injection.
(2) $z \cdot S_{[F]} \in \mathbb{B}_{\mathbb{H}^\infty(\mathbb{U})} - \mathrm{Inn}$. ([3]).
(3) If $w = f_{[F]}(z) = z \cdot S_{[F]}(z)$, $(z \in U)$, then

$$F(w) = -\log\left(\frac{w}{z}\right) = -\log\left(\frac{w}{f_{[F]}^{-1}(w)}\right),\quad w \in \mathrm{Im}(z \cdot S_{[F]}(z)).$$

We would like to characterize the conformal family CONF without any reference to the family of the parameters RP$/2\pi i\mathbb{Z}$.

Theorem 5.2. *The family* CONF *consists of all the holomorphic functions $f(z) \in H(U)$ that satisfy the following:*
(a) $f : U \to \mathrm{Im}(f) \subseteq U$ is a conformal mapping.
(b) $f(0) = 0$.
(c) The function:

$$F(w) = -\log\left(\frac{w}{f^{-1}(w)}\right),\quad w \in \mathrm{Im}(f),$$

can be analytically be defined on all of U (not just on $\mathrm{Im}(f)$), and it satisfies $\Re F(w) \ge 0$ for all $w \in \mathrm{Im}(f)$ and $\Re F(f^{-1}(w)) = 0$ for all $w \in \partial\,\mathrm{Im}(f)$. (d) For a fixed $z \in U$, the function $\frac{z \cdot t}{f^{-1}(z \cdot t)}$ is a contraction in $t \in U$ for which $z \cdot t \in \mathrm{Im}(f)$. It is a contraction with respect to the Euclidean metric.

Proof.
Let us denote the family of all the mappings $f \in H(U)$ that satisfy (a), (b), (c) and (d) by A. We need to prove that CONF = A where the definition of CONF is given in Definition 5.1.

(I) CONF $\subseteq$ A:

Let $f_{[F]}(z) = z \cdot S_{[F]}(z) \in$ CONF. By fact (1) after Definition 5.1, $f_{[F]} : U \to \mathrm{Im}(f_{[F]}) \subseteq U$ is an injection. Clearly $f_{[F]}(0) = 0$. By fact (3) after Definition 5.1,

$$F(w) = -\log\left(\frac{w}{f_{[F]}^{-1}(w)}\right), \ \forall\, w \in \mathrm{Im}(f).$$

Finally, by Definition 3.1 (iii) we have: (iii) The function of $t \in U$ given by $\exp(-F(z \cdot t))$ is a contraction (with respect to the Euclidean metric) where $z \in U$ is fixed. But $\exp(-F(z \cdot t)) = \frac{z \cdot t}{f^{-1}(z \cdot t)}$ which proved (d) and completes the proof of CONF $\subseteq$ A.

(II) A $\subseteq$ CONF:

Let $f : U \to \mathrm{Im}(f) \subseteq U$ be an elelment of A. Then $f(0) = 0$ and $|f(z)| \le 1$ for all $z \in U$. By the Schwarz Lemma we get $|f(z)| \le |z|$ for all $z \in U$. Hence for all $w \in U$ where $w = f(z)$ we have

$$\left|\frac{w}{z}\right| \le 1 \text{ so } -\log\left|\frac{w}{z}\right| \ge 0.$$

If we define

$$F_f(w) = -\log\left(\frac{w}{z}\right) = -\log\left(\frac{w}{f^{-1}(w)}\right),$$

then $\Re F_f(w) \ge 0$. Now (ii) and (iii) in Definition 3.1 follow. Hence $f \in$ CONF. $\square$

Here is an interesting consequence on the conformal mappings of the family CONF:

Corollary 5.3. *Let $f, g \in$ CONF and let $a, b \in \mathbb{R}_{\geq \not{0}}$, $0 < a + b \le 1$. Then $\exists\, h_{a,b} \in$ CONF such that we have the following multiplicative relation among these three conformal mappings:*

$$\left(\frac{h_{a,b}^{-1}(w)}{w}\right) = \left(\frac{f^{-1}(w)}{w}\right)^a \left(\frac{g^{-1}(w)}{w}\right)^b.$$

Proof.

