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Some New Series Expansions of a Special Type of Functions Involving the Logarithmic Function

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Abstract. In this article, we present new series expansions for a certain family of functions that depend on the logarithmic function. A general result is demonstrated by considering a tunable intermediate function. This result has the interest of unifying several important results in the literature, including a well-known series expansion established by Srinivasa Ramanujan. Several precise examples are given and discussed in detail. In addition, we recover the so-called Seidel formula and derive new product expansions, with an emphasis on the so-called Einstein function. Some inequalities involving logarithmic functions are also applications of our series expansion approach. Selected results are supported by graphical work.

1. Introduction

The logarithmic function, denoted $\log(x)$, plays a crucial role in several mathematical contexts, including calculus, number theory, and computer science. See [13], and the references therein. Understanding its properties, especially its various series expansions, is fundamental to many mathematical analyses. The classical (Taylor) expansion of $\log(x)$ is given by

$$\log(x) = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k}(x-1)^k.$$

It is valid for $x \in (0, 2]$ only (see [8], among others). From this expansion, we immediately get

$$x - 1 - \log(x) = \sum_{k=2}^{+\infty} \frac{1}{k}(1-x)^k. \tag{1}$$

A consequence of this result is the following inequality: $\log(x) \leq x - 1$ for $x \in (0, 2]$. However, as discussed in [4], we know that it actually holds for $x > 0$. Thus, the use of the classical logarithmic expansion is somehow inadequate for a full understanding of this inequality. There is a kind

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of "gap in understanding". A solution is given in [4]. Using a telescoping technique and careful development, the following series expansion is demonstrated:

$$x - 1 - \log(x) = \sum_{k=1}^{+\infty} 2^{k-1}(x^{2^{-k}} - 1)^2,$$

and it is valid for $x > 0$. From this result, we immediately observe that $\log(x) \leq x - 1$ for $x > 0$; the constraint $x \in (0, 2]$ is relaxed. This key inequality is now fully understandable using the series expansion tool. In addition, the underlying telescoping technique provides an original alternative proof, making it very interesting from a mathematical point of view. It can also be used for other purposes; the proof of natural logarithmic inequalities is just one example.

On the other hand, from a completely different perspective, a famous result of Srinivasa Ramanujan ensures that, for $x > 0$ with $x \neq 1$, we have

$$\frac{1}{\log(x)} + \frac{1}{1-x} = \sum_{k=1}^{+\infty} \frac{1}{2^k(1 + x^{2^{-k}})}.$$

See [12, page 364]. The proof is based on an iteration technique, noting that $1/(1-x) = (1/2)[1/(1+\sqrt{x}) + 1/(1-\sqrt{x})]$. It received special attention in [2, Chapter 31, Entry 29, page 399] and was the object of an in-depth study in [5]. As a new visual note, this extension can be reformulated as

$$\frac{1}{x-1} - \frac{1}{\log(x)} = -\sum_{k=1}^{+\infty} \frac{1}{2^k(1 + x^{2^{-k}})}.$$

Based on this form, doing a parallel with the formula in Equation (1), a functional pattern seems to be present. In fact, both expansions can be expressed as

$$\phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} a_k(x), \tag{2}$$

where $\phi(t) = t$ and $\phi(t) = 1/t$, respectively, and $a_k(x) = 2^{k-1}(x^{2^{-k}} - 1)^2$ and $a_k(x) = -1/2^k(1 + x^{2^{-k}})$, respectively. Given this, a unified approach seems possible.

In this article, we formalize such an approach. It aims to generate a wide range of new series expansions of certain functions that depend on the logarithmic function, i.e., functions of the form $\phi(x-1) - \phi[\log(x)]$. The proof is based on telescoping techniques inspired by [4] and precise factorization developments. The results established have interesting consequences, including the derivation of old and new product expansions. Among other things, a new product expansion of the Einstein function, i.e., $E_2(x) = x/(e^x - 1)$ (see [1]), is established. In addition, inequalities involving logarithmic functions are obtained almost immediately, in the spirit of [3, 6, 7, 9–11, 16]. Some graphics illustrate the results.

The following sections structure the article: Section 2 is devoted to the general result on the series expansion of $\phi(x-1) - \phi[\log(x)]$ and emphasizes several examples. Section 3 deals with some of its implications, including product expansions and inequalities. A conclusion is given in Section 4.

2. Results

Our general and specific findings are presented in this section.

2.1. A general result.

The theorem below suggests a series expansion for the difference function $\phi(x-1)-\phi[\log(x)]$, defined with a certain function ϕ.

Theorem 2.1. *Let $x>0$ and ϕ be a continuous function such that $|\phi(x-1)|<+\infty$ and $|\phi[\log(x)]|<+\infty$. Then we have*

$$\phi(x-1)-\phi[\log(x)]=\sum_{k=1}^{+\infty}\alpha_k(\phi)(x),$$

where

$$\alpha_k(\phi)(x)=\phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]-\phi\left[2^{k}(x^{2^{-k}}-1)\right].$$

Proof. Introducing an integer $n\geq 1$ and using the telescoping technique, we get

$$\begin{aligned}\phi(x-1)-\phi\left[2^{n}(x^{2^{-n}}-1)\right]&=\phi\left[2^{0}(x^{2^{-0}}-1)\right]-\phi\left[2^{n}(x^{2^{-n}}-1)\right]\\&=\sum_{k=1}^{n}\left\{\phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]-\phi\left[2^{k}(x^{2^{-k}}-1)\right]\right\}\\&=\sum_{k=1}^{n}\alpha_k(\phi)(x).\end{aligned}$$

Since $\lim_{n\to+\infty}2^n(x^{2^{-n}}-1)=\lim_{n\to+\infty}2^n(e^{2^{-n}\log(x)}-1)=\lim_{n\to+\infty}2^n\left\{\left[1+2^{-n}\log(x)\right]-1\right\}=\log(x)$, thanks to the continuity of ϕ, we obtain

$$\begin{aligned}\phi(x-1)-\phi[\log(x)]&=\phi(x-1)-\phi\left[\lim_{n\to+\infty}2^{n}(x^{2^{-n}}-1)\right]\\&=\lim_{n\to+\infty}\left\{\phi(x-1)-\phi\left[2^{n}(x^{2^{-n}}-1)\right]\right\}\\&=\lim_{n\to+\infty}\sum_{k=1}^{n}\alpha_k(\phi)(x)=\sum_{k=1}^{+\infty}\alpha_k(\phi)(x).\end{aligned}$$

This ends the proof of the theorem. □

From this theorem, if ϕ is bijective, then the following series expansion of the logarithmic function holds:

$$\log(x)=\phi^{-1}\left\{\phi(x-1)-\sum_{k=1}^{+\infty}\alpha_k(\phi)(x)\right\}.$$

It can be useful in several mathematical contexts.

In fact, this theorem has the advantage of being general and tunable thanks to the function ϕ. It also unifies several results in the literature, as will be developed in the next part.

2.2. **Specific results.** The proposition below shows some consequences of Theorem 2.1, including some new results.

