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New Iterative Algorithm for Solving Constrained Convex Minimization Problem and Split Feasibility Problem

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Abstract. The purpose of this paper is to introduce a new iterative algorithm to approximate the fixed points of almost contraction mappings and generalized α-nonexpansive mappings. Also, we show that our proposed iterative algorithm converges weakly and strongly to the fixed points of almost contraction mappings and generalized α-nonexpansive mappings. Furthermore, it is proved analytically that our new iterative algorithm converges faster than one of the leading iterative algorithms in the literature for almost contraction mappings. Some numerical examples are also provided and used to show that our new iterative algorithm has better rate of convergence than all of S, Picard-S, Thakur and M iterative algorithms for almost contraction mappings and generalized α-nonexpansive mappings. Again, we show that the proposed iterative algorithm is stable with respect to T and data dependent for almost contraction mappings. Some applications of our main results and new iterative algorithm are considered. The results in this article are improvements, generalizations and extensions of several relevant results existing in the literature.

1. Introduction

Fixed point theory is concerned with solution of the equation

$$T\ell = \ell, \tag{1.1}$$

where T could be a nonlinear operator defined on a metric space. Any ℓ that solves (1.1) is called the fixed point of T and the collection all such elements is denoted by $F(T)$. Fixed point theory is

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an area in nonlinear analysis that has become very attractive and interesting with a large number of applications in various fields of mathematics and other branches of science. Fixed point theory has remained not only a field with a huge development, but also a very helpful means for solving various problems in different fields of mathematics. It is well known that fixed point theorems are used for proving the existence and uniqueness to various mathematical models like differential, integral and partial differential equations and variational inequalities problems etc., representing phenomena arising in different fields such as steady state temperature distribution, chemical equations, neutron transport theory, economic theories, epidemics and flow of fluids. Furthermore, it as also significant in the field of computer science, image processing, artificial intelligence, decision making, population dynamics, computer science, operational research, industrial engineering, pattern recognition, medicine, group health underwriting, management and many others.

Existence theorem is concerned with establishing sufficient conditions in which the equation (1.1) will have solution, but does not necessarily show how to find such solution. On the other hand, iteration method of fixed points is concerned with approximation or computation of sequences which converge to the solution of (1.1). When existence of a fixed point of an operator is guaranteed, obtaining constructive technique for finding such a fixed point is also paramount.

In 2003, Berinde [6] introduced the concept of weak contraction mappings which is also known as almost contraction mappings. He showed that the class of almost contraction mappings is more general than the class of Zamfirescu mappings [41] which includes contraction mappings, Kannan mappings [22] and Chatterjea mappings [10].

Throughout this paper, let Ω denote a Banach space and Λ a nonempty closed convex subset of Ω. Let $\mathbb{R}$ stand for set of real numbers.

Definition 1.1. A mapping $T : \Lambda \to \Lambda$ is called almost contraction if there exists a constant $\gamma \in (0, 1)$ and some constant $L \geq 0$, such that

$$\|T\ell - T\zeta\| \leq \gamma\|\ell - \zeta\| + L\|\ell - T\ell\|, \ \ \forall \ell, \zeta \in \Lambda. \tag{1.2}$$

Definition 1.2. A mapping $T : \Lambda \to \Lambda$ is said to be Suzuki generalized nonexpansive if for all $\ell, \zeta \in \Lambda$, we have

$$\frac{1}{2}\|\ell - T\ell\| \leq \|\ell - \zeta\| \Longrightarrow \|T\ell - T\zeta\| \leq \|\ell - \zeta\|.$$

Suzuki generalized nonexpansive mappings is also known as mappings satisfying condition (C). In [33], Suzuki showed that the class of Suzuki generalized nonexpansive mappings is more general than the class of nonexpansive mappings and obtained some fixed points and convergence theorems.

Definition 1.3. A mapping $T : \Lambda \to \Lambda$ is said to be α-nonexpansive if there exists $\alpha \in [0, 1)$ such that

$$\|T\ell - T\zeta\|^2 \leq \alpha\|T\ell - \zeta\|^2 + \alpha\|\ell - T\zeta\|^2 + (1 - 2\alpha)\|\ell - \zeta\|^2,$$

for all $\ell, \zeta \in \Lambda$.

The class of α-nonexpansive mappings was introduced in 2011 by Aoyama and Kohsaka [3] as generalization of nonexpansive mappings and further obtained some convergence results. It is worthy noting that nonexpansive mappings are continuous on their domains, but Suzuki-type generalized nonexpansive mappings and α-nonexpansive mappings need not be continuous (see [33]). Clearly, every nonexpansive mapping is an α-nonexpansive mapping with $\alpha = 0$ (i.e., 0-nonexpansive) and every α-nonexpansive mapping with a nonempty fixed point set is quasinonexpansive.

Definition 1.4. A mapping $T : \Lambda \to \Lambda$ is said to be generalized α-nonexpansive if there exists $\alpha \in [0, 1)$ such that

$$\begin{aligned} \frac{1}{2}\|\ell - T\ell\| &\leq \|\ell - \zeta\| \text{ implies} \\ \|T\ell - T\zeta\| &\leq \alpha\|T\ell - \zeta\| + \alpha\|T\zeta - \ell\| + (1 - 2\alpha)\|\ell - \zeta\| \end{aligned}$$

for all $\ell, \zeta \in \Lambda$.

In [26], Pant and Shukla introduced a wider class of nonexpansive mappings in Banach spaces known as generalized α-nonexpansive mappings which contains the class of Suzuki generalized nonexpansive mappings.

It is well known that the case of contraction mappings is simple and carries most of the good behavior using Picard iterative algorithm. But when we move to the case of nonexpansive mappings, the Picard iterative algorithm need not converge to a fixed point. Apparently, the conclusion of Banach contraction principle fails for nonexpansive mappings even if Λ is compact. As an example, one may consider a geometric rotation on the unit circle in the plane $\mathbb{R}^2$.

The limitation of Picard iterative algorithm gave many researchers in nonlinear analysis the room to construct more efficient iterative algorithms for approximating the fixed points of nonexpansive mappings and other classes of mappings which are more general than the class of nonexpansive mappings.

Some notable iterative algorithms in the existing literature are: Mann [24], Ishikawa [21], Noor [25], Argawal et al. [2], Abbas and Nazir [1], SP [27], S* [20], CR [12], Normal-S [28], Picard-S [17], Thakur [36], Thakur New [37], M [39], M* [38], Garodia and Uddin [16], Two-Step Mann [35] iterative algorithms and many others.

In 2007, the S iterative algorithm was introduced by Argawal et al. [2] as follows:

$$\begin{cases} \psi_0 \in \Lambda, \\ \mu_s = (1 - \beta_s)\psi_s + \beta_s T\psi_s, & \forall s \geq 1, \\ \psi_{s+1} = (1 - \delta_s)T\psi_s + \delta_s T\mu_s, \end{cases} \tag{1.3}$$

where $\{\delta_s\}$ and $\{\beta_s\}$ are sequences in [0,1].

In 2014, the Picard-S iterative algorithm was introduced by Gursoy and Karakaya [17] as follows:

$$\begin{cases} u_0 \in \Lambda, \\ \varphi_s = (1-\beta_s)u_s + \beta_s T u_s, \\ \varrho_s = (1-\delta_s)T u_s + \delta_s T \varphi_s, \\ u_{s+1} = T \varrho_s, \end{cases} \quad \forall s \geq 1, \tag{1.4}$$

where $\{\delta_s\}$ and $\{\beta_s\}$ are sequences in [0,1]. The authors showed with the aid of an example that Picard-S iterative algorithm (1.4) converges at a rate faster than all of Picard, Mann, Ishikawa, Noor, SP, CR, S, S*, Abbas and Nazir, Normal-S and Two-Step Mann iterative algorithms for contraction mappings.

In 2016, Thakur et al. [37] introduced the following three steps iterative algorithm:

$$\begin{cases} \omega_0 \in \Lambda, \\ \rho_s = (1-\beta_s)\omega_s + \beta_s T \omega_s, \\ v_s = T((1-\delta_s)\omega_s + \delta_s \rho_s), \\ \omega_{s+1} = T v_s, \end{cases} \quad \forall s \geq 1, \tag{1.5}$$

where $\{\delta_s\}$ and $\{\beta_s\}$ are sequences in [0,1]. With the help of numerical example, they proved that (1.5) is faster than Picard, Mann, Ishikawa, Agarwal, Noor and Abbas iterative algorithm for suzuki generalized nonexpansive mappings.

In 2018, Ullah and Arshad [39] introduced M iterative algorithm as follows:

$$\begin{cases} m_0 \in \Lambda, \\ c_s = (1-\delta_s)m_s + \delta_s T m_s, \\ d_s = T c_s, \\ m_{s+1} = T d_s, \end{cases} \quad \forall s \geq 1, \tag{1.6}$$

where $\{\delta_s\}$ is a sequence in [0,1]. Numerically they showed that M iterative algorithm (1.2) converges faster than S iterative algorithm (1.3) and Picard-S iterative algorithm (1.4) for Suzuki generalized nonexpansive mappings. Also, they noted that the speed of convergence of Picard-S iterative algorithm (1.4) and Thakur iterative algorithm (1.5) are almost same.