By the proof of Theorem 5.2 it follows that,

$$f, g \in \text{CONF} \Leftrightarrow -\log\left(\frac{w}{f^{-1}(w)}\right), -\log\left(\frac{w}{g^{-1}(w)}\right) \in \text{RP}.$$

We note that in $f^{-1}(w)$ we have $w \in \mathrm{Im}(f)$ and in $g^{-1}(w)$ we have $w \in \mathrm{Im}(g)$. By property (5) after Definition 5.1 we know that the two-dimensional Lebesgue measures of the sets $U - \mathrm{Im}(f)$ and $U - \mathrm{Im}(g)$ are zero. Hence $\mathrm{Im}(f) \cap \mathrm{Im}(g)$ is an open subset of U and the two-dimensional Lebesgue measure of the set $U - (\mathrm{Im}(f) \cap \mathrm{Im}(g))$ is zero. So $\mathrm{Im}(f) \cap \mathrm{Im}(g)$ is a large open subset of U on which both holomorphic functions

$$-\log\left(\frac{w}{f^{-1}(w)}\right), \text{ and } -\log\left(\frac{w}{g^{-1}(w)}\right),$$

are defined and belong to RP. I.e. these are the restrictions of RP functions to the intersection of the images of the conformal mappings f and g. Since:

$$a\left\{-\log\left(\frac{w}{f^{-1}(w)}\right)\right\} + b\left\{-\log\left(\frac{w}{g^{-1}(w)}\right)\right\} = -\log\left(\frac{w}{f^{-1}(w)}\right)^a \left(\frac{w}{g^{-1}(w)}\right)^b \in \text{RP},$$

it follows by the proof of Theorem 5.2 that $\exists\, h_{a,b} \in$ CONF such that:

$$\left(\frac{w}{f^{-1}(w)}\right)^{a}\left(\frac{w}{g^{-1}(w)}\right)^{b}=\left(\frac{w}{h_{a,b}^{-1}(w)}\right).$$

Our proof is now completed. □

One can conclude more surprising properties on the conformal mappings of the family CONF.

6. The geometry of the image of conformal mappings in CONF

Let $F \in$ RP. We chose an arbitrary singular inner function $S_0(z) \in$ SInn and we generated an infinite sequence of singular inner functions using the following recursion: $S_{n+1}(z) = \exp\left(-F(z \cdot S_n(z))\right)$ for $n \in \mathbb{Z}_{\geq \nvDash}$. The limit $S(z) = \lim_{n\to\infty} S_n(z)$ exists for all $z \in U$. The convergence is uniform on compact subsets of U. $S(z)$ satisfies the fixed-point equation $S(z) = \exp\left(-F(z \cdot S(z))\right)$, $z \in U$. The function $g(z) = z \cdot S(z)$ is a conformal mapping $U \to \mathrm{Im}(g) \subseteq U$. To see that we defined the following holomorphic function defined on U.

$$f: U \to \mathbb{C}, \quad \mho(\precapprox) = \precapprox \exp\left(\mathbb{F}(\precapprox)\right).$$

It easily follows by the fixed-point equation that $f(g(z)) = z$. Thus g has a left inverse f and hence g is injective. That was the first surprise. The sequence $\{S_n(z)\}_{n=1}^{\infty}$ of singular inner functions, each of which covers U infinitely many times, produced the limit $S(z)$ so that $g(z) = z \cdot S(z)$ was injective. The complete opposite behavior. g covers each point of U at most once.
In this section we will describe the image $\mathrm{Im}(g)$ by trying to identify its boundary. At first one might expect a very wild boundary because of the limit above. We will encounter here our second surprise. The boundary $\partial\mathrm{Im}(g)$ will turn out to be completely tame. It will be composed of curves which are the zero sets of certain planar harmonic functions plus very few corners in between the different zero sets. Thus a piecewise smooth closed Jordan curve.

