Eur. J. Math. Anal. 4 (2024) 20
doi: 10.28924/ada/ma.4.20

Extremal Functions and Calderon's Formulas for the Riemann-Liouville Two-Wavelet Transform

Ahmed Chana*, Abdellatif Akhlidj

Laboratory of Fundamental and Applied Mathematics, Department of Mathematics and Informatics, Faculty of Sciences Ain Chock, University of Hassan II, B.P 5366 Maarif, Casablanca, Morocco
maths.chana@gmail.com, akhlidj@hotmail.fr
**Correspondence: maths.chana@gmail.com*

Abstract. The Riemann-Liouville operator has been extensively investigated and his witnessed a remarkable development in numerous fields of harmonic analysis. Knowing the fact of the study of the time-frequency analysis are both theoritically interesting and pratically useful, we investigated several problems for this subject on the setting of the Riemann-Liouville wavelet transform. Firstly, we introduce the notion of Riemann-Liouville two-wavelet and we present generalized version of Parseval's, Plancherel's, inversion and Calderon's reproducing formulas. Next, using the theory of reproducing kernels, we give best estimates and an integral representation of the extremal functions related to the Riemann-Liouville wavelet transform on weighted Sobolev spaces.

1. Introduction

The mean operator is defined for a continuous function on $\mathbb{R}^2$, even with respect to the first variable by

$$\mathscr{R}_0(f)(x,t) = \frac{1}{2\pi}\int_0^{2\pi} f(x\sin\theta, t + x\cos\theta)d\theta.$$

Which means that $\mathscr{R}_0(f)(x,t)$ is the mean value of f on the circle centered at $(0,t)$ and radius x. The operators $\mathscr{R}_0$ play an mportant role and has many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data see [10,11], or in the linearized inverse scattering problem in acoustics see [5,7].

In [3], the authors have generalized $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ by introducing the so-called Riemann-Liouville operator defined on the space of continuous functions on $\mathbb{R}^2$, even with the respect to the

Received: 17 Aug 2024.
Key words and phrases. Riemann-Liouville operator, Wavelet transform, Time-Frequency Analysis, Extremal Functions, Weighted Sobolev spaces.

first variable by

$$\mathscr{R}_\alpha(f)(x,t) := \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1} f\left(xs\sqrt{1-y^2}, t+xy\right)\left(1-y^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1} dyds & \text{if } \alpha > 0, \\ \frac{1}{\pi}\int_{-1}^{1} f\left(r\sqrt{1-y^2}, t+xy\right)\frac{dt}{\sqrt{1-t^2}} & \text{if } \alpha = 0, \end{cases} \tag{1.1}$$

Many harmonic analysis results related to the Riemann-Liouville operator (1.1) have been established see [1–4] and the references therein. The wavelet transform has a long story which stared in 1984 with Mortel, a French petroleum engineer in connection with his study of seismic traces, the mathematical foundations were given by Grossman and Mortel in [8, 9]. Mortel defined a wavelet as a collection of functions constructed by using translation and dilatation of a single function $\psi \in L^2(\mathbb{R})$ called the mother wavelet by:

$$\psi_{b,a}(t) = \frac{1}{\sqrt{a}}\psi(\frac{t-b}{a}),$$

where $a > 0$ is called the scaling parameter, wich measure the degree of compression and $b \in \mathbb{R}$ is a translation parameter wich determines the time location of the wavelet. The theory of wavelet have applications in several research area as signal theory, time frequency analysis, geophysics and medicine see [6, 9].

A lot of attention has been given to various generalization of the classical Fourier transform, this paper focuses on the generalized Fourier transform associated with the Riemann-Liouville operator (1.1) called the Riemann-Liouville transform, more precisely we consider a system of partial differential operator Δ_1 and Δ_2 defined by

$$\begin{cases} \Delta_1 := \frac{\partial}{\partial x}; \quad \alpha \geq 0, \quad t > 0, \\ \Delta_2 := \frac{\partial^2}{\partial t^2} + \frac{2\alpha+1}{t}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \quad \alpha \geq 0. \end{cases}$$

From [3], the authors gives the connection between the eigenfunctions of this system denoted by $\varphi_{\mu,\lambda}$ with $(\mu,\lambda) \in \mathbb{C}^2$ and the Riemann-Liouville operator (1.1) as follows:

$$\varphi_{\mu,\lambda}(x,t) = \mathscr{R}_\alpha(\cos(\mu.)\exp(-i\lambda\cdot))(x,t). \tag{1.2}$$

Wavelet analysis has attracted attention for its ability to analyse rapidly changing transient signals, any application using the Fourier like transform can be formulated using wavelets to provide more time and frequency information.

The reason for the extension from one wavelet to two-wavelet comes from the extra degree of flexibility in signal analysis and imaging when the localization operators are used as time-varing filters. This paper is an attempt to fill this gap by extending one wavelet to two wavelets in the Riemann-Liouville setting. The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the Riemann-Liouville transform, in section 3 we introduce the notion of Riemann-Liouville two-wavelet and we give a generalized version of Parseval's, Plancherel's, inversion and Calderon's reproducing formulas related to this transform, the last section is devoted to give an integral representation and best estimates of extremal functions related to the Riemann-Liouville wavelet transform on weighted Sobolev spaces.

2. Harmonic Analysis Associated with the Riemann-Liouville Operator

In this section we set some notations and we recall some results in harmonic analysis related to the Riemann-Liouville operator (1.1), for more details we refer the reader to [1–4, 15]. In the following we denote by

- $\mathbb{K} :=]0,+\infty[\times\mathbb{R}$ equipped with the weighted Lebesgue measure μ_α given by

$$d\mu_\alpha(x,t) := \frac{x^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}dx\otimes dt,, \quad \alpha\geq 0,$$

where Γ is the Gamma function.