Motivated by the above results, in this paper, we construct a new four step iterative algorithm which outperforms the iterative algorithm (1.6) in terms of convergence rate for almost contraction mappings as follows:

$$\begin{cases} \ell_0 \in \Lambda, \\ g_s = (1-\beta_s)\ell_s + \beta_s T \ell_s, \\ w_s = (1-\delta_s)T\ell_s + \delta_s T g_s, \\ \zeta_s = T w_s, \\ \ell_{s+1} = T \zeta_s, \end{cases} \quad \forall s \geq 1, \tag{1.7}$$

where $\{\delta_s\}$ and $\{\beta_s\}$ are sequences in [0,1].

The purpose of this paper is to prove analytically that our new iterative algorithm converges faster than (1.6) for almost contraction mappings. In order to support our analytical proof, we use some new examples to show that our iterative algorithm (1.7) converges faster than (1.6) and a number of other leading iterative algorithms in the literature. We also prove the weak and strong convergence of new iterative algorithm (1.7) to the fixed points generalized α-nonexpansive mappings in a uniformly convex Banach spaces. Furthermore, we show that our new iterative algorithm is T-stable and data dependent. Finally, we use our new iterative algorithm (1.7) to solve a constrained convex minimization problem and a split feasibility problem.

2. Preliminaries

The following definitions, propositions and lemmas will be useful in proving our main results.

Definition 2.1. A Banach space Ω is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for $\ell, \zeta \in \Omega$ satisfying $\|\ell\| \leq 1$, $\|\zeta\| \leq 1$ and $\|\ell - \zeta\| > \epsilon$, we have $\left\|\frac{\ell+\zeta}{2}\right\| < 1-\delta$.

Definition 2.2. A Banach space Ω is said to satisfy Opial's condition if for any sequence $\{\ell_s\}$ in Ω which converges weakly to $\ell \in \Omega$ implies

$$\limsup_{s\to\infty} \|\ell_s - \ell\| < \limsup_{s\to\infty} \|\ell_s - \zeta\|, \ \forall\, \zeta \in \Omega \text{ with } \zeta \neq \ell.$$

Definition 2.3. Let $\{\ell_s\}$ be a bounded sequence in Ω. For $\ell \in \Lambda \subset \Omega$, we put

$$r(\ell, \{\ell_s\}) = \limsup_{s\to\infty} \|\ell_s - \ell\|.$$

The asymptotic radius of $\{\ell_s\}$ relative to Λ is defined by

$$r(\Lambda, \{\ell_s\}) = \inf\{r(\ell, \{\ell_s\}) : \ell \in \Lambda\}.$$

The asymptotic center of $\{\ell_s\}$ relative to Λ is given as:

$$A(\Lambda, \{\ell_s\}) = \{\ell \in \Lambda : r(\ell, \{\ell_s\}) = r(\Lambda, \{\ell_s\})\}.$$

In a uniformly convex Banach space, it is well known that $A(\Lambda, \{\ell_s\})$ consist of exactly one point.

Definition 2.4. [5] Let $\{a_s\}$ and $\{b_s\}$ be two sequences of real numbers that converge to a and b respectively, and assume that there exists

$$k = \lim_{s\to\infty} \frac{\|a_s - a\|}{\|b_s - b\|}.$$

Then,

(R_1) if $k = 0$, we say that $\{a_s\}$ converges faster to a than $\{b_s\}$ does to b.

(R_2) If $0 < k < \infty$, we say that $\{a_s\}$ and $\{b_s\}$ have the same rate of convergence.

Definition 2.5. [5] Let $\{\eta_s\}$ and $\{\phi_s\}$ be two fixed point iteration processes that converge to the same point z, the error estimates

$$\begin{aligned} \|\eta_s - z\| &\leq a_s, \ \forall s \geq 1, \\ \|\phi_s - z\| &\leq b_s, \ \forall s \geq 1, \end{aligned}$$

are available where $\{a_s\}$ and $\{b_s\}$ are two sequences of positive numbers converging to zero. Then we say that $\{\eta_s\}$ converges faster to z than $\{\phi_s\}$ does if $\{a_s\}$ converges faster than $\{b_s\}$.

Definition 2.6. [5] Let $T, \tilde{T} : \Lambda \to \Lambda$ be two operators. We say that $\tilde{T}$ is an approximate operator for T if for some $\epsilon > 0$, we have

$$\|T\ell - \tilde{T}\ell\| \leq \epsilon, \ \forall \ell \in \Lambda.$$

Definition 2.7. [18] Let $\{y_s\}$ be any sequence in Λ. Then, an iteration process $\ell_{s+1} = f(T, y_s)$, which converges to fixed point z, is said to be stable with respect to T, if for $\varepsilon_s = \|y_{s+1} - f(T, y_s)\|$, $\forall s \in \mathbb{N}$, we have

$$\lim_{s\to\infty} \varepsilon_s = 0 \Leftrightarrow \lim_{s\to\infty} y_s = z.$$

Definition 2.8. [31] A mapping $T : \Lambda \to \Lambda$ is said to satisfy condition (I) if a nondecreasing function $f : [0, \infty) \to [0, \infty)$ exists with $f(0) = 0$ and for all $r > 0$ then $f(r) > 0$ such that $\|\ell - T\ell\| \geq f(d(\ell, F(T))))$ for all $\ell \in \Lambda$, where $d(\ell, F(T)) = \inf_{z\in F(T)} \|\ell - z\|$.

Proposition 2.9. *[26] Let Λ be a nonempty subset of a Banach space Ω. Suppose $T : \Lambda \to \Lambda$ is any mapping. Then*

(i) *If T is a Suzuki generalized nonexpansive mapping, it follows that T is a generalized α-nonexpansive mapping.*

(ii) *Every generalized α-nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive mapping.*

(ii) *If T is a generalized α-nonexpansive mapping, then $F(T)$ is closed. Moreover, if Ω is strictly convex and Λ is convex, then $F(T)$ is also convex.*

(iv) *If T is a generalized α-nonexpansive mapping, then the following inequality holds:*

$$\|\ell - T\zeta\| \leq \left(\frac{3+\alpha}{1-\alpha}\right) \|\ell - T\ell\| + \|\ell - \zeta\|, \ \forall \ell, \zeta \in \Lambda.$$

Lemma 2.10. *[26] Let T be a self mapping on a subset Λ of a Banach space Ω which satisfies Opial's condition. Suppose T is a generalized α-nonexpansive mapping. If $\{\ell_s\}$ converges weakly to z and $\lim_{s\to\infty} \|T\ell_s - \ell_s\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero.*

Lemma 2.11. *[33] Let T be a self mapping on a weakly compact convex subset Λ of a Banach space Ω with the Opial's property. If T is a Suzuki generalized nonexpansive mapping, then T has a fixed point.*

Lemma 2.12. *[40] Let $\{‘_s\}$ and $\{\lambda_s\}$ be nonnegative real sequences satisfying the following inequalities:*

$$‘_{s+1} \leq (1-\sigma_s)‘_s + \lambda_s,$$

where $\sigma_s \in (0,1)$ for all $s \in \mathbb{N}$, $\sum\limits_{s=0}^{\infty} \sigma_s = \infty$ and $\lim\limits_{s\to\infty} \frac{{}_s}{\sigma_s} = 0$, then $\lim\limits_{s\to\infty} ‘_s = 0$.

Lemma 2.13. *[32] Let $\{‘_s\}$ be a nonnegative real sequence and there exits an $s_0 \in \mathbb{N}$ such that for all $s \geq s_0$ satisfying the following condition:*

$$‘_{s+1} \leq (1-\sigma_s)‘_s + \sigma_s\lambda_s,$$

where $\sigma_s \in (0,1)$ for all $s \in \mathbb{N}$, $\sum\limits_{s=0}^{\infty} \sigma_s = \infty$ and $\lambda_s \geq 0$ for all $s \in \mathbb{N}$, then

$$0 \leq \limsup_{s\to\infty} ‘_s \leq \limsup_{s\to\infty} \lambda_s.$$

Lemma 2.14. *[29] Suppose Ω is a uniformly convex Banach space and $\{\iota_s\}$ is any sequence satisfying $0 < p \leq \iota_s \leq q < 1$ for all $s \geq 1$. Suppose $\{\ell_s\}$ and $\{\zeta_s\}$ are any sequences of Ω such that $\limsup\limits_{s\to\infty} \|\ell_s\| \leq x$, $\limsup\limits_{s\to\infty} \|\zeta_s\| \leq x$ and $\limsup\limits_{s\to\infty} \|\iota_s\ell_s + (1-\iota_s)\zeta_s\| = x$ hold for some $x \geq 0$. Then $\lim\limits_{s\to\infty} \|\ell_s - \zeta_s\| = 0$.*

3. Rate of Convergence

In this section, we will prove that our new iterative algorithm (1.7) converges faster than the iterative algorithm (1.6) for almost contraction mappings.