Theorem 6.1. *The image $\mathrm{Im}(g) = \mathrm{Im}(z \cdot S(z))$ of a conformal mapping in* CONF *is a piecewise smooth closed Jordan curve. It is composed of arcs on $\mathbb{T}$ and of arcs which are subsets of the zero set of the planar harmonic function $\Re F(w) + \log|w|$, plus a small number of corners.*

Proof.
Since $f(w) = g^{-1}(w)$ and $g: U \to \mathrm{Im}(g) \subseteq U$ is conformal, in order to to identify $\mathrm{Im}(g)$, we need to identify those arcs in the closure of the unit disk $\overline{U}$ that are mapped by f into the unit circle $\mathbb{T} = \partial\mathbb{U}$. So we want to solve for all $w \in \overline{U}$ that satisfy $|f(w)| = 1$. I.e.

$$|w \exp\left(F(w)\right)| = 1.$$

We recall that $F \in$ RP and hence, by Definition 3.1, condition (ii) we know that $\Re F(e^{i\theta}) = 0$ almost everywhere on $\mathbb{T}$ with respect to the Lebesgue measure on $\mathbb{T}$. Since $|\exp\left(F(w)\right)| = \exp\left(\Re F(w)\right)$ it follows that:

$$\left|e^{i\theta}\exp\left(F\left(e^{i\theta}\right)\right)\right| = 1$$

almost everywhere on $\mathbb{T}$. So we want to solve for all $w \in U$ that satisfy $|f(w)| = 1$. This means, to find all $w \in U$ for which $|w \exp(F(w))| = 1$, that means $|w| \exp(\Re F(w)) = 1$, i.e. those $w \in U$ that satisfy:

$$\Re F(w) = -\log|w|.$$

This equation has harmonic functions on both sides. Alternatively we look for the zero set in U of the planar harmonic function $\Re F(w) + \log|w|$. This proves our theorem. $\square$

In the case of our first example:

$$F(w) = \frac{1+w}{1-w}.$$

In this case:

$$\frac{1+w}{1-w} = \frac{1-|w|^2}{|1-w|^2} + \frac{w-\overline{w}}{|1-w|^2}.$$

Our equation $\Re F(w) + \log|w| = 0$ becomes:

$$\frac{1-|w|^2}{|1-w|^2} + \log|w| = 0.$$

We note that any $w \in \mathbb{T} - \{\nVdash\}$ solves this equation. In particular both $\pm i$ are solutions. We mention those two in particular because we will see soon that they are the two zeros of the derivative of our holomorphic function in interest and hence this function is not injective exactly at those two points. We can obtain the equation of this curve either in Cartesian coordinates: $x = \Re w$, $y = \Im w$, $|w|^2 = x^2 + y^2$.

$$\frac{1-x^2-y^2}{1+x^2+y^2-2x} + \frac{1}{2}\log(x^2+y^2) = 0,$$

or better in polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{1-r^2}{1+r^2-2r\cos\theta} + \log r = 0.$$

Solving for $\cos\theta$ this is:

$$\cos\theta = \frac{1}{2r}\left\{1 + r^2 + \frac{1-r^2}{\log r}\right\}.$$

Remark 6.2. We note that:

$$\lim_{r\to 1^-} \frac{1}{2r}\left\{1 + r^2 + \frac{1-r^2}{\log r}\right\} = 0,$$

so the equation above has exactly two solutions in $[-\pi, \pi]$, and these are $\pm\frac{\pi}{2}$. These correspond to $\pm i$.

The zero set within U is the given by

$$\theta(r) = \cos^{-1}\left\{\frac{1}{2r}\left(1 + r^2 + \frac{1-r^2}{\log r}\right)\right\}.$$

This intersects the x-axis to the right of 0, for $\theta = 0$ so $\cos\theta = 1$:

$$1 = \frac{1}{2r}\left\{1 + r^2 + \frac{1-r^2}{\log r}\right\} \Rightarrow (1-r)\left\{1 - r + \frac{1+r}{\log r}\right\} = 0.$$

One solution is $r = x = 1$ and others we obtain by:

$$1 - x + \frac{1+x}{\log x} = 0 \ \text{ or } \ 1 + x + (1-x)\log x = 0.$$

We easily check that $(1 + x + (1-x)\log x)' > 0$ for $0 < x < 1$ and so there is exactly one solution $x = x_0$ of

$$1 - x + \frac{1+x}{\log x} = 0,$$

in $0 < x < 1$. A similar computation shows that the curve:

$$\frac{1 - x^2 - y^2}{1 + x^2 + y^2 - 2x} + \frac{1}{2}\log(x^2 + y^2) = 0,$$

determines x as a function of y in $[-1, 1]$. It connects $-i = (0, -1)$ to $i = (0, 1)$. If goes through $(x_0, 0)$ and is symmetric with respect to the x-axis. It is strictly monotonic decreasing from $(x_0, 0)$ to $(0, 1)$ and by symmetry with respect to the x-axis it is strictly monotonic increasing from $(0, -1)$ to $(x_0, 0)$. Thus U is divided into two parts by that zero set. The part in U to the left of the curve and the part to the right of that zero set. Since the left part contains the origin (by $x_0 > 0$) it is that left part that is the image of our conformal mapping in this example, that corresponds to the function in RP given by:

$$F(z) = \frac{1+z}{1-z}.$$

7. Combining two dynamical systems

Next we will make use of two dynamical systems. The first is the discrete dynamical system we used above. It is controlled by a simple recursion which is generated by a function in RP. The second is the continuous dynamical system of Löwner type that is controlled by the partial differential equation for $\mathcal{B}$, the class of bounded non-vanishing functions. In fact $\mathcal{B} = \text{SInn}$ the class of the singular inner functions. The notation $\mathcal{B}$ as well as its differential equation were described in Section 2 of the basic paper [1]. The notation SInn was used in [2]. We recall facts from Section 2 of [1]. Suppose $f \in \mathcal{B}$ has the Herglotz representation

$$f(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} h(\theta)d\theta\right),$$

where $h(\theta) \geq 0$. The collection of such functions is dense in the subfamily of $\mathcal{B}$ consisting of functions for which $f(0) > 0$. Changing variable by the substitution $\tau = \tau(\theta) = \int_0^\theta h(\phi)d\phi$, and putting $k(\tau) = e^{i\theta}$ leads to the formula,

$$f(z) = \exp\left(-\int_0^{t_0} \frac{1 + k(\tau)z}{1 - k(\tau)z} d\tau\right), \tag{7.1}$$

where $t_0 = \tau(2\pi) = -\log f(0)$. Conversely, if $k(\tau)$ is a measurable function of τ which satisfies $|k(\tau)| = 1$, $\tau \in \mathbb{R}$, then equation (7.1) defines a function of class $\mathcal{B}$. Given $f(z)$ as in equation (7.1), we set

$$f(z,t) = \exp\left(-\int_0^t \frac{1+k(\tau)z}{1-k(\tau)z} d\tau\right), \quad 0 \leq t \leq t_0. \tag{7.2}$$

Then $f(z,t) \in \mathcal{B}$ for all $t \in [0, t_0]$, $f(z, t_0) = f(z)$, and $f(z, 0) = 1$. It follows from equation (7.2) that for almost all t,

$$\frac{\partial f(z,t)}{\partial t} = -f(z,t) \cdot \frac{1+k(t)\cdot z}{1-k(t)\cdot z}. \tag{7.3}$$

This is the differential equation for $\mathcal{B}$.

We recall our discrete dynamical system:

Let $G \in \mathrm{RP}$. We will use the function $G(z)$ to generate a sequence $\{S_n\}_{n=0}^{\infty}$ of singular inner functions. It is controlled by the following recursion, $S_0(z) \in \mathrm{SInn}$ (an arbitrary initial point).

$$S_{n+1}(z) = \exp\left(-G(z \cdot S_n(z))\right) \text{ for } n \in \mathbb{Z}_{\geq \nvdash}. \tag{7.4}$$

We proved in Theorem 3.3, the following:

The limit $S(z) = \lim_{n\to\infty} S_n(z)$ exists and is uniform on compact subsets of U. $S \in H(U)$ satisfies $|S(z)| \leq 1\ \forall z \in U$, and satisfies the following fixed-point equation, $S(z) = \exp\left(-G(z \cdot S(z))\right)$. The function $z\cdot S(z) \in \mathrm{B}_{\mathbb{H}^\infty(\mathbb{U})}$, the unit ball of $\mathbb{H}^\infty(\mathbb{U})$. $z \cdot S(z)$ is a conformal mapping (it belongs to CONF). Thus $z \cdot S(z) : U \to \mathrm{Im}(z \cdot S(z)) \subseteq U$, but it is not an inner function, see for example [3].

Let us denote the following correspondence by F:

$$F : \mathrm{SInn} \to \mathrm{CONF}, \quad F(S_0) = z \cdot S(z).$$

One result that we will demonstrate below is that the correspondence F is, in fact, a constant. We will give two different proofs for that result. This result might seem to be surprising at first. But it is not really surprising.

