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Eur. J. Math. Anal. 2 (2022) 8
doi: 10.28924/ada/ma.2.8

Stability of Positive Weak Solution for Generalized Weighted p-Fisher-Kolmogoroff Nonlinear Stationary-State Problem

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Abstract. In the present paper, we investigate the stability results of positive weak solution for the generalized Fisher–Kolmogoroff nonlinear stationary-state problem involving weighted p-Laplacian operator $-d\Delta_{P,p}u = ka(x)u[\nu - \upsilon u]$ in Ω, $Bu = 0$ on $\partial\Omega$, where $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$, the continuous function $a(x) : \Omega \to R$ satisfies either $a(x) > 0$ or $a(x) < 0$ for all $x \in \Omega$, d, k, ν and υ are positive parameters and $\Omega \subset R^N$ is a bounded domain with smooth boundary $Bu = \delta h(x)u + (1-\delta)\frac{\partial u}{\partial n}$ where $\delta \in [0, 1]$, $h : \partial\Omega \to R^+$ with $h = 1$ when $\delta = 1$.

1. Introduction:

In this paper we study the stability results of positive weak solution for the generalized weighted p-Fisher–Kolmogoroff nonlinear stationary-state problem

$$\left.\begin{aligned} -d\Delta_{P,p}u = ka(x)f(u) = ka(x)u[\nu - \upsilon u] \quad &\text{in } \Omega, \\ Bu = 0 \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$ (see for details [6]), the continuous function $a(x) : \Omega \to R$ satisfies either $a(x) > 0$ or $a(x) < 0$ for all $x \in \Omega$, d, k, ν, ν and υ are positive parameter and $\Omega \subset R^N$ is a bounded domain with smooth boundary $Bu = \delta h(x)u + (1-\delta)\frac{\partial u}{\partial n}$ where $\delta \in [0, 1]$, $h : \partial\Omega \to R^+$ with $h = 1$ when $\delta = 1$. System (1.1) is the generalized weighted p-Fisher–Kolmogoroff nonlinear stationary-state problem [21], where d is the diffusion coefficient, k is the is the linear reproduction rate and u is the population density. Situations where d is space-dependent are arising in more and more modelling situations of biomedical importance from diffusion of genetically engineered organisms in heterogeneous environments to the effect of white and grey

Received: 12 Sep 2021.
Key words and phrases. stability; weak solution; p-Laplacian.

matter in the growth and spread of brain tumours. Problem (1.1) arises from the population biology of one species.

Systems of type (1.1) have received considerable attention in the last decade (see, e.g., [18,19,24] and the references therein). It has been shown that for some certian values of ν, υ, system (1.1) has a rich mathematical structure. In [8, 23] the system (1.1) is considered under the hypothesis $P(x) = (k/d) = 1, p = 2$ and $f(u) = u$.This corresponds to the Emden-Fowler stationary-state problem of polytropic index of order one. While in [9, 19], system (1.1) is considered under the hypothesis $P(x) = (k/d) = 1, p = 2$ and $f(u) = u - u^2$,where u is the population denisty of degree two.This corresponds to the Logestic nonlinear stationary-state problem. Due to the appearance of weighted p-Laplacian operator in (1.1) and the particular cases; the extensions are challenging and nontrivial.

Many authors are interested in the study of stability and instability of nonnegative solutions of linear [2] , semilinear (see [10,26]), semiposiotne (see [3,25]), nonlinear (see [1,16]) and singular (see [17]) systems, due to the great number of applications in reaction-diffusion problems, in autocatalytic reaction, in temperature on plasma, population dynamics, etc.; see [4, 23] and references therein. Also, in the recent past, many authors devoted their attention to study the weighted p-Laplacian nonlinear systems (see [11, 12, 14, 15]).

Tertikas in [25] have been proved the stability and instability results of positive solutions for the semilinear system

$$-\Delta u = \lambda f(u) \text{ in } \Omega, \;\; Bu = 0 \text{ on } \;\partial\Omega,$$

under various choices of the function f. In [3], the authors have been studied the uniqueness and stability of nonnegative solutions for classes of nonlinear elliptic Dirichlet problems in a ball, when the nonlinearity is monotone, negative at the origin, and either concave or convex. In the case $P(x) = a(x) = 1$, $p = 2$ and a function $\lambda f(u)$ instead of $\lambda u^\alpha + u^\beta$, system (1.1) have been studied by several authors (see [5,7,20]).

Khafagy in [13] have been studied the stability and instability of positive weak solution for the nonlinear system

$$\left.\begin{array}{ll} -\Delta_{P,p} u + a(x)|u|^{p-2}u = \lambda b(x)u^\alpha & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{1.2}$$

where $0 < \alpha < p-1$. He proved that if $0 < \alpha < p-1$ and $b(x) > 0(< 0)$ for all $x \in \Omega$, then every positive weak solution u of (1.2) is linearly stable (unstable) respectively.