Theorem 3.1. *Let Ω be a Banach space and let Λ be a nonempty closed convex subset of Ω. Let $T : \Lambda \to \Lambda$ be a mapping satisfying (1.2) with $F(T) \neq \emptyset$. Let $\{\ell_s\}$ be the iterative algorithm defined by (1.7) with sequences $\{\delta_s\}, \{\beta_s\} \in [0,1]$ such that $\sum\limits_{s=0}^{\infty} \delta_s\beta_s = \infty$, then $\{\ell_s\}$ converges strongly to a unique fixed point of T.*

Proof. Let $z \in F(T)$ and from (1.7), we have get

$$\begin{aligned}
\|g_s - z\| &= \|(1-\beta_s)\ell_s + \beta_s T\ell_s - z\| \\
&\leq (1-\beta_s)\|\ell_s - z\| + \beta_s\|T\ell_s - z\| \\
&\leq (1-\beta_s)\|\ell_s - z\| + \beta_s\gamma\|\ell_s - z\| \\
&= (1-(1-\gamma)\beta_s)\|\ell_s - z\|. \qquad (3.1)
\end{aligned}$$

Using (1.7) and (3.1), we have

$$\begin{aligned}
\|w_s - z\| &= \|(1-\delta_s)T\ell_s + \delta_s T g_s - z\| \\
&\leq (1-\delta_s)\|T\ell_s - z\| + \delta_s\|Tg_s - z\| \\
&\leq \gamma(1-\delta_s)\|\ell_s - z\| + \gamma\delta_s\|g_s - z\| \\
&\leq \gamma(1-\delta_s)\|\ell_s - z\| + \gamma\delta_s(1-(1-\gamma)\beta_s)\|\ell_s - z\| \\
&= \gamma(1-(1-\gamma)\delta_s\beta_s)\|\ell_s - z\|. \qquad (3.2)
\end{aligned}$$

From (1.7) and (3.2), we obtain

$$\begin{aligned}
\|\zeta_s - z\| &= \|Tw_s - z\| \\
&\leq \gamma\|w_s - z\| \\
&\leq \gamma^2(1-(1-\gamma)\delta_s\beta_s)\|\ell_s - z\|. \qquad (3.3)
\end{aligned}$$

Using (1.7) and (3.3), we have

$$\begin{aligned}
\|\ell_{s+1} - z\| &= \|T\zeta_s - z\| \\
&\leq \gamma\|\zeta_s - z\| \\
&\leq \gamma^3(1-(1-\gamma)\delta_s\beta_s)\|\ell_s - z\|. \qquad (3.4)
\end{aligned}$$

From (3.4), we have the following inequalities:

$$\begin{aligned}
\|\ell_{s+1} - z\| &\leq \gamma^3(1-(1-\gamma)\delta_s\beta_s)\|\ell_s - z\| \\
&\leq \gamma^3(1-(1-\gamma)\delta_{s-1}\beta_{s-1})\|\ell_{s-1} - z\| \\
&\vdots \\
\|\ell_1 - z\| &\leq \gamma^3(1-(1-\gamma)\delta_0\beta_0)\|\ell_0 - z\|. \qquad (3.5)
\end{aligned}$$

From (3.5), we get

$$\|\ell_{s+1} - z\| \leq \|\ell_0 - z\|\gamma^{3(s+1)}\prod_{t=0}^{s}(1-(1-\gamma)\delta_t\beta_t). \qquad (3.6)$$

Since $\gamma \in (0,1)$, $\delta_t, \beta_t \in [0,1]$ for all $t \in \mathbb{N}$, it follows that $(1-(1-\gamma)\delta_t\beta_t) \in (0,1)$. Since from classical analysis we know that $1 - \ell \leq e^{-\ell}$ for all $\ell \in [0,1]$, thus from (3.6), we have

$$\|\ell_{s+1} - z\| \leq \frac{\gamma^{3(s+1)}\|\ell_0 - z\|}{e^{(1-\gamma)\sum_{t=0}^{s}\delta_t\beta_t}}. \qquad (3.7)$$

If we take the limits of both sides of (3.7), we get $\lim_{s\to\infty}\|\ell_s - z\| = 0$.

□

Theorem 3.2. *Let Ω be a Banach space and let Λ be a nonempty closed convex subset of Ω. Let $T : \Lambda \to \Lambda$ be a mapping satisfying (1.2) with $F(T) \neq \emptyset$. For given $\ell_0 = m_0 \in \Lambda$, let $\{\ell_s\}$ and $\{m_s\}$ be the iterative algorithms defined by (1.7) and (1.6), respectively, with real sequences $\{\delta_s\}$ and $\{\beta_s\}$ in [0,1] such that $\delta_s \leq \delta < 1$ and $\beta_s \leq \beta < 1$, for all $s \in \mathbb{N}$ and for some $\delta, \beta > 0$. Then $\{\ell_s\}$ converges to z faster than $\{m_s\}$ does.*

Proof. From (3.6) in Theorem 3.1 together with the assumptions $\alpha_s \leq \alpha < 1$ and $\beta_s \leq \beta < 1$, for all $s \in \mathbb{N}$ and for some $\alpha, \beta > 0$, then we have

$$\begin{aligned} \|\ell_{s+1} - z\| &\leq \|\ell_0 - z\|\gamma^{3(s+1)} \prod_{t=0}^{s} (1 - (1-\gamma)\alpha_t\beta_t) \\ &= \|\ell_0 - z\|\gamma^{3(s+1)}(1 - (1-\gamma)\alpha\beta)^{s+1}. \end{aligned} \tag{3.8}$$

Similarly, from (1.6), we get

$$\begin{aligned} \|c_s - z\| &= \|(1-\delta_s)m_s + \delta_s T m_s - z\| \\ &\leq (1-\delta_s)\|m_s - z\| + \delta_s\|T m_s - z\| \\ &\leq (1-\delta_s)\|m_s - z\| + \delta_s\gamma\|m_n - z\| \\ &= (1 - (1-\gamma)\delta_s)\|m_s - z\|. \end{aligned} \tag{3.9}$$

Using (1.6) and (3.9), we get

$$\begin{aligned} \|d_s - z\| &= \|T c_s - z\| \\ &\leq \gamma\|c_s - z\| \\ &\leq \gamma(1 - (1-\gamma)\delta_s)\|m_s - z\|. \end{aligned} \tag{3.10}$$

Finally, from (1.6) and (3.10), we obtain

$$\begin{aligned} \|m_{s+1} - z\| &= \|T d_s - z\| \\ &\leq \gamma\|d_s - z\| \\ &\leq \gamma^2(1 - (1-\gamma)\delta_s)\|m_s - z\|. \end{aligned} \tag{3.11}$$

From (3.11), we have the following inequalities:

$$\begin{aligned} \|m_{s+1} - z\| &\leq \gamma^2(1 - (1-\gamma)\delta_s)\|m_s - z\| \\ &\leq \gamma^2(1 - (1-\gamma)\delta_{s-1})\|m_{s-1} - z\| \\ &\vdots \\ \|m_1 - z\| &\leq \gamma^2(1 - (1-\gamma)\delta_0)\|m_0 - z\|. \end{aligned} \tag{3.12}$$

From (3.12), we get

$$\|m_{s+1} - z\| \leq \|m_0 - z\|\gamma^{2(s+1)} \prod_{t=0}^{s} (1 - (1-\gamma)\delta_t).$$

Since $\delta_s \leq \delta < 1$ and $\beta_s \leq \beta < 1$, for all $s \in \mathbb{N}$ and for some $\delta, \beta > 0$, then we have

$$\begin{aligned} \|m_{s+1} - z\| &\leq \|m_0 - z\|\gamma^{2(s+1)} \prod_{t=0}^{s} (1 - (1-\gamma)\delta_t) \\ &= \|m_0 - z\|\gamma^{2(s+1)} (1 - (1-\gamma)\delta)^{s+1}. \end{aligned}$$

Set

$$a_s = \|\ell_0 - z\|\gamma^{3(s+1)}(1 - (1-\gamma)\delta)^{s+1},$$

and

$$b_s = \|\ell_0 - z\|\gamma^{2(s+1)}(1 - (1-\gamma)\delta)^{s+1}. \tag{3.13}$$

Hence,

$$\frac{a_s}{b_s} = \frac{\|\ell_0 - z\|\gamma^{3(s+1)}(1 - (1-\gamma)\delta\beta)^{s+1}}{\|m_0 - z\|\gamma^{2(s+1)}(1 - (1-\gamma)\delta)^{s+1}} \to 0 \text{ as } s \to \infty.$$

This implies that our new iterative algorithm (1.7) converges faster to z than M iterative algorithm (1.6). □

In order to support analytical prove in Theorem 3.2 and demonstrate the advantage of our new iterative algorithm (1.7), we give the following example.