Proposition 2.2. *The series expansions below are valid.*

(1) *For $x > 0$, we have*

$$x - 1 - \log(x) = \sum_{k=1}^{+\infty} 2^{k-1}(x^{2^{-k}} - 1)^2.$$

As mentioned in the introduction, this result is not new; it was established in [4].

(2) *For $x > 0$, we have*

$$\frac{1}{x-1} - \frac{1}{\log(x)} = -\sum_{k=1}^{+\infty} \frac{1}{2^k(1 + x^{2^{-k}})}.$$

As mentioned in the introduction, this result is not new; it is a famous result proved by Srinivasa Ramanujan, as highlighted in [2, Chapter 31, page 399] and [5].

To the best of our knowledge, the eight results below are new.

(3) *For $x > 0$, we have*

$$(x-1)^2 - [\log(x)]^2 = \sum_{k=1}^{+\infty} 2^{2(k-1)}(x^{2^{-k}} - 1)^3(3 + x^{2^{-k}}).$$

(4) *For $x \geq 1$, we have*

$$\sqrt{x-1} - \sqrt{\log(x)} = \sum_{k=1}^{+\infty} \frac{2^{(k-1)/2}(x^{2^{-k}} - 1)^{3/2}}{\sqrt{1 + x^{2^{-k}}} + \sqrt{2}}.$$

(5) *For $x > 1$, we have*

$$\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{\log(x)}} = -\sum_{k=1}^{+\infty} \frac{1}{2^{k/2}\left[\sqrt{1 + x^{2^{-k}}} + \sqrt{2}\right]}\sqrt{\frac{x^{2^{-k}} - 1}{1 + x^{2^{-k}}}}.$$

(6) *For $x > 1$, we have*

$$\log(x-1) - \log[\log(x)] = \sum_{k=1}^{+\infty} \log\left(\frac{1 + x^{2^{-k}}}{2}\right).$$

(7) *For $x > 0$, we have*

$$\sin(x-1) - \sin[\log(x)] = 2\sum_{k=1}^{+\infty} \sin\left[2^{k-2}(x^{2^{-k}} - 1)^2\right]\cos\left[2^{k-2}(x^{2^{-k}} - 1)(3 + x^{2^{-k}})\right].$$

(8) *For $x > 0$, we have*

$$\cos(x-1) - \cos[\log(x)] = -2\sum_{k=1}^{+\infty} \sin\left[2^{k-2}(x^{2^{-k}} - 1)^2\right]\sin\left[2^{k-2}(x^{2^{-k}} - 1)(3 + x^{2^{-k}})\right].$$

(9) *For $x > 0$, we have*

$$\sinh(x-1) - \sinh[\log(x)] = 2\sum_{k=1}^{+\infty} \sinh\left[2^{k-2}(x^{2^{-k}}-1)^2\right] \cosh\left[2^{k-2}(x^{2^{-k}}-1)(3+x^{2^{-k}})\right].$$

(10) *For $x > 0$, we have*

$$\cosh(x-1) - \cosh[\log(x)] = 2\sum_{k=1}^{+\infty} \sinh\left[2^{k-2}(x^{2^{-k}}-1)^2\right] \sinh\left[2^{k-2}(x^{2^{-k}}-1)(3+x^{2^{-k}})\right].$$

Proof. Let us prove each result, one by one.

(1) For $x > 0$, by applying Theorem 2.1 with $\phi(t) = t$, we obtain

$$x - 1 - \log(x) = \phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} \alpha_k(\phi)(x),$$

where

$$\begin{aligned}
\alpha_k(\phi)(x) &= \phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right] - \phi\left[2^k(x^{2^{-k}}-1)\right] \\
&= 2^{k-1}(x^{2^{-(k-1)}}-1) - 2^k(x^{2^{-k}}-1) \\
&= 2^{k-1}(x^{2^{-k}}-1)(1+x^{2^{-k}}) - 2^k(x^{2^{-k}}-1) \\
&= 2^{k-1}(x^{2^{-k}}-1)\left[(1+x^{2^{-k}})-2\right] \\
&= 2^{k-1}(x^{2^{-k}}-1)^2.
\end{aligned}$$

The used factorization arguments are the same as those in [4]. Hence, we have

$$x - 1 - \log(x) = \sum_{k=1}^{+\infty} 2^{k-1}(x^{2^{-k}}-1)^2.$$

(2) For $x > 0$, applying Theorem 2.1 to $\phi(t) = 1/t$, we establish that

$$\frac{1}{x-1} - \frac{1}{\log(x)} = \phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} \alpha_k(\phi)(x),$$

where

$$\begin{aligned}
\alpha_k(\phi)(x) &= \phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right] - \phi\left[2^k(x^{2^{-k}}-1)\right] \\
&= \frac{1}{2^{k-1}(x^{2^{-(k-1)}}-1)} - \frac{1}{2^k(x^{2^{-k}}-1)} \\
&= -\frac{2^{k-1}(x^{2^{-(k-1)}}-1) - 2^k(x^{2^{-k}}-1)}{2^{2k-1}(x^{2^{-(k-1)}}-1)(x^{2^{-k}}-1)} \\
&= -\frac{2^{k-1}(x^{2^{-k}}-1)^2}{2^{2k-1}(x^{2^{-(k-1)}}-1)(x^{2^{-k}}-1)} = -\frac{(x^{2^{-k}}-1)^2}{2^k(1+x^{2^{-k}})(x^{2^{-k}}-1)^2} \\
&= -\frac{1}{2^k(1+x^{2^{-k}})}.
\end{aligned}$$

Hence, we have

$$\frac{1}{x-1}-\frac{1}{\log(x)}=-\sum_{k=1}^{+\infty}\frac{1}{2^k(1+x^{2^{-k}})}.$$

(3) For $x>0$, applying Theorem 2.1 to $\phi(t)=t^2$, we get

$$(x-1)^2-[\log(x)]^2=\phi(x-1)-\phi[\log(x)]=\sum_{k=1}^{+\infty}\alpha_k(\phi)(x),$$

where

$$\begin{aligned}
&\alpha_k(\phi)(x)=\phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]-\phi\left[2^k(x^{2^{-k}}-1)\right]\\
&=\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]^2-\left[2^k(x^{2^{-k}}-1)\right]^2\\
&=\left[2^{k-1}(x^{2^{-(k-1)}}-1)-2^k(x^{2^{-k}}-1)\right]\left[2^{k-1}(x^{2^{-(k-1)}}-1)+2^k(x^{2^{-k}}-1)\right]\\
&=2^{k-1}(x^{2^{-k}}-1)^2 2^{k-1}(x^{2^{-k}}-1)(3+x^{2^{-k}})\\
&=2^{2(k-1)}(x^{2^{-k}}-1)^3(3+x^{2^{-k}}).
\end{aligned}$$

Hence, we have

$$(x-1)^2-[\log(x)]^2=\sum_{k=1}^{+\infty}2^{2(k-1)}(x^{2^{-k}}-1)^3(3+x^{2^{-k}}).$$