- $L^p_\alpha(\mathbb{K}), 1\leq p\leq\infty$, the space of measurable functions on $\mathbb{K}$, satisfying

$$\|f\|_{p,\mu_\alpha} := \begin{cases}\left(\int_{\mathbb{K}}|f(x,t)|^p d\mu_\alpha(x,t)\right)^{1/p}<\infty, & \text{if } p\in[1,+\infty[,\\ \operatorname{ess\,sup}_{(x,t)\in\mathbb{K}}|f(x,t)|<\infty, & \text{if } p=+\infty.\end{cases}$$

- $\hat{\mathbb{K}} := [0,+\infty[\times\mathbb{R}\cup\{(is,y);(s,y)\in[0,+\infty[\times\mathbb{R}; s\leqslant|y|\}$.
- $\mathscr{B}_{\hat{\mathbb{K}}}$ the σ-algebra defined on $\hat{\mathbb{K}}$ by

$$\mathscr{B}_{\hat{\mathbb{K}}} = \{\theta^{-1}(B), B\in\mathscr{B}([0,+\infty[\times\mathbb{R})\},$$

where θ is the bijective function given by

$$\theta(s,y) = \left(\sqrt{s^2+y^2},y\right).$$

- $d\gamma_\alpha$ the measure defined on $\mathscr{B}_{\hat{\mathbb{K}}}$ by

$$\forall A\in\mathscr{B}_{\hat{\mathbb{K}}}; \gamma_\alpha(A) = \mu_\alpha(\theta(A)).$$

and for all non-negative measurable function on $\hat{\mathbb{K}}$ we have

$$\begin{aligned}\int_{\hat{\mathbb{K}}} g(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) =& \frac{1}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}\left(\int_0^{+\infty}\int_{\mathbb{R}} g(\mu,\lambda)\left(\mu^2+\lambda^2\right)^\alpha \mu d\mu d\lambda\right.\\ &\left.+\int_{\mathbb{R}}\int_0^{|\lambda|} g(i\mu,\lambda)\left(\lambda^2-\mu^2\right)^\alpha \mu d\mu d\lambda\right).\end{aligned} \tag{2.1}$$

- $L^p_\alpha(\hat{\mathbb{K}})$ with $p\in[1,+\infty]$ the space of measurable functions on $\hat{\mathbb{K}}$ satisfying

$$\|g\|_{p,\gamma_\alpha} := \begin{cases}\left(\int_{\hat{\mathbb{K}}}|g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{p}}<\infty & \text{if } p\in[1,+\infty[,\\ \operatorname{ess\ sup}_{(\lambda,m)\in\hat{\mathbb{K}}}|g(\lambda,m)|<\infty, & \text{if } p=+\infty.\end{cases}$$

2.1. **The eigenfunctions of the partial Differential operators Δ_1 and Δ_2.** For $(\mu,\lambda)\in\hat{\mathbb{K}}$ we consider the following Cauchy problem

$$(S):\begin{cases}\Delta_1(u)(x,t)=-\lambda^2u(x,t),\\ \Delta_2(u)(x,t)=-\mu^2u(x,t)\\ u(0,0)=1;\frac{\partial u}{\partial x}(0,t)=0.\end{cases}$$

From [3], the Cauchy problem (S) admits a unique solution $\varphi_{\mu,\lambda}$ given by:

$$\varphi_{\mu,\lambda}(x,t)=j_\alpha\left(x\sqrt{\mu^2+\lambda^2}\right)\exp(-i\lambda t), \tag{2.2}$$

where j_α is the spherical Bessel function of index α see [16] for more information about the Bessel functions. The function $\varphi_{\mu,\lambda}$ is infinitely differentiable on $\mathbb{R}^2$, even with respect to each variable and we have the following important result:

$$\sup_{(x,t)\in\mathbb{R}^2}\left|\varphi_{\mu,\lambda}(x,t)\right|=1. \quad \text{for } (\mu,\lambda)\in\hat{\mathbb{K}}. \tag{2.3}$$

2.2. **The Riemann-Liouville transform.**

Definition 2.1. *The generalized Fourier transform $\mathcal{F}_\alpha$ associated with the Riemann-Liouville operator (1.1) is defined on $L^1_\alpha(\mathbb{K})$ by*

$$\mathscr{F}_\alpha(f)(\mu,\lambda)=\int_{\mathbb{K}}\varphi_{\mu,\lambda}(x,t)f(x,t)d\mu_\alpha(x,t),\ \text{for } (\mu,\lambda)\in\hat{\mathbb{K}}.$$

Some basic properties of this transform are as follows, for the proofs one can see [2–4].

Proposition 2.1.
(1) For every $f\in L^1_\alpha(\mathbb{K})$, we have

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\gamma_\alpha}\le\|f\|_{1,\mu_\alpha}. \tag{2.4}$$

(2)(Inversion formula) For $f\in\left(L^1_\alpha\cap L^2_\alpha\right)(\mathbb{K})$ such that $\mathscr{F}_\alpha(f)\in L^1_\alpha(\hat{\mathbb{K}})$ we have

$$f(x,t)=\int_{\hat{\mathbb{K}}}\overline{\varphi_{\mu,\lambda}(x,t)}\mathscr{F}_\alpha(f)(\mu,\lambda)d\gamma_\alpha(\mu,\lambda),\qquad a.e\quad (x,t)\in\mathbb{K}. \tag{2.5}$$

(3) (Parseval formula) For all $f,g\in L^2_\alpha(\mathbb{K})$ we have

$$\int_{\mathbb{K}}f(x,t)\overline{g(x,t)}d\mu_\alpha(x,t)=\int_{\hat{\mathbb{K}}}\mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\mathscr{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda), \tag{2.6}$$

in particular we have

$$\|f\|_{2,\mu\alpha}=\|\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha}. \tag{2.7}$$

(4) (Plancherel's theorem) The Reimann-Liouville transform $\mathscr{F}_\alpha$ can be extended to an isometric isomorphism from $L^2_\alpha(\mathbb{K})$ into $L^2_\alpha(\hat{\mathbb{K}})$.

2.3. **Generalized translation operator Associated with the Riemann-Liouville operator.**

Definition 2.2. *The translation operator associated with Riemann-Liouville transform is defined on $L^p_\alpha(\mathbb{K})$, for all $(x,t),(y,s)\in\mathbb{K}$, by*

$$\tau_\alpha^{(x,t)}(f)(y,s)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^\pi f\left(\sqrt{x^2+y^2+2xy\cos\theta},t+s\right)\sin^{2\alpha}\theta d\theta.$$

The following proposition summarizes some properties of the Riemann-Liouville translation operator see [2–4].