Example 3.3. Let $\Omega = \Re$ and $\Lambda = [1, 50]$. Let $T : \Lambda \to \Lambda$ be a mapping defined by $T(\ell) = \sqrt{\ell^2 - 8\ell + 40}$. Obviously, 5 is the fixed point of T. Take $\delta_s = \beta_s = \frac{3}{4}$, with an initial value of $\ell_1 = 50$.

By writing all the codes in MATLAB (R2015a) for Example 3.3, we obtain the following comparison Table 1 and Figure 1.

TABLE 1. Comparison of convergence behaviour of our new iterative algorithm with S, Picard-S, Thakur and M iterative algorithms.

Step	S	Picard-S	Thakur	M	New
1	50.00000000	50.00000000	50.00000000	50.00000000	50.00000000
2	44.16905011	40.46668490	40.46648707	39.77487312	36.79428091
3	38.40054569	31.13624438	31.13566491	29.79220887	24.07958149
4	32.71513008	22.15533283	22.15389446	20.25245189	12.59321471
5	27.14503094	13.88761070	13.88380778	11.71208997	5.60936561
6	21.74399379	7.46589475	7.45557218	6.06597569	5.00355869
7	16.60935306	5.14776230	5.14203305	5.02641919	5.00001569
8	11.93484164	5.00348330	5.00331403	5.00042732	5.00000000
9	8.12786414	5.00007676	5.00007301	5.00000684	5.00000000
10	5.84725921	5.00000169	5.00000161	5.00000011	5.00000000
11	5.12789697	5.00000004	5.00000004	5.00000000	5.00000000
12	5.01483168	5.00000000	5.00000000	5.00000000	5.00000000
13	5.00164168	5.00000000	5.00000000	5.00000000	5.00000000

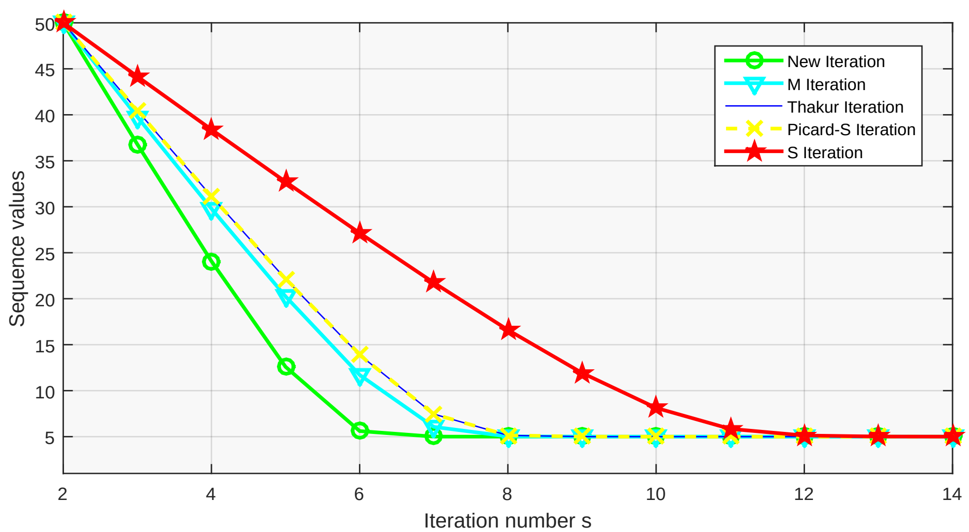

FIGURE 1. Graph corresponding to Table 1.

4. Convergence Results

In this section, we will prove the weak and strong convergence of our new iterative algorithm (1.7) for generalized α–nonexpansive mappings in the framework of uniformly convex Banach spaces.

Firstly, we will state and prove the following lemmas which will be useful in obtaining our main results.

Lemma 4.1. *Let Ω be a Banach space and Λ be a nonempty closed convex subset of Ω. Let $T : \Lambda \to \Lambda$ be a generalized α–nonexpansive mapping with $F(T) \neq \emptyset$. If $\{\ell_s\}$ is the iterative algorithm defined by (1.7), then $\lim_{s\to\infty} \|\ell_s - z\|$ exists for all $z \in F(T)$.*

Proof. Let $z \in F(T)$. By Proposition 2.9(ii), we know that every Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping. Then, from (1.7), we have

$$\begin{aligned} \|g_s - z\| &= \|(1-\beta_s)\ell_s + \beta_s T\ell_s - z\| \\ &\leq (1-\beta_s)\|\ell_s - z\| + \beta_s\|T\ell_s - z\| \\ &\leq (1-\beta_s)\|\ell_s - z\| + \beta_s\|\ell_s - z\| \\ &= \|\ell_s - z\|. \end{aligned} \tag{4.1}$$

Using (1.7) and (4.1), we obtain

$$\begin{aligned} \|w_s - z\| &= \|(1-\delta_s)T\ell_s + \delta_s Tg_s - z\| \\ &\leq (1-\delta_s)\|T\ell_s - z\| + \delta_s\|Tg_s - z\| \\ &\leq (1-\delta_s)\|\ell_s - z\| + \delta_s\|g_s - z\| \\ &\leq (1-\delta_s)\|\ell_s - z\| + \delta_s\|\ell_s - z\| \\ &= \|\ell_s - z\|. \end{aligned} \tag{4.2}$$

Again, using (1.7) and (4.2), we get

$$\begin{aligned} \|\zeta_s - z\| &= \|Tw_s - z\| \\ &\leq \|w_s - z\| \\ &\leq \|\ell_s - z\|. \end{aligned} \tag{4.3}$$

Lastly, from (1.7) and (4.3), we have

$$\begin{aligned} \|\ell_s - z\| &= \|T\zeta_s - z\| \\ &\leq \|\zeta_s - z\| \\ &\leq \|\ell_s - z\|. \end{aligned} \tag{4.4}$$

This implies that $\{\|\ell_s - z\|\}$ is bounded and nondecreasing for all $z \in F(T)$. Hence, $\lim_{s\to\infty} \|\ell_s - z\|$ exists. □

Lemma 4.2. *Let Ω be a uniformly convex Banach space and Λ be a nonempty closed convex subset of Ω. Let $T : \Lambda \to \Lambda$ be a generalized α–nonexpansive mapping. Suppose $\{\ell_s\}$ is the iterative algorithm defined by (1.7). Then, $F(T) \neq \emptyset$ if and only if $\{\ell_s\}$ is bounded and $\lim_{s\to\infty} \|T\ell_s - \ell_s\| = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $z \in F(T)$. Then, by Lemma 4.1, $\lim_{s\to\infty} \|\ell_s - z\|$ exists and $\{\ell_s\}$ is bounded. Put

$$\lim_{s\to\infty} \|\ell_s - z\| = x. \tag{4.5}$$

From (4.4) and (4.5), we obtain

$$\limsup_{s\to\infty} \|g_s - z\| \leq \limsup_{s\to\infty} \|\ell_s - z\| = x. \tag{4.6}$$

From Proposition 2.9(ii), we know that every generalized α–nonexpansive mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping. So that we have

$$\limsup_{s\to\infty} \|T\ell_s - z\| \leq \limsup_{s\to\infty} \|\ell_s - z\| = x. \tag{4.7}$$

Again, using (1.7), we get

$$\begin{aligned}
\|\ell_{s+1} - z\| &= \|T\zeta_s - z\| \\
&\leq \|\zeta_s - z\| \\
&= \|Tw_s - z\| \\
&\leq \|w_s - z\| \\
&= \|(1-\delta_s)T\ell_s + \delta_s T g_s - z\| \\
&\leq (1-\delta_s)\|T\ell_s - z\| + \delta_s\|Tg_s - z\| \\
&\leq (1-\delta_s)\|\ell_s - z\| + \delta_s\|g_s - z\| \\
&= \|\ell_s - z\| - \delta_s\|\ell_s - z\| + \delta_s\|g_s - z\|. \qquad (4.8)
\end{aligned}$$

From (4.8), we have

$$\frac{\|\ell_{s+1} - z\| - \|\ell_s - z\|}{\delta_s} \leq \|g_s - z\| - \|\ell_s - z\|. \tag{4.9}$$

Since $\delta_s \in [0,1]$, then from (4.9), we have

$$\|\ell_{s+1} - z\| - \|\ell_s - z\| \leq \frac{\|\ell_{s+1} - z\| - \|\ell_s - z\|}{\delta_s} \leq \|g_s - z\| - \|\ell_s - z\|,$$

which implies that

$$\|\ell_{s+1} - z\| \leq \|g_s - z\|.$$

Therefore, from (4.5), we obtain

$$x \leq \liminf_{s\to\infty} \|g_s - z\|. \tag{4.10}$$

From (4.6) and (4.10) we obtain

$$\begin{aligned} x &= \lim_{s\to\infty} \|g_n - z\| \\ &= \lim_{s\to\infty} \|(1-\beta_s)\ell_s + \beta_s T\ell_s - z\| \\ &= \lim_{s\to\infty} \|(1-\beta_s)(\ell_s - z) + \beta_s(T\ell_s - z)\| \\ &= \lim_{s\to\infty} \|\beta_s(T\ell_s - z) + (1-\beta_s)(\ell_s - z)\|. \end{aligned} \tag{4.11}$$