(4) For $x\geq 1$, using Theorem 2.1 with $\phi(t)=\sqrt{t}$, we obtain

$$\sqrt{x-1}-\sqrt{\log(x)}=\phi(x-1)-\phi[\log(x)]=\sum_{k=1}^{+\infty}\alpha_k(\phi)(x),$$

where

$$\begin{aligned}
\alpha_k(\phi)(x)&=\phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]-\phi\left[2^k(x^{2^{-k}}-1)\right]\\
&=\sqrt{2^{k-1}(x^{2^{-(k-1)}}-1)}-\sqrt{2^k(x^{2^{-k}}-1)}\\
&=\frac{2^{k-1}(x^{2^{-(k-1)}}-1)-2^k(x^{2^{-k}}-1)}{\sqrt{2^{k-1}(x^{2^{-(k-1)}}-1)}+\sqrt{2^k(x^{2^{-k}}-1)}}\\
&=\frac{2^{k-1}(x^{2^{-k}}-1)^2}{\sqrt{2^{k-1}(x^{2^{-k}}-1)(1+x^{2^{-k}})}+\sqrt{2[2^{k-1}(x^{2^{-k}}-1)]}}\\
&=\frac{2^{(k-1)/2}(x^{2^{-k}}-1)^{3/2}}{\sqrt{1+x^{2^{-k}}}+\sqrt{2}}.
\end{aligned}$$

Hence, we have

$$\sqrt{x-1}-\sqrt{\log(x)}=\sum_{k=1}^{+\infty}\frac{2^{(k-1)/2}(x^{2^{-k}}-1)^{3/2}}{\sqrt{1+x^{2^{-k}}}+\sqrt{2}}.$$

(5) For $x > 1$, applying Theorem 2.1 to $\phi(t) = 1/\sqrt{t}$, we have

$$\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{\log(x)}} = \phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} \alpha_k(\phi)(x),$$

where

$$\begin{aligned}
\alpha_k(\phi)(x) &= \phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right] - \phi\left[2^k(x^{2^{-k}}-1)\right] \\
&= \frac{1}{\sqrt{2^{k-1}(x^{2^{-(k-1)}}-1)}} - \frac{1}{\sqrt{2^k(x^{2^{-k}}-1)}} \\
&= -\frac{\sqrt{2^{k-1}(x^{2^{-(k-1)}}-1)} - \sqrt{2^k(x^{2^{-k}}-1)}}{\sqrt{2^{2k-1}(x^{2^{-(k-1)}}-1)(x^{2^{-k}}-1)}} \\
&= -\frac{2^{k-1}(x^{2^{-(k-1)}}-1) - 2^k(x^{2^{-k}}-1)}{\left[\sqrt{2^{k-1}(x^{2^{-(k-1)}}-1)} + \sqrt{2^k(x^{2^{-k}}-1)}\right]\sqrt{2^{2k-1}(x^{2^{-(k-1)}}-1)(x^{2^{-k}}-1)}} \\
&= -\frac{2^{k-1}(x^{2^{-k}}-1)^2}{\left[\sqrt{2^{k-1}(x^{2^{-k}}-1)(1+x^{2^{-k}})} + \sqrt{2[2^{k-1}(x^{2^{-k}}-1)]}\right]\sqrt{2^{2k-1}(x^{2^{-k}}-1)^2(1+x^{2^{-k}})}} \\
&= -\frac{1}{2^{k/2}\left[\sqrt{1+x^{2^{-k}}} + \sqrt{2}\right]}\sqrt{\frac{x^{2^{-k}}-1}{1+x^{2^{-k}}}}.
\end{aligned}$$

Hence, we have

$$\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{\log(x)}} = -\sum_{k=1}^{+\infty} \frac{1}{2^{k/2}\left[\sqrt{1+x^{2^{-k}}} + \sqrt{2}\right]}\sqrt{\frac{x^{2^{-k}}-1}{1+x^{2^{-k}}}}.$$

(6) For $x > 1$, it follows from Theorem 2.1 with $\phi(t) = \log(t)$ that

$$\log(x-1) - \log[\log(x)] = \phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} \alpha_k(\phi)(x),$$

where

$$\begin{aligned}
\alpha_k(\phi)(x) &= \phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right] - \phi\left[2^k(x^{2^{-k}}-1)\right] \\
&= \log\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right] - \log\left[2^k(x^{2^{-k}}-1)\right] \\
&= \log\left[\frac{2^{k-1}(x^{2^{-(k-1)}}-1)}{2^k(x^{2^{-k}}-1)}\right] = \log\left[\frac{(x^{2^{-k}}-1)(1+x^{2^{-k}})}{2(x^{2^{-k}}-1)}\right] \\
&= \log\left(\frac{1+x^{2^{-k}}}{2}\right).
\end{aligned}$$

Hence, we have

$$\log(x-1)-\log\left[\log(x)\right]=\sum_{k=1}^{+\infty}\log\left(\frac{1+x^{2^{-k}}}{2}\right).$$

(7) For $x>0$, applying Theorem 2.1 to $\phi(t)=\sin(t)$, we obtain

$$\sin(x-1)-\sin[\log(x)]=\phi(x-1)-\phi[\log(x)]=\sum_{k=1}^{+\infty}\alpha_k(\phi)(x),$$

where

$$\begin{aligned}\alpha_k(\phi)(x)&=\phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]-\phi\left[2^k(x^{2^{-k}}-1)\right]\\&=\sin\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]-\sin\left[2^k(x^{2^{-k}}-1)\right].\end{aligned}$$

Using the standard trigonometric formula $\sin(t)-\sin(u)=2\sin[(t-u)/2]\cos[(t+u)/2]$, we get

$$\begin{aligned}&\alpha_k(\phi)(x)=2\sin\left[2^{k-2}(x^{2^{-(k-1)}}-1)-2^{k-1}(x^{2^{-k}}-1)\right]\cos\left[2^{k-2}(x^{2^{-(k-1)}}-1)+2^{k-1}(x^{2^{-k}}-1)\right]\\&=2\sin\left[2^{k-2}(x^{2^{-k}}-1)^2\right]\cos\left[2^{k-2}(x^{2^{-k}}-1)(3+x^{2^{-k}})\right].\end{aligned}$$

Hence, we have

$$\sin(x-1)-\sin[\log(x)]=2\sum_{k=1}^{+\infty}\sin\left[2^{k-2}(x^{2^{-k}}-1)^2\right]\cos\left[2^{k-2}(x^{2^{-k}}-1)(3+x^{2^{-k}})\right].$$

(8) For $x>0$, it follows from Theorem 2.1 with $\phi(t)=\cos(t)$ that

$$\cos(x-1)-\cos[\log(x)]=\phi(x-1)-\phi[\log(x)]=\sum_{k=1}^{+\infty}\alpha_k(\phi)(x),$$

where

$$\begin{aligned}\alpha_k(\phi)(x)&=\phi\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]-\phi\left[2^k(x^{2^{-k}}-1)\right]\\&=\cos\left[2^{k-1}(x^{2^{-(k-1)}}-1)\right]-\cos\left[2^k(x^{2^{-k}}-1)\right].\end{aligned}$$