Definition 1.1. We recall that, if u be any positive weak solution of (1.1), then the linearized equation of (1.1) about u is given by

$$\left.\begin{array}{c} -(p-1)div[P(x)|\nabla u|^{p-2}\nabla\phi] - (k/d)a(x)[\nu - 2\upsilon u]\phi = \mu\phi, x \in \Omega, \\ B\phi = 0, \quad x \in \partial\Omega, \end{array}\right\} \tag{1.3}$$

where μ is the eigenvalue corresponding to the eigenfunction ϕ.

Definition 1.2. [3] A solution u of (1.1) is called stable solution if all eigenvalues of (1.3) are strictly positive, which can be implied if the principal eigenvalue $\mu_1 > 0$. Otherwise u unstable.

2. Main Results

The main goal of this section is to prove the stability and instability of the positive weak solution u of (1.1). Our main results are formulate in the following theorems.

Theorem 2.1. *If $\alpha + 1 < p < \beta + 1$ and $a(x) > 0$ for all $x \in \Omega$, then every positive weak solution of (1.1) is linearly stable.*

Proof. Let u_0 be any positive weak solution of (1.1), then the linearized equation bout u_0 is

$$\left.\begin{array}{c} -(p-1)div[P(x)|\nabla u_0|^{p-2}\nabla\phi] - (k/d)a(x)[\nu - 2\upsilon u_0]\phi = \mu\phi, \ x \in \Omega \\ B\phi = 0, \quad x \in \partial\Omega. \end{array}\right\} \tag{2.1}$$

Let μ_1 be the first eigenvalue of (2.1) and let $\psi(x) \geq 0$ be the corresponding eigenfunction. Multiplying (1.1) by ψ and integrating over Ω, we have

$$-\int_\Omega \psi div[P(x)|\nabla u_0|^{p-2}\nabla u_0]dx = (k/d)\int_\Omega a(x)[\nu u_0 - \upsilon u_0^2]\psi dx. \tag{2.2}$$

The first term of the L.H.S. of (2.2) may be written in the form

$$\begin{aligned} \int_\Omega \psi div[P(x)|\nabla u_0|^{p-2}\nabla u_0]dx &= \int_\Omega \psi\nabla u_0\nabla[P(x)|\nabla u_0|^{p-2}]dx \\ &+ \int_\Omega \psi[P(x)|\nabla u_0|^{p-2}]div(\nabla u_0)dx. \end{aligned}$$

Applying Green's first identity, we have

$$\begin{aligned} \int_\Omega \psi div[P(x)|\nabla u_0|^{p-2}\nabla u_0]dx &= \int_\Omega \psi\nabla u_0\nabla[P(x)|\nabla u_0|^{p-2}]dx \\ &- \int_\Omega \nabla[\psi(P(x)|\nabla u_0|^{p-2})\nabla u_0 dx \\ &+ \int_{\partial\Omega} \psi[P(x)|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n}ds, \\ &= -\int_\Omega \nabla\psi[P(x)|\nabla u_0|^{p-2}]\nabla u_0 dx \\ &+ \int_{\partial\Omega} \psi[P(x)|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n}ds. \end{aligned} \tag{2.3}$$

From (2.3) in (2.2), we have

$$\begin{aligned}(k/d)\int_{\Omega} a(x)[\nu u_0 - \upsilon u_0^2]\psi dx &= \int_{\Omega} \nabla\psi[P(x)|\nabla u_0|^{p-2}]\nabla u_0 dx \\ &\quad - \int_{\partial\Omega} \psi[P(x)|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n} ds \\ &\quad + \int_{\Omega} a(x)\psi|u_0|^{p-2}u_0]dx. \end{aligned} \tag{2.4}$$

Also, Multiplying (2.1) by $(-u_0)$ and integrating over Ω, we have

$$\begin{aligned}-\mu_1\int_{\Omega} u_0\psi dx &= (p-1)\int_{\Omega} u_0 div[P(x)|\nabla u_0|^{p-2}\nabla\psi]dx \\ &\quad -(p-1)\int_{\Omega} u_0 a(x)|u_0|^{p-2}\psi \\ &\quad +\lambda\int_{\Omega} a(x)[\nu - 2\upsilon u_0]\psi dx. \end{aligned} \tag{2.5}$$

The first term of the L.H.S. of (2.5) may be written in the form

$$\begin{aligned}\int_{\Omega} u_0 div[P(x)|\nabla u_0|^{p-2}\nabla\psi]dx &= \int_{\Omega} u_0[P(x)|\nabla u_0|^{p-2}]\nabla\cdot\nabla\psi dx \\ &\quad + \int_{\Omega} u_0\nabla\psi\nabla[P(x)|\nabla u_0|^{p-2}]dx. \end{aligned}$$

Using Green's first identity, one have

$$\begin{aligned}\int_{\Omega} u_0 div[P(x)|\nabla u_0|^{p-2}\nabla\psi]dx &= -\int_{\Omega} \nabla[u_0 P(x)|\nabla u_0|^{p-2}]\nabla\psi \\ &\quad + \int_{\Omega} u_0\nabla[P(x)|\nabla u_0|^{p-2}]\nabla\psi dx \\ &\quad + \int_{\partial\Omega} u_0[P(x)|\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n} ds, \\ &= -\int_{\Omega} [P(x)|\nabla u_0|^{p-2}]\nabla u_0\nabla\psi \\ &\quad + \int_{\partial\Omega} u_0[P(x)|\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n} ds. \end{aligned} \tag{2.6}$$