Proposition 2.2. *For all $(x,t),(y,s)\in\mathbb{K}$, $f\in L^p_\alpha(\mathbb{K})$ we have:*
(1)

$$\int_{\mathbb{K}}\tau_\alpha^{(x,t)}(f)(y,s)d\mu_\alpha(y,s)=\int_{\mathbb{K}}f(y,s)d\mu_\alpha(y,s). \tag{2.8}$$

(2) for $f\in L^p_\alpha(\mathbb{K})$ with $p\in[1;+\infty]$ $\quad\tau_\alpha^{(x,t)}(f)\in L^p_\alpha(\mathbb{K})$ and we have

$$\left\|\tau_\alpha^{(x,t)}(f)\right\|_{p,\mu_\alpha}\leq\|f\|_{p,\mu_\alpha}. \tag{2.9}$$

(3) For $f\in L^1_\alpha(\mathbb{K})$, $\quad\tau_\alpha^{(x,-t)}(f)\in L^1_\alpha(\mathbb{K})$ and we have

$$\mathscr{F}_\alpha\left(\tau_\alpha^{(x,-t)}(f)\right)(\mu,\lambda)=\varphi_{\mu,\lambda}(x,t)\mathcal{F}_\alpha(f)(\mu,\lambda),\quad\forall(\mu,\lambda)\in\hat{\mathbb{K}}. \tag{2.10}$$

By using the generalized translation, we define the generalized convolution product of f,g by

$$(f*_\alpha g)(x,t)=\int_{\mathbb{K}}\tau_\alpha^{(x,-t)}(\check{f})(y,s)g(y,s)d\mu_\alpha(y,s).$$

where $\check{f}(y,s)=f(y,-s)$.
With this convolution product $(\mathbb{K},*_\alpha)$ is a hypergroup in the sense of Jewett [13].
We have the following results for the proofs, we refer the reader to [2–4].

Proposition 2.3.
(1)(Young's inequality) for all $p,q,r\in[1;+\infty]$ such that: $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ and for all $f\in L^p_\alpha(\mathbb{K})$, $g\in L^q_\alpha(\mathbb{K})$ the function $f*_\alpha g$ belongs to the space $L^r_\alpha(\mathbb{K})$ and we have

$$\|f*_\alpha g\|_{r,\mu_\alpha}\leq\|f\|_{p,\mu_\alpha}\|g\|_{q,\mu_\alpha} \tag{2.11}$$

(2) For $f,g\in L^2_\alpha(\mathbb{K})$ the function $f*_\alpha g$ belongs to $L^2_\alpha(\mathbb{K})$ if and only if the function $\mathscr{F}_\alpha(f)\mathscr{F}_\alpha(g)$ belongs to $L^2_\alpha(\hat{\mathbb{K}})$ and in this case we have

$$\mathscr{F}_\alpha(f*_\alpha g)=\mathscr{F}_\alpha(f)\mathscr{F}_\alpha(g). \tag{2.12}$$

(3) For $f,g\in L^2_\alpha(\mathbb{K})$ then we have

$$\int_{\mathbb{K}}|f*_\alpha g(x,t)|^2\,d\mu_\alpha(x,t)=\int_{\hat{\mathbb{K}}}|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2\,|\mathscr{F}_\alpha(g)(\mu,\lambda)|^2\,d\gamma_\alpha(\mu,\lambda), \tag{2.13}$$

where both integrals are simultaneously finite or infinite.

3. Calderón's Reproducing Formula for the Riemann-Liouville Two-wavelet Transform

Using the harmonic analysis associated with the Riemann-Liouville transform, the main purpose of this section is to define the wavelet transform associated with the Riemann-Liouville operator and to give generalized Parseval's, Plancherel's, inversion and Calderon's reproducing formulas related to this transform which generalizes all the results proved in [4].

Notation: we denote by

•$L^p_\alpha(\mathbb{R}_+\times\mathbb{K})$,$1\le p\le+\infty$ the space of measurable functions on $\mathbb{R}_+\times\mathbb{K}$ satisfying

$$\|f\|_{p,\theta_\alpha}:=\begin{cases}\left(\int_0^{+\infty}\int_{\mathbb{K}}|f(a,x,t)|^p\mathrm{d}\theta_\alpha(a,x,t)\right)^{\frac{1}{p}}<\infty, & \text{if } p\in[1,+\infty[,\\ \operatorname*{ess\,sup}_{(a,x,t)\in\mathbb{R}_+\times\mathbb{K}}|f(a,x,t)|<\infty, & \text{if } p=+\infty.\end{cases}$$

.

where θ_α is the measure defined on $\mathbb{R}_+\times\mathbb{K}$ by

$$\mathrm{d}\theta_\alpha(a,x,t):=a^{2\alpha+2}\mathrm{d}a\otimes\mathrm{d}\mu_\alpha(x,t).$$

Definition 3.1. *Let $\psi_1,\psi_2\in L^2_\alpha(\mathbb{K})$, the pair (ψ_1,ψ_2) is said to be a Rieman-Liouville two-wavelet on $\mathbb{K}$ if for almost all $(\mu,\lambda)\in\hat{\mathbb{K}}$ we have*

$$0<C_{\psi_1,\psi_2}:=\int_0^\infty\overline{\mathscr{F}_\alpha(\psi_1)\left(\frac{\mu}{a},\frac{\lambda}{a}\right)}\mathscr{F}_\alpha(\psi_2)\left(\frac{\mu}{a},\frac{\lambda}{a}\right)\frac{da}{a}<+\infty.\tag{3.1}$$

Remark 3.1. *Its clear that if $\psi=\psi_1=\psi_2$, we have*

$$C_{\psi_1,\psi_2}=C_\psi:=\int_0^\infty\left|\mathscr{F}_\alpha(\psi)\left(\frac{\mu}{a},\frac{\lambda}{a}\right)\right|^2\frac{da}{a}<+\infty,\tag{3.2}$$

in this case we say that ψ is a Riemann-Liouville wavelet in $L^2_\alpha(\mathbb{K})$.