Using the standard trigonometric formula $\cos(t)-\cos(u)=-2\sin[(t-u)/2]\sin[(t+u)/2]$, we obtain

$$\begin{aligned}&\alpha_k(\phi)(x)=-2\sin\left[2^{k-2}(x^{2^{-(k-1)}}-1)-2^{k-1}(x^{2^{-k}}-1)\right]\sin\left[2^{k-2}(x^{2^{-(k-1)}}-1)+2^{k-1}(x^{2^{-k}}-1)\right]\\&=-2\sin\left[2^{k-2}(x^{2^{-k}}-1)^2\right]\sin\left[2^{k-2}(x^{2^{-k}}-1)(3+x^{2^{-k}})\right].\end{aligned}$$

Hence, we have

$$\cos(x-1)-\cos[\log(x)]=-2\sum_{k=1}^{+\infty}\sin\left[2^{k-2}(x^{2^{-k}}-1)^2\right]\sin\left[2^{k-2}(x^{2^{-k}}-1)(3+x^{2^{-k}})\right].$$

(9) For $x > 0$, applying Theorem 2.1 to $\phi(t) = \sinh(t)$, we establish that

$$\sinh(x-1) - \sinh[\log(x)] = \phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} \alpha_k(\phi)(x),$$

where

$$\begin{aligned}\alpha_k(\phi)(x) &= \phi\left[2^{k-1}(x^{2^{-(k-1)}} - 1)\right] - \phi\left[2^k(x^{2^{-k}} - 1)\right]\\ &= \sinh\left[2^{k-1}(x^{2^{-(k-1)}} - 1)\right] - \sinh\left[2^k(x^{2^{-k}} - 1)\right].\end{aligned}$$

Using the standard hyperbolic formula $\sinh(t) - \sinh(u) = 2\sinh[(t-u)/2]\cosh[(t+u)/2]$, we get

$$\begin{aligned}\alpha_k(\phi)(x) &= 2\sinh\left[2^{k-2}(x^{2^{-(k-1)}} - 1) - 2^{k-1}(x^{2^{-k}} - 1)\right]\cosh\left[2^{k-2}(x^{2^{-(k-1)}} - 1) + 2^{k-1}(x^{2^{-k}} - 1)\right]\\ &= 2\sinh\left[2^{k-2}(x^{2^{-k}} - 1)^2\right]\cosh\left[2^{k-2}(x^{2^{-k}} - 1)(3 + x^{2^{-k}})\right].\end{aligned}$$

Hence, we have

$$\sinh(x-1) - \sinh[\log(x)] = 2\sum_{k=1}^{+\infty} \sinh\left[2^{k-2}(x^{2^{-k}} - 1)^2\right]\cosh\left[2^{k-2}(x^{2^{-k}} - 1)(3 + x^{2^{-k}})\right].$$

(10) For $x > 0$, using Theorem 2.1 with $\phi(t) = \cosh(t)$, we find that

$$\cosh(x-1) - \cosh[\log(x)] = \phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} \alpha_k(\phi)(x),$$

where

$$\begin{aligned}\alpha_k(\phi)(x) &= \phi\left[2^{k-1}(x^{2^{-(k-1)}} - 1)\right] - \phi\left[2^k(x^{2^{-k}} - 1)\right]\\ &= \cosh\left[2^{k-1}(x^{2^{-(k-1)}} - 1)\right] - \cosh\left[2^k(x^{2^{-k}} - 1)\right].\end{aligned}$$

Using the standard trigonometric formula $\cosh(t) - \cosh(u) = 2\sinh[(t-u)/2]\sinh[(t+u)/2]$, we obtain

$$\begin{aligned}\alpha_k(\phi)(x) &= 2\sinh\left[2^{k-2}(x^{2^{-(k-1)}} - 1) - 2^{k-1}(x^{2^{-k}} - 1)\right]\sinh\left[2^{k-2}(x^{2^{-(k-1)}} - 1) + 2^{k-1}(x^{2^{-k}} - 1)\right]\\ &= 2\sinh\left[2^{k-2}(x^{2^{-k}} - 1)^2\right]\sinh\left[2^{k-2}(x^{2^{-k}} - 1)(3 + x^{2^{-k}})\right].\end{aligned}$$

Hence, we have

$$\cosh(x-1) - \cosh[\log(x)] = 2\sum_{k=1}^{+\infty} \sinh\left[2^{k-2}(x^{2^{-k}} - 1)^2\right]\sinh\left[2^{k-2}(x^{2^{-k}} - 1)(3 + x^{2^{-k}})\right].$$

All the claimed expansions are established, ending the proof. □

Thus, Proposition 2.2 is derived from Theorem 2.1 by using various functions ϕ, namely $\phi(t) = t$, $\phi(t) = 1/t$, $\phi(t) = t^2$, $\phi(t) = \sqrt{t}$, $\phi(t) = 1/\sqrt{t}$, $\phi(t) = \log(t)$, $\phi(t) = \sin(t)$, $\phi(t) = \cos(t)$, $\phi(t) = \sinh(t)$ and $\phi(t) = \cosh(t)$, one for each sub-result, in order. In addition, some extended results can be proved. For example, for $x > 0$, based on the proof Theorem 2.1 with the function $\phi(t) = \sqrt{|t|}$, we can extend the item numbered 4 as

$$\sqrt{|x-1|} - \sqrt{|\log(x)|} = \mathrm{sign}(x-1) \sum_{k=1}^{+\infty} \frac{2^{(k-1)/2}|x^{2^{-k}} - 1|^{3/2}}{\sqrt{1 + x^{2^{-k}}} + \sqrt{2}},$$

where $\mathrm{sign}(x-1) = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } x = 1, \\ -1 & \text{if } x < 1. \end{cases}$ Analogous extension of the item numbered 5 is possible.

With a little mathematical effort, we can get similar series expansions by considering the translated version of ϕ, i.e., $\phi(t; a) = \phi(t + a)$ for some $a \in \mathbb{R}$. Of course, other interesting functions can also be examined for ϕ, such as $\phi(t) = \mathrm{arctanh}(t)$, which benefits from an interesting addition formula, among other things.

It is important to note that the convergence of the series expansions in Proposition 2.2 has been checked on the basis of theoretical and practical work. Let us illustrate graphically the convergence of the series expansion in the item numbered 3. To do this, we consider the following truncated-series function:

$$\varphi(x; m) = (x-1)^2 - [\log(x)]^2 - \sum_{k=1}^{m} 2^{2(k-1)}(x^{2^{-k}} - 1)^3(3 + x^{2^{-k}}),$$

where m denotes an integer such that $m \geq 1$. Figure 1 displays the plots of $\varphi(x; m)$ for $m = 1, 2, \ldots, 15$ and four arbitrary values of x.