Remark 7.1. We clearly have $\forall\, n \in \mathbb{Z}_{\geq \nvdash}$, $F(S_n) = z \cdot S(z)$. So the correspondence F is certainly constant on the sequence $\{S_n(z)\}_{n=0}^{\infty}$ which is the output of our recursion, that generates the discrete dynamical system. So we can view F as a correspondence $\mathrm{SInn}/\{\{S_n(z)\}_{n=0}^{\infty}\} \to \mathrm{CONF}$. However, since we will prove that F is a constant correspondence (given a $G \in \mathrm{RP}$) we will conclude that the truly interesting correspondence is not $\mathrm{SInn} \to \mathrm{CONF}$, but is $T : \mathrm{RP} \to \mathrm{CONF}$, $T(G(z)) = z \cdot S(z)$.

We combine the continuous dynamical system that was described in equation (7.3), with our G-discrete dynamical system ($G \in \mathrm{RP}$) that was described in equation (7.4), as follows:

$$F : \{f(z,t)\,|\,0 \leq t \leq t_0\} \to \mathrm{CONF}, \quad F(f(z,t)) = z \cdot S(z,t).$$

Here the starting point of the recursion is $S_0(z,t) = f(z,t)$ and $S_{n+1}(z,t) = \exp\left(-G(z \cdot S_n(z,t))\right)$ for $n \in \mathbb{Z}_{\geq \nvdash}$. $S(z,t) = \lim_{n\to\infty} S_n(z,t)$ for $z \in U$ (as was mentioned above), also $S(z,t) = \exp\left(-G(z \cdot S(z,t))\right)$ for $z \in U$, and $z \cdot S(z,t) \in \mathrm{CONF}$ for each $0 \leq t \leq t_0$.

8. More results

Those results will be summarized in three theorems and one corollary. We begin with the corresponding computations. By the differential equation for $\mathcal{B}$, in equation (7.3) and by the recursion, in equation (7.4) we have,

$$\frac{\partial S_1(z,t)}{\partial t} = \frac{\partial}{\partial S_0}\{\exp(-G(z\cdot S_0))\}\cdot\frac{\partial S_0(z,t)}{\partial t} =$$
$$= S_1(z,t)\cdot\left\{-z\cdot\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_0(z,t)}\right\}\cdot\frac{\partial S_0(z,t)}{\partial t} =$$
$$= S_1(z,t)\cdot\left\{-z\cdot\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_0(z,t)}\right\}\cdot\left\{-S_0(z,t)\cdot\frac{1+k(t)z}{1-k(t)z}\right\} =$$
$$= S_0(z,t)\cdot S_1(z,t)\cdot z\cdot\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_0(z,t)}\cdot\left\{\frac{1+k(t)z}{1-k(t)z}\right\}.$$

Next,

$$\frac{\partial S_2(z,t)}{\partial t} = \frac{\partial}{\partial S_1}\{\exp(-G(z\cdot S_1))\}\cdot\frac{\partial S_1(z,t)}{\partial t} =$$
$$= S_2(z,t)\cdot\left\{-z\cdot\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_1(z,t)}\right\}\cdot\frac{\partial S_1(z,t)}{\partial t} =$$
$$= S_2(z,t)\cdot\left\{-z\cdot\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_1(z,t)}\right\}\cdot S_0(z,t)\cdot S_1(z,t)\cdot z\cdot\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_0(z,t)}\cdot\left\{\frac{1+k(t)z}{1-k(t)z}\right\} =$$
$$= -S_0(z,t)\cdot S_1(z,t)\cdot S_2(z,t)\cdot z^2\cdot\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_0(z,t)}\cdot\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_1(z,t)}\cdot\frac{1+k(t)z}{1-k(t)z}.$$

Inductive arguments prove:

Theorem 8.1.

$$\frac{\partial S_n(z,t)}{\partial t} = (-1)^{n+1}\cdot\left\{\prod_{j=0}^{n} S_j(z,t)\right\}\cdot z^n\cdot\left\{\prod_{j=0}^{n-1}\left(\frac{\partial G(w)}{\partial w}|_{w=z\cdot S_j(z,t)}\right)\right\}\cdot\left\{\frac{1+k(t)z}{1-k(t)z}\right\}.$$