From (2.6) in (2.5) we have

$$\begin{aligned}-\mu_1\int\limits_{\Omega} u_0\psi dx &= (p-1)[\int\limits_{\partial\Omega} u_0[P(x)|\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n}ds \\ &\quad -\int\limits_{\Omega}[P(x)|\nabla u_0|^{p-2}]\nabla u_0\nabla\psi] \\ &\quad +(k/d)\int\limits_{\Omega} a(x)[\nu u_0-2\upsilon u_0^2]\psi dx. \end{aligned} \tag{2.7}$$

Multiplying (2.4) by $(p-1)$ and adding with (2.7), we have

$$\begin{aligned}-\mu_1\int\limits_{\Omega} u_0\psi dx &= (p-1)[\int\limits_{\partial\Omega} u_0[P(x)|\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n}ds \\ &\quad -\int\limits_{\partial\Omega}\psi[P(x)|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n}ds] \\ &\quad +(k/d)\int\limits_{\Omega} a(x)[\nu u_0-2\upsilon u_0^2]\psi dx \\ &\quad -(p-1)(k/d)\int\limits_{\Omega} a(x)[\nu u_0-\upsilon u_0^2]\psi dx. \end{aligned}$$

Hence

$$\begin{aligned}-\mu_1\int\limits_{\Omega} u_0\psi dx &= (p-1)\int\limits_{\partial\Omega}[P(x)|\nabla u_0|^{p-2}][u_0\frac{\partial\psi}{\partial n}-\psi\frac{\partial u_0}{\partial n}]ds \\ &\quad +(k/d)\int\limits_{\Omega} a(x)\nu u_0[1-(p-1)]\psi dx \\ &\quad +(k/d)\int\limits_{\Omega} a(x)\upsilon u_0^2[(p-1)-2]\psi dx. \end{aligned} \tag{2.8}$$

Now, when $\delta=1$, we have $Bu_0=u_0=0$ for $s\in\partial\Omega$ and also we have $\psi=0$ for $s\in\partial\Omega$. Then

$$\int\limits_{\partial\Omega}[P(x)|\nabla u_0|^{p-2}][u_0\frac{\partial\psi}{\partial n}-\psi\frac{\partial u_0}{\partial n}]ds=0. \tag{2.9}$$

Also, when $\delta\neq 1$, we have

$$\frac{\partial u_0}{\partial n}=-\frac{\delta h u_0}{1-\delta} \qquad \text{and} \qquad \frac{\partial\psi}{\partial n}=-\frac{\delta h\psi}{1-\delta},$$

which implies again the result given by (2.9).

Hence

$$-\mu_1\int\limits_{\Omega} u_0\psi dx=(k/d)\int\limits_{\Omega} a(x)[\nu u_0[2-p]+\upsilon u_0^2[p-3]]\psi dx. \tag{2.10}$$

Since $2 < p < 3$ and $a(x) > 0$ for all x, then (2.10) becomes

$$-\mu_1 \int_\Omega u_0 \psi dx < 0, \tag{2.11}$$

so $\mu_1 > 0$ and the result follows. □

Theorem 2.2. *If $2 < p < 3$ and $a(x) < 0$ for all $x \in \Omega$, then every positive weak solution of (1.1) is unstable.*

Proof. As in the proof of Theorem 1., we have

$$-\mu_1 \int_\Omega u_0 \psi dx > 0, \tag{2.12}$$

so $\mu_1 < 0$ and the result follows. □

3. Applications and related results

Here we introduce some examples to demonstrate the effectiveness of our results.

Example 3.1. *Consider the Emden-Fowler steady-state problem of polytropic index of order one [8],*

$$\left.\begin{array}{ll} -\Delta u = \lambda a(x) u & in\ \ \Omega, \\ Bu = 0 & on\ \ \partial\Omega, \end{array}\right\} \tag{3.1}$$

with $a(x) > 0$ for all $x \in \Omega$.Here $P(x) = 1, (k/d) = \lambda, p = 2$. Then according to Theorem 1., every positive weak solution of (3.1) is unstable.

Example 3.2. *Consider the population denisty steady-state problem of degree two [19],*

$$\left.\begin{array}{ll} -\Delta_p u = \lambda a(x)[u - u^2] & in\ \ \Omega, \\ Bu = 0 & on\ \ \partial\Omega, \end{array}\right\} \tag{3.2}$$

with $a(x) > 0$ for all $x \in \Omega$.Hence, according to Theorem 1., every positive weak solution of (3.1) is stable.

Example 3.3. *Consider the chemotaxis steady-state problem of degree two [9, 19],*

$$\left.\begin{array}{ll} -\Delta_p u = \lambda a(x)[-u + u^2] & in\ \ \Omega, \\ Bu = 0 & on\ \ \partial\Omega, \end{array}\right\} \tag{3.3}$$

with $a(x) > 0$ for all $x \in \Omega$.Hence, according to Theorem 1., every positive weak solution of (3.1) is unstable.

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