Let $a>0$, we define the dilatation operator D_a of a measurable function ψ on $\mathbb{C}^2$ by

$$D_a(\psi)(x,t)=a^{\alpha+3/2}\psi(ax,at),\quad (x,t)\in\mathbb{C}^2.$$

the dilatation operator D_a satisfies the following properties

• *For all $\psi\in L^p_\alpha(\mathbb{K})$ we have $D_a(\psi)\in L^p_\alpha(\mathbb{K})$ and*

$$\|D_a(\psi)\|_{p,\mu_\alpha}=a^{(\frac{1}{2}-\frac{1}{p})(2\alpha+3)}\|\psi\|_{p,\mu_\alpha}.\tag{3.3}$$

• *For all $\psi\in L^2(\mathbb{K})$ we have*

$$\mathscr{F}_\alpha(D_a(\psi))(\mu,\lambda)=\frac{1}{a^{\alpha+3/2}}\mathscr{F}_\alpha(\psi)\left(\frac{\mu}{a},\frac{\lambda}{a}\right).\tag{3.4}$$

Let ψ be a Riemann-Liouville wavelet on $\mathbb{K}$ in $L^p(\mathbb{K})$ with $1\le p\le\infty$, for all $a>0$,$(x,t)\in\mathbb{K}$ we define the function

$$\psi_{a,x,t}(y,s)=\tau_\alpha^{(x,-t)}(D_a(\psi))(y,s).\tag{3.5}$$

By using the relations (2.9) and (3.3) we find that $\psi_{a,x,t} \in L^p_\alpha(\mathbb{K})$ and

$$\|\psi_{a,x,t}\|_{p,\mu_\alpha} \leq a^{(\frac{1}{2}-\frac{1}{p})(2\alpha+3)}\|\psi\|_{p,\mu_\alpha}.. \tag{3.6}$$

Definition 3.2. *([4]) Let ψ be a Riemann-Liouville wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$ the continuous wavelet transform S^α_ψ associated with the Riemann-Liouville operator is defined for a function $f \in L^2_\alpha(\mathbb{K})$ and $(a, x, t) \in \mathbb{R}_+ \times \mathbb{K}$ by*

$$S^\alpha_\psi(f)(a, x, t) := \int_{\mathbb{K}} f(y, s)\overline{\psi_{a,x,t}(y, s)}d\mu_\alpha(y, s). \tag{3.7}$$

Remark 3.2. *The Riemann-Liouville wavelet transform (3.7) can be written as*

$$S^\alpha_\psi(f)(a, x, t) = (\overline{D_a(\check{\psi})} *_\alpha f)(x, t) = \langle f, \psi_{a,x,t}\rangle_\alpha. \tag{3.8}$$

The following result gives the relation between the Riemann-Liouville transform $\mathscr{F}_\alpha$ and the Riemann-Liouville wavelet transform S^α_ψ.

Proposition 3.1. *Let ψ be a Riemann-Liouville wavelet for all $f \in L^2(\mathbb{K})$ we have*

$$\mathscr{F}_\alpha\left[S^\alpha_\psi(f)(a, .)\right](\mu, \lambda) = \frac{1}{a^{\alpha+3/2}}\mathscr{F}_\alpha(f)(\mu, \lambda)\overline{\mathscr{F}_\alpha(\psi)\left(\frac{\mu}{a}, \frac{\lambda}{a}\right)}. \tag{3.9}$$

Proof. Is a consequence of the convolution theorem (2.12) and the relations (3.4),(3.8). □

The following theorem generalizes the Parseval's formula for the Riemann-Liouville wavelet transform $S^\alpha_\psi(f)$ proved in [4].

Theorem 3.1. *Let (ψ_1, ψ_2) be a Rieman-Liouville two-wavelet on $\mathbb{K}$ for all $f, g \in L^2(\mathbb{K})$ we have*

$$\int_0^{+\infty}\int_{\mathbb{K}} S^\alpha_{\psi_1}(f)(a, x, t)\overline{S^\alpha_{\psi_2}(g)(a, x, t)}\mathrm{d}\theta_\alpha(a, x, t) = C_{\psi_1,\psi_2}\int_{\mathbb{K}} f(y, s)\overline{g(y, s)}d\mu_\alpha(y, s), \tag{3.10}$$

where C_{ψ_1,ψ_2} is the constant given by the relation (3.1).

Proof. By using the relations (2.5),(2.12),(3.4), (3.8) and Fubini's theorem we get

$$\int_0^{+\infty}\int_{\mathbb{K}} S^\alpha_{\psi_1}(f)(a, x, t)\overline{S^\alpha_{\psi_1}(g)(a, x, t)}\mathrm{d}\theta_\alpha(a, x, t)$$

$$= \int_0^{+\infty}[\int_{\mathbb{K}} (\overline{D_a(\check{\psi}_1)} *_\alpha f)(x, t)\overline{(\overline{D_a(\check{\psi}_2)} *_\alpha g)(x, t)}d\mu_\alpha(x, t)]a^{2\alpha+2}\mathrm{d}a$$

$$= \int_0^{+\infty}[\int_{\hat{\mathbb{K}}} \mathscr{F}_\alpha(\overline{D_a(\check{\psi}_1)})(\mu, \lambda)\mathscr{F}_\alpha(f)(\mu, \lambda)\overline{\mathscr{F}_\alpha(\overline{D_a(\check{\psi}_2)})(\mu, \lambda)\mathscr{F}_\alpha(g)(\mu, \lambda)}d\gamma_\alpha(\lambda, m)]a^{2\alpha+2}\mathrm{d}a$$

$$= C_{\psi_1,\psi_2}\int_{\hat{\mathbb{K}}} \mathscr{F}_\alpha(f)(\mu, \lambda)\overline{\mathscr{F}_\alpha(g)(\mu, \lambda)}d\gamma_\alpha(\mu, \lambda),$$

by using Parseval's formula for the Riemann-Liouville transform (2.5) we find the disered result. □

In the following we establish an inversion formula for the Riemann-Liouville two-wavelet transform.