Theorem 8.2. *There exists a unique $S(z)\in H(U)$ such that it is the only fixed-point of the function $\exp(-G(z\cdot w))$, i.e. $S(z)=\exp(-G(z\cdot S(z)))$. Moreover, $\forall\, S_0(z)\in \mathrm{SInn}$, the recursion $S_{n+1}(z)=\exp(-G(z\cdot S_n(z)))$, $n\in\mathbb{Z}_{\geq \not{\prime}}$, defines a sequence of singular inner functions $\{S_n(z)\}_{n=0}^{\infty}$. This sequence converges uniformly on compact subsets of U to the fixed-point $S(z)$, i.e. $S(z)=\lim_{n\to\infty}S_n(z)$ uniformly on compact subsets of U.*
So $S(z)$ is determined by the recursion but independently of the initial singular inner function $S_0(z)$.
Thus $\forall\, S_0(z),\, T_0(z)\in \mathrm{SInn}$, $S_{n+1}(z)=\exp(-G(z\cdot S_n(z)))$, $T_{n+1}(z)=\exp(-G(z\cdot T_n(z)))$ and we have: $\lim_{n\to\infty}S_n(z)=\lim_{n\to\infty}T_n(z)=S(z)$, uniformly on compact subsets of U.

Proof.
We will outline two proofs. The first proof is using the Banach fixed-point theorem. Namely, $\forall\, z\in U$, the function of $w\in U$ given by: $\exp(-G(z\cdot w))$ is a contraction and so by the theorem of Banach it has a unique fixed-point $w=S(z)$. Moreover, this fixed-point is the limit of the

sequence, defined by the recursion $w_{n+1} = \exp(-G(z \cdot w_n))$, independently of the initial point w_0. From this we get our conclusions.

A second proof uses the differential equation of $\mathcal{B}$, namely we start at the beginning of chain $S_0(z,t) = f(z,t)$ and generate the sequence of singular inner functions by our recursion: $S_{n+1}(z) = \exp(-G(z \cdot S_n(z)))$. We obtain the limit uniformly on compact subsets of U, $S(z,t) = \lim_{n\to\infty} S_n(z,t)$. $S(z,t)$ is a fixed-point

$$S(z,t) = \exp(-G(z \cdot S(z,t))).$$

We apply the operator $\frac{\partial}{\partial t}$ to both sides of the fixed-point equation (justified by our assumptions on $S_0(z,t)$). We obtain:

$$\frac{\partial S(z,t)}{\partial t} = -z \cdot S(z,t) \cdot \left\{ \frac{\partial G(w)}{\partial w}|_{w=z \cdot S(z,t)} \right\} \cdot \frac{\partial S(z,t)}{\partial t}.$$

We claim that $\frac{\partial S(z,t)}{\partial t} = 0$ for all t. For if there were an open non-empty interval of t, over which $\frac{\partial S(z,t)}{\partial t} \neq 0$, then by the equation above:

$$\left\{ w \cdot \frac{\partial G(w)}{\partial w}|_{w=z \cdot S(z,t)} \right\} = -1.$$

So $z \cdot S(z,t)$ can be one of a discrete set which are the zeros of the non-zero holomorphic function

$$w \cdot \frac{\partial G(w)}{\partial w} + 1.$$

Hence $z \cdot S(z,t) \notin \text{CONF}$, a contradiction. Hence indeed $S(z,t) = S(z)$ is independent of t. Since the beginning of the chain $\{f(z,t)\}$ equals the first element of the sequence of the singular inner functions, $S_0(z,t) = f(z,t)$, this again, implies the conclusions of Theorem 3.3. □

Theorem 8.3.

$$\lim_{n\to\infty} \left\{ \prod_{j=0}^{n} S_j(z,t) \right\} \cdot z^n \cdot \left\{ \prod_{j=0}^{n-1} \left(\frac{\partial G(w)}{\partial w}|_{w=z \cdot S_j(z,t)} \right) \right\} = 0.$$

Proof.
By Theorem 6.1 and Theorem 8.1 where we use $\lim_{n\to\infty} \frac{\partial S_n(z,t)}{\partial t} = 0$. □