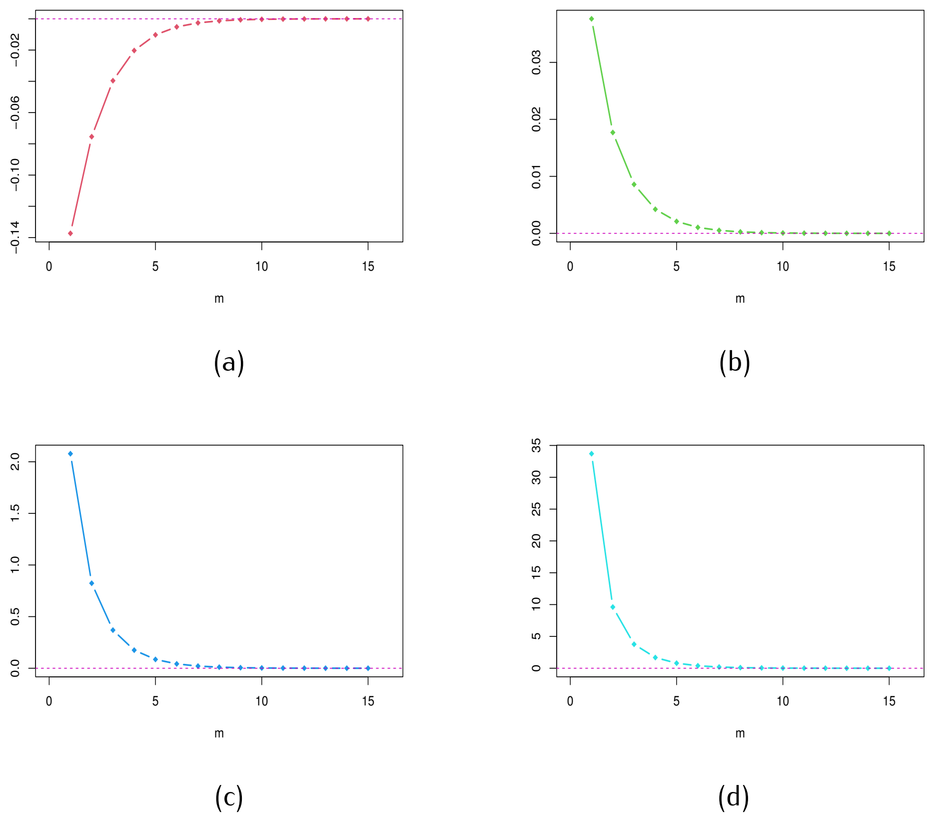

Figure 1. Plots of $\varphi(x; m)$ for $m = 1, 2, \dots, 15$ and (a) $x = 0.5$, (b) $x = 1.5$, (c) $x = 4$, and (d) $x = 18$.

From this figure we see that, for all the values of x considered, $\varphi(x; m)$ converges very quickly to 0; it begins to be very close to the $y = 0$ axis from $m = 7$.

Let us mention that the proof of the item numbered 2 has a different construction than the one in [2, Chapter 31, page 399], even though it is based on the same functional basis.

Also, from Proposition 2.2, several expansions of the logarithmic function can be deduced. For instance, based on the item numbered 4, for $x \geq 1$, we have

$$\log(x) = \left[\sqrt{x-1} - \sum_{k=1}^{+\infty} \frac{2^{(k-1)/2}(x^{2^{-k}} - 1)^{3/2}}{\sqrt{1 + x^{2^{-k}}} + \sqrt{2}}\right]^2.$$

Such expansions are innovative, to the best of our knowledge.

Concerning the item numbered 9 in Proposition 2.2, one can remark that

$$\sinh(x-1) - \sinh[\log(x)] = \frac{1}{2x} - \frac{x}{2} - \sinh(1-x)$$

and, by using the series expansion of the hyperbolic sine function, the following alternative expression is obtained:

$$\sinh(x-1) - \sinh[\log(x)] = \frac{1}{2x} - \frac{x}{2} - \sum_{k=0}^{+\infty} \frac{(1-x)^{2k+1}}{(2k+1)!}.$$

Clearly, it can be more manageable to use in comparison to the one in the item numbered 9, depending on the context. The same remark holds for the item numbered 10; we have

$$\cosh(x-1) - \cosh[\log(x)] = -\frac{1}{2x} - \frac{x}{2} + \sum_{k=0}^{+\infty} \frac{(1-x)^{2k}}{(2k)!}.$$

Thus, the items numbered 9 and 10 are mainly interesting because of their originality, i.e., the trigonometric and hyperbolic functions involved, respectively.

3. Applications

Some consequences of our results are described in this part.

3.1. Product expansions.

Proposition 2.2 can be used for many purposes, including product expansions. The result below illustrates this claim with an example.

Proposition 3.1. *The infinite product expansions below are valid.*

(1) *For $x > 1$, we have*

$$\frac{x-1}{\log(x)} = \prod_{k=1}^{+\infty} \frac{1+x^{2^{-k}}}{2}.$$

This expansion is, in fact, valid for $x \in (0, +\infty) \setminus \{1\}$, as discussed later. It "almost" corresponds to the so-called Seidel formula (see [15]).

(2) *For $x > 1$, we have*

$$\frac{\log(x)}{x-1} = \prod_{k=1}^{+\infty} \frac{2}{1+x^{2^{-k}}}.$$

This expansion is, in fact, valid for $x \in (0, +\infty) \setminus \{1\}$, as discussed later.

(3) *For $x > 0$, we have*

$$\frac{e^x-1}{x} = \prod_{k=1}^{+\infty} \frac{1+e^{2^{-k}x}}{2}.$$

This expansion is, in fact, valid for $x \in \mathbb{R} \setminus \{0\}$, as discussed later.

(4) *For $x > 0$, we have the following product expansion of the Einstein function:*

$$E_2(x) = \frac{x}{e^x-1} = \prod_{k=1}^{+\infty} \frac{2}{1+e^{2^{-k}x}}.$$

This expansion is, in fact, valid for $x \in \mathbb{R} \setminus \{0\}$, as discussed later.

Proof. Let us prove each result, one by one.

(1) We propose two different proofs.