Theorem 3.2. *Let (ψ_1, ψ_2) be a Riemann-Liouville two-wavelet such that $C_{\psi_1,\psi_2} \neq 0$ for all $f \in L^1_\alpha(\mathbb{K})$ such that $\mathcal{F}_\alpha(f) \in L^1_\alpha(\hat{\mathbb{K}}) \cap L^\infty_\alpha(\hat{\mathbb{K}})$ we have*

$$f(\cdot) = \frac{1}{C_{\psi_1,\psi_2}} \int_0^\infty \left(\int_{\mathbb{K}} S^\alpha_{\psi_1}(f)(a, x, t)\psi_{2,a,x,t}(\cdot)d\mu_\alpha(x, t) \right) a^{2\alpha+2}\mathrm{d}a.$$

Proof. Let $f, g \in L^2_\alpha(\mathbb{K})$, by using the relation (3.10), Fubini's theorem we find that

$$\int_{\mathbb{K}} f(y, s)\overline{g(y, s)}d\mu_\alpha(y, s) = \frac{1}{C_{\psi_1,\psi_2}} \int_0^{+\infty} \int_{\mathbb{K}} S^\alpha_{\psi_1}(f)(a, x, t)\overline{S^\alpha_{\psi_2}(g)(a, x, t)}\mathrm{d}\theta_\alpha(a, x, t)$$

$$= \frac{1}{C_{\psi_1,\psi_2}} \int_{\mathbb{K}} [\int_0^\infty \left(\int_{\mathbb{K}} S^\alpha_{\psi_1}(f)(a, x, t)\psi_{2,a,x,t}(y, s)d\mu_\alpha(x, t) \right) a^{2\alpha+2}\mathrm{d}a]\overline{g(y, s)}d\mu_\alpha(y, s)$$

which gives the result. □

The rest of this subsection is devoted to give a Calderón's reproducing formula for the Reimann-Liouville two-wavelet (ψ_1, ψ_2) under the following condition

$$C_{\psi_1,\psi_2} \neq 0 \quad and \quad \mathscr{F}_\alpha(D_a(\psi_1)), \mathscr{F}_\alpha(D_a(\psi_2)) \in L^\infty_\alpha(\hat{\mathbb{K}}), \tag{3.11}$$

Proposition 3.2. *For $0 < \varepsilon < \delta < \infty$, we put*

$$G_{\varepsilon,\delta}(x, t) := \frac{1}{C_{\psi_1,\psi_2}} \int_\varepsilon^\delta \left(\overline{D_a(\check{\psi}_2)} *_\alpha D_a(\psi_1) \right)(x, t)a^{2\alpha+2}\mathrm{d}a$$

and

$$K_{\varepsilon,\delta}(\lambda, m) := \frac{1}{C_{\psi_1,\psi_2}} \int_\varepsilon^\delta \mathscr{F}_\alpha(\psi_1)\left(\frac{\mu}{a}, \frac{\lambda}{a}\right) \overline{\mathscr{F}_\alpha(\psi_2)\left(\frac{\mu}{a}, \frac{\lambda}{a}\right)} \frac{da}{a}.$$

Under the condition (3.11) we have

$$G_{\varepsilon,\delta} \in L^2_\alpha(\mathbb{K}), K_{\varepsilon,\delta} \in L^1_\alpha(\hat{\mathbb{K}}) \cap L^\infty_\alpha(\hat{\mathbb{K}})$$

and

$$\mathscr{F}_\alpha(G_{\varepsilon,\delta})(\lambda, m) = K_{\varepsilon,\delta}(\lambda, m) \tag{3.12}$$

Proof. By using Hölder's inequality for the measure $a^{2\alpha+2}\mathrm{d}a$ we obtain

$$\|G_{\varepsilon,\delta}\|^2_{2,\mu_\alpha} \leq \frac{\delta^{2\alpha+3} - \varepsilon^{2\alpha+3}}{C^2_{\psi_1,\psi_2}} \int_\varepsilon^\delta \left(\int_{\mathbb{K}} \left| \left(\overline{D_a(\check{\psi}_2)} *_\alpha D_a(\psi_1) \right)(x, t) \right|^2 d\mu_\alpha(x, t) \right) a^{2\alpha+2}\mathrm{d}a,$$

By using the relations (2.13) and (3.4) we find that

$$\|G_{\varepsilon,\delta}\|^2_{2,\mu_\alpha} \leq \frac{\delta^{2\alpha+3} - \varepsilon^{2\alpha+3}}{C^2_{\psi_1,\psi_2}} \|\mathscr{F}_\alpha(D_a(\psi_2))\|^2_{\infty,\gamma_\alpha} \|\psi_1\|^2_{2,\mu_\alpha} \int_\varepsilon^\delta \frac{da}{a} < \infty.$$

Which prove that $G_{\varepsilon,\delta} \in L^2_\alpha(\mathbb{K})$, the result $K_{\varepsilon,\delta} \in L^1_\alpha(\hat{\mathbb{K}}) \cap L^\infty_\alpha(\hat{\mathbb{K}})$ can be easily checked, on the other hand by using the relations (2.5), (2.10), (3.4) and Fubini's theorem we find that

$$G_{\varepsilon,\delta}(x, t) = \int_{\hat{\mathbb{K}}} \overline{\varphi_{\mu,\lambda}(x, t)} K_{\varepsilon,\delta}(\mu, \lambda)d\gamma_\alpha(\mu, \lambda),$$

inversion formula (2.5) gives the relation (3.12). □

We can now state the main result of this section

Theorem 3.3. *(First Calderón's reproducing formula)*
Let (ψ_1, ψ_2) be a Reimann-Liouville two-wavelet satisfying the condition (3.11) and let $0 < \varepsilon < \delta < \infty$ then for all $f \in L^2_\alpha(\mathbb{K})$, the function $f_{\varepsilon,\delta}$ given by

$$f_{\varepsilon,\delta}(x,t) = \frac{1}{C_{\psi_1,\psi_2}} \int_\varepsilon^\delta \left(\int_{\mathbb{K}} S^\alpha_{\psi_1}(f)(a,y,s)\psi_{2,a,x,t}(y,s)d\mu_\alpha(y,s) \right) a^{2\alpha+2}da,$$

belongs to $L^2_\alpha(\mathbb{K})$ and satisfies

$$\lim_{\varepsilon\to 0,\delta\to\infty} \|f_{\varepsilon,\delta} - f\|_{2,\mu_\alpha} = 0. \tag{3.13}$$