In particular, if we start our recursion from its fixed-point $S_0(z,t) = S(z)$, then our sequence is stationary, $S_j(z,t) = S(z)$ for all $j \in \mathbb{Z}_{\geq \not{\,}}$, and the formula of Theorem 5.2 gives us,

Corollary 8.4.

$$\lim_{n\to\infty} (z \cdot S(z))^n \cdot \left\{ \frac{\partial G(w)}{\partial w}|_{w=z \cdot S(z)} \right\}^n = 0 \quad \forall z \in U,$$

equivalently

$$\lim_{n\to\infty} \left\{ \left(w \cdot \frac{\partial G(w)}{\partial w} \right) |_{w=z \cdot S(z)} \right\}^n = 0 \quad \forall z \in U,$$

equivalently

$$\left|\left\{\left(w\cdot\frac{\partial G(w)}{\partial w}\right)|_{w=z\cdot S(z)}\right\}\right|<1\quad\forall\, z\in U.$$

The last inequality can be written as follows:

$$z\cdot S(z)\cdot G'(z\cdot S(z))\in \mathrm{B}_{\mathbb{H}^\infty(\mathbb{U})} \text{ where } \left|z\cdot S(z)\cdot G'(z\cdot S(z))\right|<1\quad\forall\, z\in U.$$

We end our paper with the example $G(w)=\frac{1+w}{1-w}$. On the next we will present the formulas we proved, for this particular case.

9. An example

Let us consider

$$G(w)=\frac{1+w}{1-w}\in \mathrm{RP}.$$

Let $S_0(z,t)=f(z,t)$ and

$$S_{n+1}(z,t)=\exp\left(-\frac{1+z\cdot S_n(z,t)}{1-z\cdot S_n(z,t)}\right)\quad\text{for }\ n\in\mathbb{Z}_{\geq\not{\vdash}}$$

$S(z,t)=\lim_{n\to\infty}S_n(z,t)$ uniformly on compact subsets of U, so that

$$S(z,t)=\exp\left(-\frac{1+z\cdot S(z,t)}{1-z\cdot S(z,t)}\right)\quad\forall\, z\in U$$

and $z\cdot S(z,t)\in \mathrm{CONF}$, $\forall\, 0\leq t\leq t_0$. We have the following results:

$$\frac{\partial S_n(z,t)}{\partial t}=(-1)^{n+1}\cdot\left\{\prod_{j=0}^{n}S_j(z,t)\right\}\cdot\left\{\frac{(2z)^n}{\prod_{j=0}^{n-1}(1-z\cdot S_j(z,t))^2}\right\}\cdot\left\{\frac{1+k(t)z}{1-k(t)z}\right\}. \tag{9.5}$$

This follows by Theorem 2.1.

There exists a unique $S(z)\in H(U)$ such that it is the only fixed-point of the function $\exp\left(-\frac{1+z\cdot w}{1-z\cdot w}\right)$, i.e.

$$S(z)=\exp\left(-\frac{1+z\cdot S(z)}{1-z\cdot S(z)}\right).$$

Moreover $\forall\, S_0(z)\in \mathrm{SInn}$, the recursion

$$S_{n+1}(z,t)=\exp\left(-\frac{1+z\cdot S_n(z,t)}{1-z\cdot S_n(z,t)}\right)\quad\text{for }\ n\in\mathbb{Z}_{\geq\not{\vdash}}$$

defines a sequence of singular inner functions $\{S_n(z)\}_{n=0}^{\infty}$. This sequence converges uniformly on compact subsets of U to the fixed-point $S(z)$,i.e. $S(z)=\lim_{n\to\infty}S_n(z)$ uniformly on compact subsets of U.

So $S(z)$ is determined by the recursion independently of the initial singular inner function $S_0(z)$. This follows by Theorem 3.3.

$$\lim_{n\to\infty}\frac{(2z)^n\cdot\prod_{j=0}^{n}S_j(z,t)}{\prod_{j=0}^{n-1}(1-z\cdot S_j(z,t))^2}=0. \tag{9.6}$$

This follows by Theorem 8.2.

$$\left|\frac{2z\cdot S(z)}{(1-z\cdot S(z))^2}\right|<1 \quad \forall\, z\in U, \tag{9.7}$$

equivalently $(1-|z|\cdot|S(z)|)^2>2\Re\{z\cdot S(z)\}\ \forall\, z\in U$. This follows by Corollary 5.3.

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