From (4.5), (4.7), (4.11) and Lemma 2.14, we obtain

$$\lim_{s\to\infty} \|T\ell_s - \ell_s\| = 0. \tag{4.12}$$

Conversely, assume that $\{\ell_s\}$ is bounded and $\lim_{s\to\infty} \|T\ell_s - \ell_s\| = 0$. Let $z \in A(\Lambda, \{\ell_s\})$, by Definition 2.3 and Proposition 2.9(iv), we have

$$\begin{aligned} (Tz, \{\ell_s\}) &= \limsup_{s\to\infty} \|\ell_s - Tz\| \\ &\leq \limsup_{s\to\infty} \left(\frac{(3+\alpha)}{(1-\alpha)} \|T\ell_s - \ell_s\| + \|\ell_s - z\| \right) \\ &= \limsup_{s\to\infty} \|\ell_s - z\| \\ &= r(z, \{\ell_s\}). \end{aligned} \tag{4.13}$$

This implies that $z \in A(\Lambda, \{\ell_s\})$. Since Ω is uniformly convex, $A(\Lambda, \{\ell_s\})$ is singleton, thus we have $Tz = z$. □

Theorem 4.3. *Let Ω, Λ, T be same as in Lemma 4.2. Suppose tat Ω satisfies Opial's condition and $F(T) \neq \emptyset$. Then, the sequence $\{\ell_s\}$ defined by (1.7) converges weakly to a fixed point of T.*

Proof. Let $z \in F(T)$, then by Lemma 4.1, we have $\lim_{s\to\infty} \|\ell_s - z\|$ exists. Now we show that $\{\ell_s\}$ has weak sequential limit in $F(T)$. Let ℓ and ζ be weak limits of the subsequences $\{\ell_{s_j}\}$ and $\{\ell_{s_k}\}$ of $\{\ell_s\}$, respectively. By Lemma 4.2, we have $\lim_{s\to\infty} \|T\ell_s - \ell_s\| = 0$ and from Lemma 2.10, $I - T$ is demiclosed at zero. It follows that $(I - T)\ell = 0$ implies $\ell = T\ell$, similarly $T\zeta = \zeta$.

Next we show uniqueness. Suppose $\ell \neq \zeta$, then by Opial's property, we obtain

$$\begin{aligned} \lim_{s\to\infty} \|\ell_s - \ell\| &= \lim_{s_j\to\infty} \|\ell_{s_j} - \ell\| \\ &< \lim_{s_j\to\infty} \|\ell_{s_j} - \zeta\| \\ &= \lim_{s\to\infty} \|\ell_s - \zeta\| \\ &= \lim_{s_k\to\infty} \|\ell_{s_k} - \zeta\| \\ &< \lim_{s_k\to\infty} \|\ell_{s_k} - \ell\| \\ &= \lim_{s\to\infty} \|\ell_s - \ell\|, \end{aligned} \tag{4.14}$$

which is a contradiction, so $\ell = \zeta$. Hence, $\{\ell_s\}$ converges weakly to a fixed point of T. □

Theorem 4.4. *Let Ω, Λ, T be same as in Lemma 4.2. Then, the iterative algorithm $\{\ell_s\}$ defined by (1.7) converges strongly to a point of $F(T)$ if and only if $\liminf_{s\to\infty} d(\ell_s, F(T)) = 0$, where $d(\ell_s, F(T)) = \inf\{\|\ell - z\| : z \in F(T)\}$.*

Proof. Necessity is obvious. Assume that $\liminf_{s\to\infty} d(\ell_s, F(T)) = 0$. From Lemma 4.1, we have $\lim_{s\to\infty} \|\ell_s - z\|$ exists for all $z \in F(T)$, it follows that $\liminf_{s\to\infty} d(\ell_s, F(T))$ exists. But by hypothesis, $\liminf_{s\to\infty} d(\ell_s, F(T)) = 0$, thus $\lim_{s\to\infty} d(\ell_s, F(T)) = 0$. Next we prove that $\{\ell_s\}$ is a Cauchy sequence in Λ. Since $\liminf_{s\to\infty} d(\ell_s, F(T)) = 0$, then given $\varepsilon > 0$, there exists $s_0 \in \mathbb{N}$ such that, for all $s, n \geq s_0$, we have

$$\begin{aligned} d(\ell_s, F(T)) &\leq \frac{\epsilon}{2}, \\ d(\ell_n, F(T)) &\leq \frac{\epsilon}{2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|\ell_s - \ell_n\| &\leq \|\ell_s - z\| + \|\ell_n - z\| \\ &\leq d(\ell_s, F(T)) + d(\ell_n, F(T)) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\{\ell_s\}$ is a Cauchy sequence in Λ. Since Λ is closed, therefore there exists a point $\ell_1 \in \Lambda$ such that $\lim_{s\to\infty} \ell_s = \ell_1$. Since $\lim_{s\to\infty} d(\ell_s, F(T)) = 0$, it implies that $\lim_{s\to\infty} d(\ell_1, F(T)) = 0$. Hence, $\ell_1 \in F(T)$ since $F(T)$ closed. $\square$

Theorem 4.5. *Let Ω, Λ, T be same as in Lemma 4.2. If T satisfies condition (I), then the iterative algorithm $\{\ell_s\}$ defined by (1.7) converges strongly to a fixed point of T.*

Proof. We have shown in Lemma 4.2 that

$$\lim_{s\to\infty} \|T\ell_s - \ell_s\| = 0. \tag{4.15}$$

Using condition (I) in Definition 2.8 and (4.15), we get

$$\lim_{s\to\infty} f(d(\ell_s, F(T))) \leq \lim_{s\to\infty} \|T\ell_s - \ell_s\| = 0, \tag{4.16}$$

i.e., $\lim_{s\to\infty} f(d(\ell_s, F(T))) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, we have

$$\lim_{s\to\infty} d(\ell_s, F(T)) = 0. \tag{4.17}$$

From Theorem 4.4, then sequence $\{\ell_s\}$ converges strongly to a point of $F(T)$. $\square$

5. Numerical Result

In this section, we provide an example of generalized α-nonexpansive mapping which is not Suzuki generalized nonexpansive mapping. With the aid of the provided example, we will prove that our new iterative algorithm (1.7) outperforms a number of iterative algorithms in the existing literature in terms of convergence.

Example 5.1. Let $\Lambda = [0, \infty)$ be endowed with the usual norm $|\cdot|$ and let $T : \Lambda \to \Lambda$ be defined as:

$$T\ell = \begin{cases} 0, & \text{if } \ell \in [0, \frac{1}{5}), \\ \frac{3\ell}{4}, & \text{if } \ell \in [\frac{1}{5}, \infty). \end{cases} \tag{5.1}$$

Firstly, we show that T does not satisfy condition (C). To see this, let $\ell = \frac{1}{15}$ and $\zeta = \frac{1}{5}$, then

$$\frac{1}{2}|\ell - T\ell| = \frac{1}{30} < \frac{2}{15} = |\ell - \zeta|.$$

But

$$|T\ell - T\zeta| = \frac{3\zeta}{4} = \frac{3}{20} > \frac{2}{15} = |\ell - \zeta|.$$

Hence, T does not satisfy condition (C), which implies that T is not a Suzuki generalized nonexpansive mapping.