Proof 1: use of Proposition 2.2.: For $x > 1$, it follows from the item numbered 6 in Proposition 2.2 that

$$\log(x-1) - \log\left[\log(x)\right] = \sum_{k=1}^{+\infty} \log\left(\frac{1+x^{2^{-k}}}{2}\right).$$

By the continuity of the logarithmic function over its domain, it can be rewritten as

$$\log\left[\frac{x-1}{\log(x)}\right] = \log\left[\prod_{k=1}^{+\infty} \frac{1+x^{2^{-k}}}{2}\right].$$

Composing with the exponential function, we get the desired result, i.e.,

$$\frac{x-1}{\log(x)} = \prod_{k=1}^{+\infty} \frac{1+x^{2^{-k}}}{2}.$$

Proof 2: iterative scheme.: To provide an alternative proof, we now revisit the original proof of the Seidel formula in [15]. We can write $x-1 = (1+\sqrt{x})(\sqrt{x}-1) = (1+x^{2^{-1}})(x^{2^{-1}}-1)$ and, with the same principle, we can write the last term as $x^{2^{-1}}-1 = (1+x^{2^{-2}})(x^{2^{-2}}-1)$, and the same for $x^{2^{-2}}-1$, etc. So, for any integer $n \geq 1$, we have

$$\begin{aligned} x-1 &= (1+x^{2^{-1}})(x^{2^{-1}}-1) = (1+x^{2^{-1}})(1+x^{2^{-2}})(x^{2^{-2}}-1) \\ &= \ldots = \left[\prod_{k=1}^{n}(1+x^{2^{-k}})\right](x^{2^{-n}}-1) = \left[\prod_{k=1}^{n} \frac{1+x^{2^{-k}}}{2}\right] 2^n(x^{2^{-n}}-1). \end{aligned}$$

Therefore, by considering the limit when $n \to +\infty$, we obtain

$$x-1 = \left[\lim_{n\to+\infty} \prod_{k=1}^{n} \frac{1+x^{2^{-k}}}{2}\right]\left[\lim_{n\to+\infty} 2^n(x^{2^{-n}}-1)\right] = \left[\prod_{k=1}^{+\infty} \frac{1+x^{2^{-k}}}{2}\right] \log(x),$$

which implies that

$$\frac{x-1}{\log(x)} = \prod_{k=1}^{+\infty} \frac{1+x^{2^{-k}}}{2}.$$

Thus, Proof 1 offers an alternative to these known developments by using Proposition 2.2. Proof 2, however, has the advantage of being valid for $x \in (0,+\infty) \setminus \{1\}$, not just $x > 1$. We will show later why this is actually not a problem.

(2) For $x > 1$, by using the previous result, we get

$$\begin{aligned} \frac{\log(x)}{x-1} &= \frac{1}{(x-1)/\log(x)} = \frac{1}{\prod_{k=1}^{+\infty}\left[(1+x^{2^{-k}})/2\right]} \\ &= \prod_{k=1}^{+\infty} \frac{1}{(1+x^{2^{-k}})/2} = \prod_{k=1}^{+\infty} \frac{2}{1+x^{2^{-k}}}. \end{aligned}$$

(3) For $y > 0$, by applying the result in the item numbered 1 with $x = e^y > 1$, we establish that

$$\frac{e^y - 1}{y} = \frac{x-1}{\log(x)} = \prod_{k=1}^{+\infty} \frac{1 + x^{2^{-k}}}{2} = \prod_{k=1}^{+\infty} \frac{1 + e^{2^{-k}y}}{2}.$$

(4) For $x > 0$, by using the result in the previous item, we have

$$E_2(x) = \frac{x}{e^x - 1} = \frac{1}{[(e^x-1)/x]} = \frac{1}{\prod_{k=1}^{+\infty}[(1 + e^{2^{-k}x})/2]}$$
$$= \prod_{k=1}^{+\infty} \frac{1}{(1 + e^{2^{-k}x})/2} = \prod_{k=1}^{+\infty} \frac{2}{1 + e^{2^{-k}x}}.$$

This ends the proof. □

From the item numbered 2, for $x \geq 1$, we get the following product expansion of the logarithmic function:

$$\log(x) = (x-1)\prod_{k=1}^{+\infty} \frac{2}{1 + x^{2^{-k}}}, \tag{3}$$

which corresponds to the Seidel formula restricted to $(1, +\infty)$. We can complete it for $x \in (0, 1)$ based on the case $x \geq 1$. Indeed, for $x \in (0, 1)$, since $1/x > 1$ and $\sum_{k=1}^{+\infty} 2^{-k} = 1$, we get

$$\log(x) = -\log\left(\frac{1}{x}\right) = -\left(\frac{1}{x} - 1\right)\prod_{k=1}^{+\infty} \frac{2}{1 + x^{-2^{-k}}}$$
$$= (x-1)\frac{1}{x}\left[\prod_{k=1}^{+\infty} \frac{2}{1 + x^{2^{-k}}}\right]\left[\prod_{k=1}^{+\infty} x^{2^{-k}}\right] = (x-1)\left[\prod_{k=1}^{+\infty} \frac{2x^{2^{-k}}}{1 + x^{2^{-k}}}\right]\left[x^{-1+\sum_{k=1}^{+\infty} 2^{-k}}\right]$$
$$= (x-1)\prod_{k=1}^{+\infty} \frac{2}{1 + x^{2^{-k}}}.$$

We thus find the Seidel formula in its entirety, as mentioned in the second proof of the item numbered 1 in Proposition 3.1. The advantages of this decomposition are that it has no constraints on the natural domain of definition, i.e., $x > 0$, to satisfy $\log(x) = -\log(1/x)$, which is not the case for most series expansions of $\log(x)$, and also that $\log(x)$ and $x - 1$ have the same sign according to $x \in (0, 1)$ and $x > 1$. It is also underexploited in the literature to determine sharp logarithmic inequalities. We will emphasize this aspect in the next section. Thus, in a sense, the results in Theorem 2.1 unify three known results, one by Srinivasa Ramanujan in [12], one by Ludwig Seidel in [15], and a more recent one by David M. Bradley in [4].

As an additional numerical contribution, let us illustrate this expansion by considering the following function:

$$\zeta(x; m) = \log(x) - (x-1)\prod_{k=1}^{m} \frac{2}{1 + x^{2^{-k}}},$$

where m denotes an integer such that $m \geq 1$. Figure 2 displays the plots of $\zeta(x;m)$ for $m = 1, 2, \ldots, 15$ and four arbitrary values of x, including $x = 0.5 \in (0, 1)$ to check the previous statement.