Proof. It is easy to see that

$$f_{\varepsilon,\delta} = f *_\alpha G_{\varepsilon,\delta}$$

then by using the relations (2.7) and (3.12) we find that

$$\|f_{\varepsilon,\delta} - f\|^2_{2,\mu_\alpha} = \int_{\hat{\mathbb{K}}} |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2(1 - K_{\varepsilon,\delta}(\mu,\lambda))^2 d\gamma_\alpha(\mu,\lambda),$$

the relation (3.13) follows from the admissibility condition (3.1) and the dominated convergence theorem. □

4. Extremal Functions Associated with the Riemann-Liouville Wavelet Transform

By using the theory of reproducing kernels [18, 19], the main purpose of this section is to study the extremal functions associated with the Riemann-Liouville wavelet transform and to give an integral representation and best estimate of these functions on weighted Sobolev spaces.

4.1. Sobolev type spaces Associated with the Riemann-Liouville Transform.

Let $s > 0$, we define the Sobolev spaces associated with the Riemann-Liouville transform as

$$H^s_\alpha(\mathbb{K}) := \left\{ f \in L^2_\alpha(\mathbb{K}) / \left(1 + \mu^2 + 2\lambda^2\right)^{s/2} \mathscr{F}_\alpha(f) \in L^2_\alpha(\hat{\mathbb{K}}) \right\}.$$

The space $H^s_\alpha(\mathbb{K})$ provided with the inner product

$$\langle f, g\rangle_{H^s_\alpha} := \int_{\hat{\mathbb{K}}} \left(1 + \mu^2 + 2\lambda^2\right)^s \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\mathscr{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda), \tag{4.1}$$

and the norm

$$\|f\|^2_{H^s_\alpha} := \langle f, f\rangle_{H^s_\alpha} = \int_{\hat{\mathbb{K}}} \left(1 + \mu^2 + 2\lambda^2\right)^s |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda), \tag{4.2}$$

is a Hilbert space.

Definition 4.1. *Let ψ be a Riemann-Liouville wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$, we introduce the inner product in the Hilbert space $H^s_\alpha(\mathbb{K})$ for any fixed $\beta > 0$ by*

$$\langle f, g\rangle_{H^s_{\psi,\beta}} := \beta\langle f, g\rangle_{H^s_\alpha} + \langle S^\alpha_\psi(f), S^\alpha_\psi(g)\rangle_{\theta_\alpha}, \tag{4.3}$$

the norm associated to this inner product is defined by

$$\|f\|^2_{H^s_{\psi,\beta}} := \beta\|f\|^2_{H^s_\alpha} + \|S^\alpha_{\psi_1}(f)\|^2_{2,\theta_\alpha}. \tag{4.4}$$

We have the following result

Proposition 4.1. *Let $s > \frac{2\alpha+3}{2}$, ψ be aRiemann-Liouville wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$ and $\beta > 0$ then we have*

$$f \in H^s_{\psi,\beta}(\mathbb{K}) \Rightarrow \mathscr{F}_\alpha(f) \in L^1_\alpha(\hat{\mathbb{K}}) \tag{4.5}$$

Proof. Let $f \in H^s_{\psi,\beta}(\mathbb{K})$, by using the relations (2.9), (3.9), (4.2) and (4.4) we find that

$$\|f\|^2_{H^s_{\psi,\beta}} = \int_{\hat{\mathbb{K}}} \left[\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi\right] |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) \tag{4.6}$$

by using Hölder's inequality, the relation (2.1) and the fact that $s > \frac{2\alpha+3}{3}$ we find that

$$\|\mathcal{F}_\alpha(f)\|_{1,\gamma_\alpha} \le \|f\|_{H^s_{\psi,\beta}} \left(\int_{\hat{\mathbb{K}}} \frac{d\gamma_\alpha(\mu,\lambda)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi}\right)^{\frac{1}{2}} < \infty$$

wich give the result. □

Theorem 4.1. *Let $s > \frac{2\alpha+3}{2}$, ψ be a Riemann-Liouville wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$ and $\beta > 0$ then the space $(H^s_{\psi,\beta}(\mathbb{K}), \langle,\rangle_{H^s_{\psi,\beta}})$ is a reproducing kernel Hilbert space with kernel function given by*

$$\mathscr{K}_{\psi,\beta}[(x,t),(y,z)] = \int_{\hat{\mathbb{K}}} \frac{\varphi_{\mu,-\lambda}(x,t)\varphi_{\mu,\lambda}(y,z)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi} d\gamma_\alpha(\mu,\lambda) \tag{4.7}$$

that is for every $(y,z) \in \mathbb{K}$,

1. *the function $(x,t) \to \mathscr{K}_{r,h}[(x,t),(y,z)] \in H^s_{\psi,\beta}(\mathbb{K})$.*
2. *For every $f \in H^s_{\psi,\beta}(\mathbb{K})$ and $(y,z) \in \mathbb{K}$ we have*

$$f(y,z) = \left\langle f, \mathscr{K}_{\psi,\beta}[\cdot,(y,z)]\right\rangle_{H^s_{\psi,\beta}}.$$