Now we show that T is a generalized α-nonexpansive mapping with $\alpha = \frac{1}{3}$ (i.e., generalized $\frac{1}{3}$-nonexpansive). We consider the following cases:

Case (a): When $\ell, \zeta \in [0, \frac{1}{5})$, we have

$$\frac{1}{3}|T\ell - \zeta| + \frac{1}{3}|\ell - T\zeta| + \frac{1}{3}|\ell - \zeta| \geq 0 = |T\ell - T\zeta|.$$

Case (b): When $\ell, \zeta \in [\frac{1}{5}, \infty)$, we obtain

$$\begin{aligned} \frac{1}{3}|T\ell - \zeta| + \frac{1}{3}|\ell - T\zeta| + \frac{1}{3}|\ell - \zeta| &= \frac{1}{3}\left|\frac{3\ell}{4} - \zeta\right| + \frac{1}{3}\left|\ell - \frac{3\zeta}{4}\right| + \frac{1}{3}|\ell - \zeta| \\ &\geq \frac{1}{3}\left|\left(\frac{3\ell}{4} - \zeta\right) + \left(\ell - \frac{3\zeta}{4}\right)\right| + \frac{1}{3}|\ell - \zeta| \\ &= \frac{7}{12}|\ell - \zeta| + \frac{1}{3}|\ell - \zeta| \\ &= \frac{11}{12}|\ell - \zeta| \\ &\geq \frac{3}{4}|\ell - \zeta| = |T\ell - T\zeta|. \end{aligned}$$

Case (c): When $\ell \in [\frac{1}{5}, \infty)$ and $\zeta \in [0, \frac{1}{5})$, we get

$$\begin{aligned} \frac{1}{3}|T\ell - \zeta| + \frac{1}{3}|\ell - T\zeta| + \frac{1}{3}|\ell - \zeta| &= \frac{1}{3}\left|\frac{3\ell}{4} - \zeta\right| + \frac{1}{3}|\ell| + \frac{1}{3}|\ell - \zeta| \\ &\geq \frac{1}{3}\left|\frac{3\ell}{4} - \zeta\right| + \frac{1}{3}|\ell - \zeta| \\ &\geq \frac{7\ell}{12} = |T\ell - T\zeta|. \end{aligned}$$

Hence, T is generalized α-nonexpansive mapping with $\alpha = \frac{1}{3}$ (i.e., generalized $\frac{1}{3}$-nonexpansive) with $F(T) = \{0\}$.

With the aid of MATLAB (R2015a), we obtain the following comparison Table 2 and Figure 2 for various iterative algorithms with control sequences $\delta_s = 0.65$, $\beta_s = 0.8$ and initial guess $\ell_1 = 50$.

Table 2. Comparison of convergence behaviour of our new iterative algorithm with S, Picard-S, Thakur and M iterative algorithms.

Step	S	Picard-S	Thakur	M	New
1	50.00000000	50.00000000	50.00000000	50.00000000	50.00000000
2	32.62500000	24.46875000	24.46875000	23.55468750	18.35156250
3	21.28781250	11.97439453	11.97439453	11.09646606	6.73559692
4	13.89029766	5.85996932	5.85996932	5.22747581	2.47217456
5	9.06341922	2.86772249	2.86772249	2.46263118	0.00000000
6	5.91388104	1.40339169	1.40339169	1.16013016	0.00000000
7	3.85880738	0.00000000	0.00000000	0.00000000	0.00000000
8	2.51787182	0.00000000	0.00000000	0.00000000	0.00000000
9	1.64291136	0.00000000	0.00000000	0.00000000	0.00000000

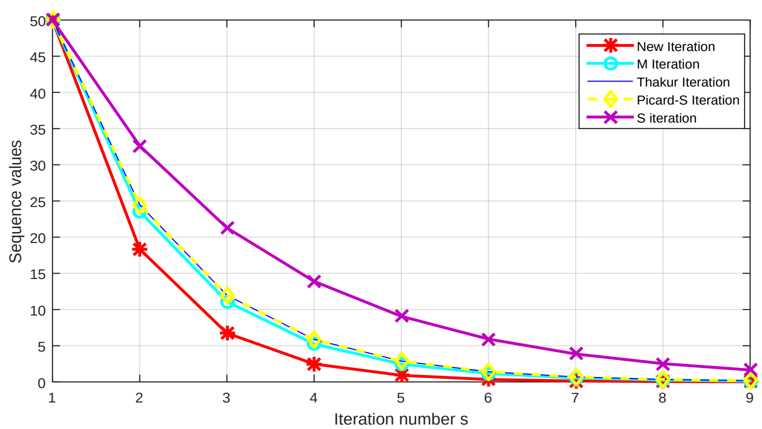

Figure 2. Graph corresponding to Table 2.

From the above Table 2 and Figure 2, it is clear that our new iterative algorithm (1.7) outperforms a number of existing iterative algorithms.

6. Stability result

Our aim in this section is to show that our new iterative algorithm (1.7) is T–Stable.

Theorem 6.1. *Let Ω be a Banach space and Λ be a nonempty closed convex subset of Ω. Let T be a mapping satisfy (1.2). Let $\{\ell_s\}$ be the iterative algorithm defined by (1.7) with sequences δ_s and $\beta_s \in [0,1]$ such that $\sum_{s=0}^{\infty} \delta_s\beta_s = \infty$. Then the iterative algorithm (1.7) is T–stable.*

Proof. Let $\{y_s\} \subset \Omega$ be an arbitrary sequence in Λ and suppose that the sequence iteratively generated by (1.7) is $\ell_{s+1} = f(G, y_s)$ converging to a unique point z and that $\varepsilon_s = \|y_{s+1} - f(T, y_s)\|$. To prove that (1.7) is T-stable, we have to show that $\lim_{s\to\infty} \varepsilon_s = 0 \Leftrightarrow \lim_{s\to\infty} y_s = z$.

Let $\lim_{s\to\infty} \varepsilon_s = 0$. Then from (1.7) and (1.6), we obtain

$$\begin{aligned}
\|y_{s+1} - z\| &= \|y_{s+1} - f(T, y_s) + f(T, y_s) - z\| \\
&\leq \|y_{s+1} - f(T, y_s)\| + \|f(T, y_s) - z\| \\
&= \varepsilon_s + \|f(T, y_s) - z\| \\
&= \varepsilon_s + \|T(T((1-\delta_s)Ty_s + \delta_s T((1-\beta_s)y_s + \beta_s Ty_s))) - z\| \\
&= \gamma^3(1 - (1-\gamma)\delta_s\beta_s)\|y_s - z\| + \varepsilon_s. \qquad (6.1)
\end{aligned}$$

For all $s \geq 1$, put

$$\begin{aligned}
\theta_s &= \|y_s - z\|, \\
\sigma_s &= (1-\gamma)\delta_s\beta_s \in (0,1), \\
\lambda_s &= \varepsilon_s.
\end{aligned}$$

Since $\lim_{s\to\infty} \varepsilon_s = 0$, this implies that $\frac{\lambda_s}{\sigma_s} = \frac{\varepsilon_s}{(1-\gamma)\delta_s\beta_s} \to 0$ as $s \to \infty$. Apparently, all the conditions of Lemma 2.12 are fulfilled. Hence, from Lemma 2.12 we have $\lim_{s\to\infty} y_s = z$.

Conversely, let $\lim_{s\to\infty} y_s = z$. The we have

$$\begin{aligned}
\varepsilon_s &= \|y_{s+1} - f(T, y_s)\| \\
&= \|y_{s+1} - z + z - f(T, y_s)\| \\
&\leq \|y_{s+1} - z\| + \|f(T, y_s) - z\| \\
&\leq \|y_{s+1} - z\| + \gamma^3(1 - (1-\gamma)\delta_s\beta_s)\|y_s - z\|. \qquad (6.2)
\end{aligned}$$

From (6.2), it follows that $\lim_{s\to\infty} \varepsilon_s = 0$. Hence, our new iterative algorithm (1.7) is stable with respect to T. $\square$

7. Data Dependence result

In this section, we obtain data dependence result for the mapping T satisfying (1.2) by utilizing our new iterative algorithm (1.7).

Theorem 7.1. *Let $\tilde{T}$ be an approximate operator of a mapping T satisfying (1.2). Let $\{\ell_s\}$ be an iterative sequence generated by (1.7) for T and define an iterative algorithm as follows:*

$$\begin{cases} \tilde{\ell}_0 \in \Lambda, \\ \tilde{g}_s = (1-\beta_s)\tilde{\ell}_s + \beta_s \tilde{T}\tilde{\ell}_s, \\ \tilde{w}_s = (1-\delta_s)\tilde{T}\tilde{\ell}_s + \delta_s \tilde{T}\tilde{g}_s, \qquad \forall s \geq 1, \\ \tilde{\zeta}_s = \tilde{T}\tilde{w}_s, \\ \tilde{\ell}_{s+1} = \tilde{T}\tilde{\zeta}_s, \end{cases} \tag{7.1}$$

where $\{\delta_s\}$ and $\{\beta_s\}$ are sequences in $[0,1]$ satisfying the following conditions:
(i) $\frac{1}{2} \leq \delta_s\beta_s$, $\forall\, s \in \mathbb{N}$,
(ii) $\sum\limits_{s=0}^{\infty} \delta_s\beta_s = \infty$.
If $Tz = z$ and $\tilde{T}\tilde{z} = \tilde{z}$ such that $\lim\limits_{s\to\infty} \tilde{\ell}_s = \tilde{z}$, we have

$$\|z - \tilde{z}\| \leq \frac{7\epsilon}{1-\gamma},$$

where $\epsilon > 0$ is a fixed number.