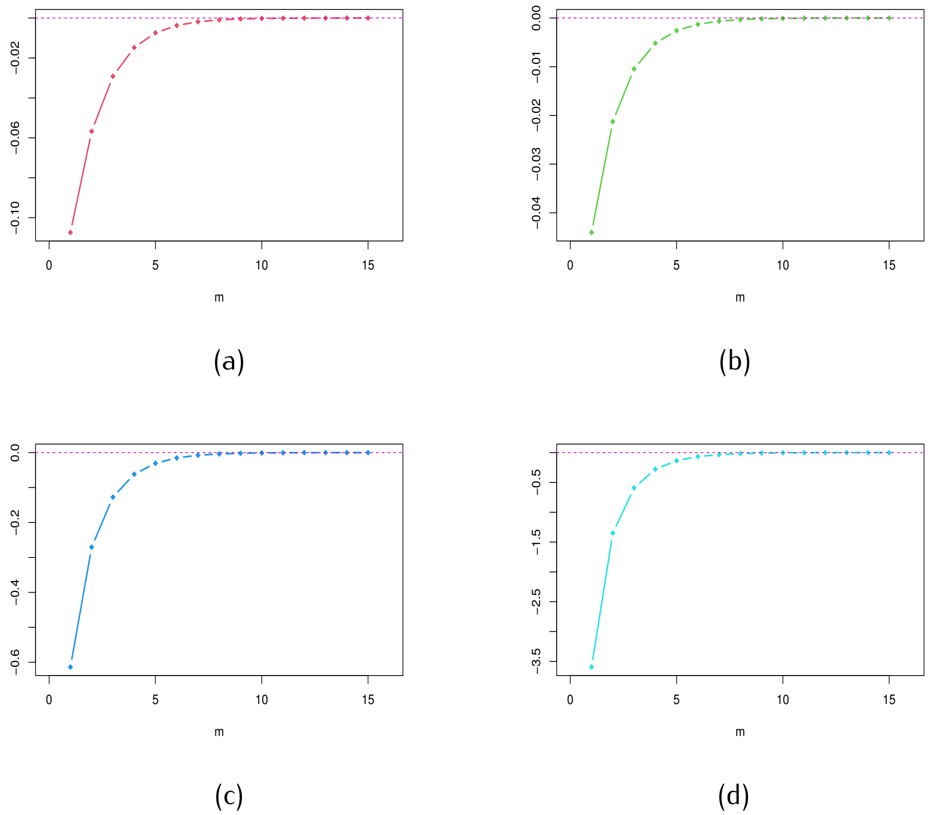

FIGURE 2. Plots of $\zeta(x;m)$ for $m = 1, 2, \ldots, 15$ and (a) $x = 0.5$, (b) $x = 1.5$, (c) $x = 4$, and (d) $x = 18$.

This figure shows that $\zeta(x;m)$ converges very quickly to 0 for all the considered values of x; it starts very close to the axis $y = 0$ from $m = 7$.

Sophisticated infinite product formulas can be derived from the natural properties of the logarithmic function and Equation (3). In particular, the formulas below are true.

- For $x > 0$, we have $x = e^{\log(x)}$, which is equivalent to

$$x = e^{(x-1)\prod_{k=1}^{+\infty} \frac{2}{1+x^{2^{-k}}}}.$$

- For $x > 0$ and $y \in \mathbb{R}$, we have $\log(x^y) = y \log(x)$, which yields

$$(x^y - 1) \prod_{k=1}^{+\infty} \frac{2}{1 + x^{2^{-k}y}} = y(x-1) \prod_{k=1}^{+\infty} \frac{2}{1 + x^{2^{-k}}}.$$

Similarly, for $x > 0$ and $y \in \mathbb{R} \setminus \{0\}$, we have $\log(x) = (1/y)\log(x^y)$, giving the following expansions:

$$\log(x) = \frac{1}{y}(x^y - 1)\prod_{k=1}^{+\infty} \frac{2}{1 + x^{2^{-k}y}}.$$

In particular, for $y = 2^m$ with an arbitrary $m > 0$, we have

$$\log(x) = \frac{1}{2^m}(x^{2^m} - 1)\prod_{k=1}^{+\infty} \frac{2}{1 + x^{2^{m-k}}}.$$

- For $x > 0$ and $y > 0$, we have $\log(xy) = \log(x) + \log(y)$, which is equivalent to

$$(xy - 1)\prod_{k=1}^{+\infty} \frac{2}{1 + (xy)^{2^{-k}}} = (x - 1)\prod_{k=1}^{+\infty} \frac{2}{1 + x^{2^{-k}}} + (y - 1)\prod_{k=1}^{+\infty} \frac{2}{1 + y^{2^{-k}}}.$$

In addition, it is known that the Einstein function can be expressed as a series expansion involving Bernoulli numbers as

$$E_2(x) = \sum_{k=0}^{+\infty} \frac{B_k^-}{k!} x^k,$$

where

$$B_k^- = \sum_{\ell=0}^{k} \sum_{v=0}^{\ell} (-1)^v \binom{\ell}{v} \frac{v^k}{\ell + 1}$$

and $\binom{\ell}{v} = \ell!/[v!(\ell - v)!]$.

In some sense, item numbered 4 completes this result by investigating a simple product expansion for $x > 0$, given as

$$E_2(x) = \prod_{k=1}^{+\infty} \frac{2}{1 + e^{2^{-k}x}}. \quad (4)$$

It is interesting to note that this formula is also valid for $x < 0$. Indeed, in this case, we can remark that

$$E_2(x) = e^{-x}E_2(-x) = e^{-x}\prod_{k=1}^{+\infty} \frac{2}{1 + e^{-2^{-k}x}} = e^{-x}\left[\prod_{k=1}^{+\infty} \frac{2}{1 + e^{2^{-k}x}}\right]\left[\prod_{k=1}^{+\infty} e^{2^{-k}x}\right]$$
$$= \left[\prod_{k=1}^{+\infty} \frac{2}{1 + e^{2^{-k}x}}\right]\left[e^{-x + x\sum_{k=1}^{+\infty} 2^{-k}}\right] = \prod_{k=1}^{+\infty} \frac{2}{1 + e^{2^{-k}x}}.$$

Thus, for $x \in \mathbb{R}\setminus\{0\}$, the formula in Equation (4) is true. To the best of our knowledge, this special representation of the Einstein function is a new result.

The next part is about some inequalities derived from our findings.

3.2. **Inequalities.** Inequalities of various kinds can be derived from our previous results. The proposition below proposes an original one.

Proposition 3.2. *For $x \geq 1$, we have*

$$\sqrt{x-1} - \sqrt{\log(x)} \leq \frac{1}{4} \sum_{k=1}^{+\infty} 2^{k/2}(x^{2^{-k}} - 1)^{3/2}.$$

This last series expansion converges.

Proof. Thanks to the item numbered 4 in Proposition 2.2, for $x \geq 1$, we have

$$\sqrt{x-1} - \sqrt{\log(x)} = \sum_{k=1}^{+\infty} \frac{2^{(k-1)/2}(x^{2^{-k}} - 1)^{3/2}}{\sqrt{1 + x^{2^{-k}}} + \sqrt{2}}.$$

Since $x \geq 1$, for any integer $k \geq 1$, we have $x^{2^{-k}} \geq 1$, so $\sqrt{1 + x^{2^{-k}}} \geq \sqrt{2}$, which implies that $\sqrt{1 + x^{2^{-k}}} + \sqrt{2} \geq 2\sqrt{2}$. Furthermore, it is clear that $2^{(k-1)/2}(x^{2^{-k}} - 1)^{3/2} \geq 0$. Therefore, we obtain

$$\sqrt{x-1} - \sqrt{\log(x)} \leq \frac{1}{2\sqrt{2}} \sum_{k=1}^{+\infty} 2^{(k-1)/2}(x^{2^{-k}} - 1)^{3/2} = \frac{1}{4} \sum_{k=1}^{+\infty} 2^{k/2}(x^{2^{-k}} - 1)^{3/2}.$$

The convergence of this series expansion can be shown by the equivalence technique. More precisely, when $k \to +\infty$, we have

$$2^{k/2}(x^{2^{-k}} - 1)^{3/2} = 2^{-k}[2^k(x^{2^{-k}} - 1)]^{3/2} \sim 2^{-k}[\log(x)]^{3/2},$$

and 2^{-k} is the term of a convergent geometric series. The desired result is demonstrated. □

This inequality may be more interesting for the lower bound of the series term than for the upper bound of $\sqrt{x-1} - \sqrt{\log(x)}$. In fact, it is difficult to capture the analytic function associated with the series term.