Proof. Let $(y,z) \in \mathbb{K}$, by using the fact that $s > \frac{2\alpha+3}{2}$ and the relation (2.3) we find that the function

$$(\mu,\lambda) \to \frac{\varphi_{\mu,\lambda}(y,z)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi}$$

belongs to $L^1_\alpha(\hat{\mathbb{K}}) \cap L^2_\alpha(\hat{\mathbb{K}})$, by using Plancherel's theorem for the Riemann-Liouville transform there exist a unique function in $L^2_\alpha(\mathbb{K})$, wich we denote by $\mathscr{K}_{\psi,\beta}[\cdot,(y,z)]$ such that

$$\mathscr{F}_\alpha\left(\mathscr{K}_{\psi,\beta}[\cdot,(y,z)]\right) = \frac{\varphi_{\mu,\lambda}(y,z)}{r\left[1+\lambda^2\left(1+m^2\right)\right]^s + C_h}, \tag{4.8}$$

by using the relation (2.5) we find that

$$\mathscr{K}_{\psi,\beta}[(x,t),(y,z)] = \int_{\hat{\mathbb{K}}} \frac{\varphi_{\mu,-\lambda}(x,t)\varphi_{\mu,\lambda}(y,z)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi} d\gamma_\alpha(\mu,\lambda),$$

furthermore by using the relations (2.3), (4.6) and (4.8) we find that

$$\left\|\mathscr{K}_{\psi,\beta}[\cdot,(y,z)]\right\|^2_{H^s_{\psi,\beta}} \leq \int_{\hat{\mathbb{K}}} \frac{d\gamma_\alpha(\mu,\lambda)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi} < \infty.$$

wich proves that $\mathscr{K}_{\psi,\beta}[\cdot,(y,z)] \in H^s_{\psi,\beta}(\mathbb{K})$. Let $f \in H^s_{\psi,\beta}(\mathbb{K})$, by using the relations the relations (3.10),(4.1),(4.3),(and (4.8) we find that

$$\left\langle f, \mathscr{K}_{\psi,\beta}[\cdot,(y,z)]\right\rangle_{H^s_{r,h}} = \int_{\hat{\mathbb{K}}} \overline{\varphi_{\mu,\lambda}(y,z)}\mathscr{F}_\alpha(f)(\mu,\lambda)d\gamma_\alpha(\mu,\lambda),$$

inversion formula (2.5) gives the disered result. □

In the following we give the main result of this section.

Theorem 4.2. *Let ψ be a Riemann-Liouville wavelet in $L^2_\alpha(\mathbb{K})$, $s > \frac{2\alpha+3}{2}$, $g \in L^2_\alpha\left(\mathbb{R}_+ \times \mathbb{K}\right)$ and $\beta > 0$ then the infimum*

$$\inf_{f \in H^s_\alpha(\mathbb{K})} \left\{\beta\|f\|^2_{H^s_\alpha} + \left\|S^\alpha_\psi(f) - g\right\|^2_{2,\theta_\alpha}\right\} \tag{4.9}$$

is attained by a unique function $f^*_{g,\psi,\beta}$ given explicitly by

$$f^*_{g,\psi,\beta}(x,t) = \int_0^{+\infty}\int_{\mathbb{K}} g(a,y,z)\phi_{\psi,\beta}\left(a,(y,z),(x,t)\right) \mathrm{d}\theta_\alpha(a,y,z), \tag{4.10}$$

where $\phi_{\psi,\beta}$ is given by

$$\phi_{\psi,\beta}\left(a,(y,z),(x,t)\right) = a^{-\frac{2\alpha+3}{2}} \int_{\hat{\mathbb{K}}} \frac{\varphi_{\mu,-\lambda}(x,t)\varphi_{\mu,\lambda}(y,z)\mathscr{F}_\alpha(\psi)\left(\frac{\mu}{a},\frac{\lambda}{a}\right)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi} d\gamma_\alpha(\mu,\lambda). \tag{4.11}$$

Proof. The existence and unicity of the extremal function $f^*_{g,\psi,\beta}$ solution of the problem (4.9) is assured in [18, 19], moreover this solution is given by

$$f^*_{g,\psi,\beta}(x,t) = \left\langle g, S^\alpha_\psi\left(\mathscr{K}_{\psi,\beta}[\cdot,(x,t)]\right)\right\rangle_{\theta_\alpha}, \tag{4.12}$$

where $\mathscr{K}_{\psi,\beta}$ is the kernel given by (4.8), by using the relations (2.6), (2.10), (3.4) and (4.8) we find that

$$S^\alpha_\psi\left(\mathscr{K}_{\psi,\beta}[\cdot,(x,t)]\right)(a,y,z) = a^{-\frac{2\alpha+3}{2}} \int_{\hat{\mathbb{K}}} \frac{\varphi_{\mu,\lambda}(x,t)\varphi_{\mu,-\lambda}(y,z)\mathscr{F}_\alpha(\psi)\left(\frac{\mu}{a},\frac{\lambda}{a}\right)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi} d\gamma_\alpha(\mu,\lambda), \tag{4.13}$$

by using the relations (4.12) and (4.13) we find the desired result. □

We have the following results

Theorem 4.3. *Let $s > \frac{2\alpha+3}{2}$, ψ be a Riemann-Liouville wavelet in on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$, , and $g \in L^2_\alpha\left(\mathbb{R}_+ \times \mathbb{K}\right)$, $\beta > 0$ then we have*

$$(i) \quad f^*_{g,\psi,\beta}(x,t) = \int_0^{+\infty}\int_{\hat{\mathbb{K}}} \frac{\varphi_{\mu,-\lambda}(x,t)\mathscr{F}_\alpha(\psi)\left(\frac{\mu}{a},\frac{\lambda}{a}\right)\mathscr{F}_\alpha(g(a,.))(\mu,\lambda)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi} a^{\frac{2\alpha+1}{2}} da \otimes d\gamma_\alpha(\mu,\lambda). \tag{4.14}$$

$$(ii) \quad \mathscr{F}_\alpha(f^*_{g,\psi,\beta})(\mu,\lambda) = \int_0^{+\infty} \frac{\mathscr{F}_\alpha(\psi)\left(\frac{\mu}{a},\frac{\lambda}{a}\right)\mathscr{F}_\alpha(g(a,.))(\mu,\lambda)}{\beta\left(1+\mu^2+2\lambda^2\right)^s + C_\psi} a^{\frac{2\alpha+1}{2}} da \tag{4.15}$$

$$(iii) \quad \|f^*_{g,\psi,\beta}\|_{H^s_\alpha} \leq \frac{\|g\|_{2,\theta_\alpha}}{\sqrt{2\beta}}. \tag{4.16}$$

Proof. (i) Is a consequence of (4.10), (4.11) and Fubini's theorem.