Proof. Using (1.7), (1.2) and (7.1), we have

$$\begin{aligned} \|\ell_{s+1} - \tilde{\ell}_{s+1}\| &= \|T\zeta_s - \tilde{T}\tilde{\zeta}_s\| \\ &= \|T\zeta_s - T\tilde{\zeta}_s + T\tilde{\zeta}_s - \tilde{T}\tilde{\zeta}_s\| \\ &\leq \|T\zeta_s - T\tilde{\zeta}_s\| + \|T\tilde{\zeta}_s - \tilde{T}\tilde{\zeta}_s\| \\ &\leq \gamma\|\zeta_s - \tilde{\zeta}_s\| + L\|\zeta_s - T\zeta_s\| + \epsilon. \end{aligned} \tag{7.2}$$

From (1.7), (1.2) and (7.1), we have

$$\begin{aligned} \|\zeta_s - \tilde{\zeta}_s\| &= \|Tw_s - \tilde{T}\tilde{w}_s\| \\ &= \|Tw_s - T\tilde{w}_s + T\tilde{w}_s - \tilde{T}\tilde{w}_s\| \\ &\leq \|Tw_s - T\tilde{w}_s\| + \|T\tilde{w}_s - \tilde{T}\tilde{w}_s\| \\ &\leq \gamma\|w_s - \tilde{w}_s\| + L\|w_s - Tw_s\| + \epsilon. \end{aligned} \tag{7.3}$$

Putting (7.3) into (7.2), we have

$$\begin{aligned} \|\ell_{s+1} - \tilde{\ell}_{s+1}\| &\leq \gamma^2\|w_s - \tilde{w}_s\| + \gamma L\|w_s - Tw_s\| \\ &\quad + \gamma\epsilon + L\|\zeta_s - T\zeta_s\| + \epsilon. \end{aligned} \tag{7.4}$$

Again, using (1.7), (1.2) and (7.1), we get

$$\begin{aligned} \|w_s - \tilde{w}_s\| &= (1-\delta_s)\|T\ell_s - \tilde{T}\tilde{\ell}_s\| + \delta_s\|Tg_s - \tilde{T}\tilde{g}_s\| \\ &\leq (1-\delta_s)\{\|T\ell_s - T\tilde{\ell}_s\| + \|T\tilde{\ell}_s - \tilde{T}\tilde{\ell}_s\|\} \\ &\quad +\delta_s\{\|Tg_s - T\tilde{g}_s\| + \|T\tilde{g}_s - \tilde{T}\tilde{g}_s\|\} \\ &\leq (1-\delta_s)\{\gamma\|\ell_s - \tilde{\ell}_s\| + L\|\ell_s - T\ell_s\| + \epsilon\} \\ &\quad +\delta_s\{\gamma\|g_s - \tilde{g}_s\| + L\|g_s - Tg_s\| + \epsilon\}. \end{aligned} \tag{7.5}$$

Using (1.7), (1.2) and (7.1), we get

$$\begin{aligned} \|g_s - \tilde{g}_s\| &\leq (1-\beta_s)\|\ell_s - \tilde{\ell}_s\| + \beta_s\|T\ell_s - \tilde{T}\tilde{\ell}_s\| \\ &\leq (1-\beta_s)\|\ell_s - \tilde{\ell}_s\| + \beta_s\{\|T\ell_s - T\tilde{\ell}_s\| + \|T\tilde{\ell}_s - \tilde{T}\tilde{\ell}_s\|\} \\ &\leq (1-\beta_s)\|\ell_s - \tilde{\ell}_s\| + \beta_s\{\gamma\|\ell_s - \tilde{\ell}_s\| + L\|\ell_s - T\ell_s\| + \epsilon\} \\ &= [1-(1-\gamma)\beta_s]\|\ell_s - \tilde{\ell}_s\| + \beta_s L\|\ell_s - T\ell_s\| + \beta_s\epsilon \end{aligned} \tag{7.6}$$

Using (7.6) and (7.5), we have

$$\begin{aligned} \|w_s - \tilde{w}_s\| &\leq (1-\delta_s)\{\gamma\|\ell_s - \tilde{\ell}_s\| + L\|\ell_s - T\ell_s\| + \epsilon\} \\ &\quad +\delta_s\{\gamma[1-(1-\gamma)\beta_s]\|\ell_s - \tilde{\ell}_s\| + \gamma\beta\|\ell_s - T\ell_s\| + \gamma\beta_s\epsilon\} \\ &= \gamma[1-(1-\gamma)\delta_s\beta_s]\|\ell_s - \tilde{\ell}_s\| + (1-\delta_s)L\|\ell_s - T\ell_s\| \\ &\quad +(1-\delta_s)\epsilon + \gamma\delta_s\beta_s L\|\ell_s - T\ell_s\| + \gamma\delta_s\beta_s\epsilon. \end{aligned} \tag{7.7}$$

Substituting (7.7) into (7.4), we obtain

$$\begin{aligned} \|\ell_{s+1} - \tilde{\ell}_{s+1}\| &\leq \gamma^3[1-(1-\gamma)\delta_s\beta_s]\|\ell_s - \tilde{\ell}_s\| + \gamma^2(1-\delta_s)L\|\ell_s - T\ell_s\| \\ &\quad +\gamma^2(1-\delta_s)\epsilon + \gamma^3\delta_s\beta_s L\|\ell_s - T\ell_s\| + \gamma^3\delta_s\beta_s\epsilon \\ &\quad +\gamma L\|w_s - Tw_s\| + \gamma\epsilon + L\|\zeta_s - T\zeta_s\| + \epsilon. \end{aligned} \tag{7.8}$$

Since $\gamma, \gamma^2, \gamma^3 \in (0,1)$ and $\delta_s, \beta_s \in [0,1]$, then (7.8) becomes

$$\begin{aligned} \|\ell_{s+1} - \tilde{\ell}_{s+1}\| &\leq [1-(1-\gamma)\delta_s\beta_s]\|\ell_s - \tilde{\ell}_s\| + L\|\ell_s - T\ell_s\| \\ &\quad +\delta_s\beta_s L\|\ell_s - T\ell_s\| + L\|w_s - Tw_s\| \\ &\quad +L\|\zeta_s - T\zeta_s\| + \delta_s\beta_s\epsilon + 3\epsilon. \end{aligned} \tag{7.9}$$

By our assumption (i) that $\frac{1}{2} \leq \delta_s\beta_s$, we have

$$1 - \delta_s\beta_s \leq \delta_s\beta_s \Rightarrow 1 = 1 - \delta_s\beta_s + \delta_s\beta_s \leq \delta_s\beta_s + \delta_s\beta_s = 2\delta_s\beta_s.$$

This yields

$$\begin{aligned}
\|\ell_{s+1}-\tilde{\ell}_{s+1}\| &\le [1-(1-\gamma)\delta_s\beta_s]\|\ell_s-\tilde{\ell}_s\| + 3\delta_s\beta_s L\|\ell_s - T\ell_s\| \\
&\quad +2\delta_s\beta_s L\|w_s - Tw_s\| + 2\delta_s\beta_s L\|\zeta_s - T\zeta_s\| + 7\delta_s\beta_s\epsilon \\
&= (1-(1-\gamma)\delta_s\beta_s)\|\ell_s-\tilde{\ell}_s\| \\
&\quad +\delta_s\beta_s(1-\gamma)\times\left\{\frac{3L\|\ell_s - T\ell_s\| + 2L\|w_s - Tw_s\|}{(1-\gamma)}\right. \\
&\quad \left. + \frac{2L\|\zeta_s - T\zeta_s\| + 7\epsilon}{(1-\gamma)}\right\}. \qquad (7.10)
\end{aligned}$$

Set

$$\begin{aligned}
\theta_s &= \|\ell_s-\tilde{\ell}_s\| \\
\sigma_s &= (1-\gamma)\delta_s\beta_s \in (0,1) \\
\lambda_s &= \left\{\frac{3L\|\ell_s - T\ell_s\| + 2L\|w_s - Tw_s\| + 2L\|\zeta_s - T\zeta_s\| + 7\epsilon}{(1-\gamma)}\right\}
\end{aligned}$$

From Theorem 3.1, we know that $\lim_{s\to\infty}\ell_s = z$ and since $Tz = z$, it follows that

$$\lim_{s\to\infty}\|\ell_s - T\ell_s\| = \lim_{s\to\infty}\|w_s - Tw_s\| = \lim_{s\to\infty}\|\zeta_s - G\zeta_s\| = 0.$$

Using Lemma 2.13, we get

$$0 \le \limsup_{s\to\infty}\|\ell_s-\tilde{\ell}_s\| \le \limsup_{s\to\infty}\frac{7\epsilon}{(1-\gamma)}. \qquad (7.11)$$

Since by Theorem 3.1, we have that $\lim_{s\to\infty}\ell_s = z$ and from our hypothesis $\lim_{s\to\infty}\tilde{\ell}_s = \tilde{z}$, it follows from (7.11) that

$$\|z-\tilde{z}\| \le \frac{7\epsilon}{(1-\gamma)}.$$

This completes the proof. □

8. Some Applications

In this section, we will prove that the sequence generated by our new iterative algorithm (1.7) converges strongly to solutions of the constrained convex minimization problem and split feasibility problem.

Now, we present the definitions of some operators that will we be important in proving our main results. Let H be a Hilbert space and let C be a nonempty closed and convex subset of H.

Definition 8.1. Let $T : C \to C$ be a mapping. Then T is said to be:

(i) nonexpansive, if

$$\|T\ell - T\zeta\| \le \|\ell - \zeta\|, \quad \text{for all } \ell, \zeta \in C;$$

(ii) Lipschitz continuous, if there exists $L > 0$ such

$$\|T\ell - T\zeta\| \leq L\|\ell - \zeta\|, \text{ for all } \ell, \zeta \in C;$$

(iii) monotone if,

$$\langle T\ell - T\zeta, \ell - \zeta\rangle \geq 0, \text{ for all } \ell, \zeta \in C; \tag{8.1}$$

(iv) ϖ-strongly monotone if there exists $\varpi > 0$, such that

$$\langle \ell - \zeta, T\ell - T\zeta\rangle \geq \varpi\|\ell - \zeta\|, \text{ for all } \ell, \zeta \in C. \tag{8.2}$$

For any $\ell \in H$, we define the map $P_C : H \to C$ satisfying

$$\|\ell - P_C\ell\| \leq \|\ell - \zeta\|, \text{ for all } \zeta \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive.

8.1. **Application to constrained convex minimization problem.**

Consider the following constrained convex minimization problem:

$$\text{minimize } \{f(\ell) : \ell \in C\}, \tag{8.3}$$

where $f : C \to \mathbb{R}$ is a real-valued function. The minimization problem (8.3) is consistent if it has a solution. Throughout this paper, we shall use Γ to stand for the solution set of the problem (8.3). It is worthy noting that f is (Fréchect) differentiable, the gradient-projection method (GPM) generates a sequence $\{\ell_s\}$ by using the recursive formula:

$$\begin{cases} \ell_0 \in C, \\ \ell_{s+1} = P_C(\ell_s - \lambda\nabla f(\ell_s)), \text{ for all } s \geq 1. \end{cases} \tag{8.4}$$

In more general form, (8.4) can be written as:

$$\begin{cases} \ell_0 \in C, \\ \ell_{s+1} = P_C(\ell_s - \lambda_s\nabla f(\ell_s)), \text{ for all } s \geq 1, \end{cases} \tag{8.5}$$

where λ and λ_s are positive real numbers.

It is well known that if ∇f is ϖ-strongly monotone and L-Lipschitzian with $\varpi, L > 0$, then the operator

$$T = P_C(I - \lambda\nabla f) \tag{8.6}$$

is a contraction; thus the sequence $\{\ell_s\}$ in (8.4) converges in norm to the unique minimizer of (8.3).

From [14, 30], we know that $z \in C$ solve the minimization problem (8.3) if and only if z solves the following fixed point equation:

$$z = P_C(I - \lambda\nabla f)z, \tag{8.7}$$

where $\lambda > 0$ is any fixed positive number. The operator $T = P_C(I - \lambda \nabla f)$ is well known to be nonexpansive (see [14, 30] and the references therein). Several authors have have considered different iterative algorithm for constrained convex minimization problems (see [4, 9, 13, 19, 34] and the references therein). We now give our main results

Theorem 8.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Supposed that the minimization problem (8.3) is consistent and let Γ denote the solution set. Supposed that the gradient ∇f is L-Lipschitzian with constant $L > 0$. Let $\{\ell_s\}$ be the sequence generated iteratively by*

$$\begin{cases} \ell_0 \in C, \\ g_s = (1-\beta_s)\ell_s + \beta_s P_C(I - \lambda \nabla f)\ell_s \\ w_s = (1-\delta_s)P_C(I - \lambda \nabla f)\ell_s + \delta_s P_C(I - \lambda \nabla f) g_s \qquad \forall s \geq 1. \\ \zeta_s = P_C(I - \lambda \nabla f) w_s \\ \ell_{s+1} = P_C(I - \lambda \nabla f)\zeta_s, \end{cases} \tag{8.8}$$

where $\{\delta_s\}$, $\{\beta_s\}$ are sequences in [0,1] and $\lambda \in \left(0, \frac{L}{2}\right)$.

Then the sequence $\{\ell_s\}$ converges strongly to a minimizer z of (8.3).

8.2. **Application to split feasibility problem.**

For modeling inverse problems which emanate from phase retrieval and medical image reconstruction, in 1994, Censor and Elfving [11] firstly introduced the following split feasibility problem (SFP) in finite-dimensional Hilbert spaces.

Let C and Q be nonempty closed convex subsets of the Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ be a bounded linear operator. Then the split feasibility problem (SFP) is formulated to

$$\text{find } z \in C \text{ such that } Az \in Q. \tag{8.9}$$

SFP has many applications, it has been found that SFP can been used in many areas such as image restoration, computer tomograph, radiation therapy treatment planning. There exists some iterative several iterative methods for solving split feasibility problems, see, for instance [8, 15, 30].

In 2002, Byrne [8] applied the forward-backward method, a type of projection gradient method to approximate (8.9). The so called CQ-iterative procedure is defined as follows:

$$\ell_{s+1} = P_C[I - \gamma A^*(1 - P_Q)A]\ell_n, \ \forall\ n \geq 1, \tag{8.10}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ with λ being the spectral radius of the of operator A^*A, P_C and P_Q denote the projections onto sets C and Q, respectively, and $A^* : H_2^* \to H_1^*$ is the adjoint of A.

We assume that the solution set Γ of the SFP (8.10) is nonempty, let

$$\Gamma = \{\ell \in C : A\ell \in Q\} = C \cap A^{-1}Q,$$

then Γ is closed, convex and nonempty set.

Lemma 8.3. *[15] Let operator $T = P_C[I - \gamma A^*(I - P_Q)A]$, where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. Then, T is said to be a nonexpansive map.*

Since by our assumption $\Gamma \neq \emptyset$, then it is clear that any $z \in C$ solves (8.9) if and only if it solves the fixed point equation:

$$T = P_C[I - \gamma A^*(I - P_Q)A]z = z, \quad z \in C.$$

Thus, $F(T) = \Gamma = C \cap A^{-1}Q$, i.e., the solution set Γ is equal the set of fixed point of the map T. For more explicit explanation, the reader can see [42, 43].

Now, to prove our main results in this part, we will consider the following scheme:

$$\begin{cases} \ell_0 \in C, \\ g_s = (1-\beta_s)\ell_s + \beta_s P_C[I - \gamma A^*(I - P_Q)A]\ell_s \\ w_s = (1-\delta_s)P_C[I - \gamma A^*(I - P_Q)A]\ell_s + \delta_s P_C[I - \gamma A^*(I - P_Q)A]g_s \\ \zeta_s = P_C[I - \gamma A^*(I - P_Q)A]w_s \\ \ell_{s+1} = P_C[I - \gamma A^*(I - P_Q)A]\zeta_s, \end{cases} \tag{8.11}$$

for all $s \geq 1$, where $\{\delta_s\}$, $\{\beta_s\}$ are sequences in $[0,1]$ and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$.

Theorem 8.4. *Let $\{\ell_s\}$ be the sequence iteratively generated by (8.11). Then, $\{\ell_s\}$ converses weakly to an element in Γ.*

Proof. Since $T = P_C[I - \gamma A^*(I - P_Q)A]$ is a nonexpansive map and by Proposition 2.9 we know that every generalized α-nonexpansive map is nonexpansive map with $\alpha = 0$ (i.e., 0-nonexpansive), so the conclusion follows from Theorem 4.3. □

Theorem 8.5. *If $\{\ell_s\}$ is the sequence generated by the iterative scheme (8.11). Then $\{\ell_s\}$ converges strongly the an element in Γ if and only if $\liminf\limits_{s\to\infty} d(\ell_s, \Gamma) = 0$.*

Proof. Since $T = P_C[I - \gamma A^*(I - P_Q)A]$ is nonexpansive map, then the conclusion of the proof follows from Theorem 4.4. □

Theorem 8.6. *If $T = P_C[I - \gamma A^*(I - P_Q)A]$ satisfies condition (I) and $\{\ell_s\}$ is the sequence iteratively defined by (8.11), then $\{\ell_s\}$ converges strongly to a point in Γ.*

Proof. The result follows from Theorem 4.5. □

9. Conclusion

In this paper, we have shown numerically and analytically that our new iterative algorithm (1.7) has a better rate of convergence than M iterative algorithm and some other well known existing iterative algorithms in the literature for almost contraction mapping and generalized α-nonexpansive mappings. Also, it is shown that our new iterative algorithm (1.7) is T–stable and data dependent which make it reliable. As some applications of our new iterative algorithm (1.7), it is used to find the solutions of constrained convex minimization problem and split feasibility problem. Now, owing to the fact that the class of generalized α-nonexpansive mappings which is considered in our paper is more general than the class of Suzuki generalized nonexpansive mappings which has been considered by Ullah and Arshad [39] for M iteration, it implies that our results generalize and improve the results in Ullah and Arshad [39] and several other related results existing in the literature.

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