(ii) Is a consquence of Fubini's theorem and the relations (2.5),(4.10) and (4.11).

(iii)By using the relation (4.2) we find that

$$\|f^*_{g,\psi,\beta}\|^2_{H^s_\alpha} = \int_{\hat{\mathbb{K}}} \left(1+\mu^2+2\lambda^2\right)^s |\mathscr{F}_\alpha(f^*_{g,\psi,\beta})(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda),$$

by using Holder's inequality and the relations (3.2),(4.15) we find that

$$\|f^*_{g,\psi,\beta}\|^2_{H^s_\alpha} \leq \frac{1}{2\beta}\int_{\hat{\mathbb{K}}}\left(\int_0^{+\infty} |\mathscr{F}_\alpha(g(a,.))(\mu,\lambda)|^2 a^{2\alpha+2} da\right) d\gamma_\alpha(\mu,\lambda)$$

, By using Fubini's theorem and Plancherel's formula (2.7) we find that

$$\|f^*_{g,\psi,\beta}\|^2_{H^s_\alpha} \leq \frac{1}{2\beta}\|g\|^2_{2,\theta_\alpha}$$

which gives the result. □

corollary 4.1. *Let $s > \frac{2\alpha+3}{2}$, ψ be a Riemann-Liouville wavelet on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$, ,and $\beta > 0$, for all $f \in H^s_\alpha(\mathbb{K})$ and $g = S^\alpha_\psi(f)$, the extremal function $f^*_{S^\alpha_\psi(f),\psi,\beta}$ satisfies the following properties*

$$(i) \quad \mathscr{F}_\alpha(f^*_{S^\alpha_\psi(f),\psi,\beta})(\mu,\lambda) = \frac{C_\psi \mathscr{F}_\alpha(f)(\mu,\lambda)}{\beta\,(1+\mu^2+2\lambda^2)^s + C_\psi}. \tag{4.17}$$

$$(ii) \quad \|f^*_{S^\alpha_\psi(f),\psi,\beta}\|_{H^s_\alpha} \leq \sqrt{\frac{C_\psi}{2\beta}}\,\|f\|_{2,\mu_\alpha} \tag{4.18}$$

Proof. (i) By using the relations (3.2), (3.9) and (4.15) we find the result.

(ii) Is a consequence of (3.9) and (4.16). □

Theorem 4.4. *(Second Calderon's reproducing formula)*

Let $s > \frac{2\alpha+3}{2}$, ψ be a Riemann-Liouville wavelet in on $\mathbb{K}$ in $L^2_\alpha(\mathbb{K})$, ,and $\beta > 0$, for all $f \in H^s_\alpha(\mathbb{K})$ and $g = S^\alpha_\psi(f)$, the extremal function $f^*_{S^\alpha_\psi(f),\psi,\beta}$ satisfies

$$\lim_{\beta\to 0^+}\left\|f^*_{S^\alpha_\psi(f),\psi,\beta} - f\right\|_{H^s_\alpha} = 0.$$

Moreover we have $f^*_{S^\alpha_\psi(f),\psi,\beta} \longrightarrow f$ uniformly when $\beta \longrightarrow 0^+$.

Proof. By using the relation (4.17) we find that

$$\mathscr{F}_\alpha(f^*_{S^\alpha_\psi(f),\psi,\beta} - f)(\mu,\lambda) = \frac{-\beta\left(1+\mu^2+2\lambda^2\right)^s \mathscr{F}_\alpha(f)(\mu,\lambda)}{\beta\,(1+\mu^2+2\lambda^2)^s + C_\psi} \tag{4.19}$$

consequently we find that

$$\left\|f^*_{S^\alpha_\psi(f),\psi,\beta}-f\right\|_{H^s_\alpha}=\int_{\hat{\mathbb{K}}}\frac{\beta^2\left(1+\mu^2+2\lambda^2\right)^{3s}|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2}{\beta\left(1+\mu^2+2\lambda^2\right)^s+C_\psi}d\gamma_\alpha(\mu,\lambda)$$

by using the dominated convergence theorem and the fact that

$$\frac{\beta^2\left(1+\mu^2+2\lambda^2\right)^{3s}|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2}{\beta\left(1+\mu^2+2\lambda^2\right)^s+C_\psi}\leq\left(1+\mu^2+2\lambda^2\right)^s|\mathscr{F}_\alpha(f)(\mu,\lambda)|^2,$$

we deduce that

$$\lim_{\beta\to0^+}\left\|f^*_{S^\alpha_\psi(f),\psi,\beta}-f\right\|_{H^s_\alpha}=0$$

on the other hand by using inversion formula (2.5) and the relation (4.19) we find that

$$f^*_{S^\alpha_\psi(f),\psi,\beta}(y,v)-f(y,v)=\int_{\hat{\mathbb{K}}}\mathscr{F}_\alpha(f^*_{S^\alpha_\psi(f),\psi,\beta}-f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(y,v)}d\gamma_\alpha(\mu,\lambda),$$

$$=\int_{\hat{\mathbb{K}}}\frac{-\beta\left(1+\mu^2+2\lambda^2\right)^s\mathscr{F}_\alpha(f)(\lambda)\overline{\varphi_{\mu,\lambda}(y,v)}}{\beta\left(1+\mu^2+2\lambda^2\right)^s+C_\psi}d\gamma_\alpha(\mu,\lambda)$$

again by dominated convergence theorem and the fact that

$$\left|\frac{-\beta\left(1+\mu^2+2\lambda^2\right)^s\mathscr{F}_\alpha(f)(\lambda)\overline{\varphi_{\mu,\lambda}(y,s)}}{\beta\left(1+\mu^2+2\lambda^2\right)^s+C_\psi}\right|\leq|\mathscr{F}_\alpha(f)(\mu,\lambda)|$$

we deduce that

$$\lim_{\beta\to0^+}\left\|f^*_{S^\alpha_\psi(f),\psi,\beta}-f\right\|_{\infty,\mu_\alpha}=0$$

which proves that $f^*_{S^\alpha_\psi(f),\psi,\beta}\longrightarrow f$ uniformly when $\beta\longrightarrow0^+$. □

Authors' contribution

The authors contributed equally to this work.

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