The proposition below is a general inequality setting based on Theorem 2.1.

Proposition 3.3. *In the framework of Theorem 2.1, the inequalities below are true.*

(1) *If ϕ is non-decreasing, then, for any sets of integers $M \subseteq \{1, 2, \ldots\}$, we have*

$$\phi(x-1) - \phi[\log(x)] \geq \sum_{k \in M} \alpha_k(\phi)(x).$$

(2) *If ϕ is non-increasing, then, for any sets of integers $M \subseteq \{1, 2, \ldots\}$, we have*

$$\phi(x-1) - \phi[\log(x)] \leq \sum_{k \in M} \alpha_k(\phi)(x).$$

Proof. For $x > 0$, let us consider the following function:

$$\psi(y) = y(x^{1/y} - 1), \quad y \in (0, +\infty).$$

Then we have

$$\psi'(y) = x^{1/y} - \frac{1}{y}x^{1/y}\log(x) - 1 = x^{1/y} - x^{1/y}\log(x^{1/y}) - 1.$$

Using the well-known logarithmic identity $\log(t) \geq t^{-1}(t-1)$ for $t > 0$, with $t = x^{1/y}$, we obtain

$$\psi'(y) \leq x^{1/y} - x^{1/y}x^{-1/y}(x^{1/y} - 1) - 1 = 0.$$

As a result, ψ is a non-increasing function. Then, for any integer $k \geq 1$, since $2^{k-1} \leq 2^k$, we have $\psi(2^k) \leq \psi(2^{k-1})$. Let us now distinguish two cases:

- If ϕ is non-decreasing, then we have $\phi\left[\psi(2^k)\right] \leq \phi\left[\psi(2^{k-1})\right]$, implying that $\alpha_k(\phi)(x) = \phi\left[\psi(2^{k-1})\right] - \phi\left[\psi(2^k)\right] \geq 0$. It follows from Theorem 2.1 that, for any sets of integers $M \subseteq \{1, 2, \dots\}$, we have

$$\phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} \alpha_k(\phi)(x) = \sum_{k\in M} \alpha_k(\phi)(x) + \sum_{k\notin M} \alpha_k(\phi)(x) \geq \sum_{k\in M} \alpha_k(\phi)(x).$$

- With similar arguments, if ϕ is non-increasing, then we have $\phi\left[\psi(2^k)\right] \geq \phi\left[\psi(2^{k-1})\right]$, implying that $\alpha_k(\phi)(x) = \phi\left[\psi(2^{k-1})\right] - \phi\left[\psi(2^k)\right] \leq 0$. It follows from Theorem 2.1 that, for any sets of integers $M \subseteq \{1, 2, \dots\}$,

$$\phi(x-1) - \phi[\log(x)] = \sum_{k=1}^{+\infty} \alpha_k(\phi)(x) = \sum_{k\in M} \alpha_k(\phi)(x) + \sum_{k\notin M} \alpha_k(\phi)(x) \leq \sum_{k\in M} \alpha_k(\phi)(x).$$

The desired inequalities are demonstrated. □

Let us exemplify this general result with an immediate application. Taking $M = \{m, \dots, n\}$, where m and n are integers such that $n \geq m \geq 1$, with regard to the item numbered 3 in Proposition 2.2 using $\phi(t) = t^2$, the following inequalities hold:

- For $x > 1$, we have

$$(x-1)^2 - [\log(x)]^2 \geq \sum_{k=m}^{n} 2^{2(k-1)}(x^{2^{-k}} - 1)^3(3 + x^{2^{-k}}).$$

- For $x \in (0, 1)$, we have

$$(x-1)^2 - [\log(x)]^2 \leq \sum_{k=m}^{n} 2^{2(k-1)}(x^{2^{-k}} - 1)^3(3 + x^{2^{-k}}).$$

We can also remark that, for any integer $k \geq 1$, if $x > 1$, then we have $(x^{2^{-k}} - 1)^3 > 0$, and if $x \in (0, 1)$, then we have $(x^{2^{-k}} - 1)^3 < 0$, and the above inequalities follow.

Another simple application is the inequality formulated in the lemma below.

Lemma 3.4. *For $x > 0$, we have*

$$\log(x) \leq 2(\sqrt{x} - 1).$$

Proof. We propose three different proofs.

Proof 1: judicious approach.: The following inequality is well known: $\log(y) \leq y - 1$ for $y > 0$. Applying it to the judicious choice $y = \sqrt{x}$, we get $\log(x) = 2\log(\sqrt{x}) \leq 2(\sqrt{x} - 1)$.

Proof 2: use of our series expansion.: We can apply the first items in Propositions 2.2 and 3.3 with $M = \{1\}$. Indeed, since $\phi(t)$ is non-decreasing (or the coefficients of the related series expansion are clearly non-negative), we have

$$x - 1 - \log(x) \geq \sum_{k \in M} 2^{k-1}(x^{2^{-k}} - 1)^2 = (\sqrt{x} - 1)^2,$$

implying that

$$\log(x) \leq x - 1 - (\sqrt{x} - 1)^2 = 2(\sqrt{x} - 1).$$

Proof 3: use of differentiation.: In the proof of Proposition 3.3, we showed that, for $x > 0$, the function $\psi(y) = y(x^{1/y} - 1)$ is non-increasing. This implies that, for any $\theta > 0$, we have $\lim_{y \to 0^+} \psi(y) \leq \psi(\theta)$, i.e.,

$$\log(x) \leq \theta(x^{1/\theta} - 1).$$

The desired result is just a special case; it is enough to take $\theta = 2$.

This completes the proof. □

This lemma is not new; it has been demonstrated with other differentiation techniques in [17], and its sharpness has also been illustrated.

Another logarithmic inequality is highlighted in the lemma below.

Lemma 3.5. *For $x > 0$ and any integers m and n such that $n \geq m \geq 1$, we have*

$$\log(x) \leq (x - 1) \prod_{k=m}^{n} \frac{2}{1 + x^{2^{-k}}}.$$

Proof. The proof is a consequence of the infinite product expansion in Equation (3). Let us distinguish the cases $x \geq 1$ and $x \in (0, 1)$.

- For $x \geq 1$ and any integer $k \geq 1$, we have $x^{2^{-k}} \geq 1$, implying that $2/(1 + x^{2^{-k}}) \leq 1$. So we have

$$\begin{aligned}\log(x) &= (x - 1)\left[\prod_{k=1}^{m-1} \frac{2}{1 + x^{2^{-k}}}\right]\left[\prod_{k=m}^{n} \frac{2}{1 + x^{2^{-k}}}\right]\left[\prod_{k=n+1}^{+\infty} \frac{2}{1 + x^{2^{-k}}}\right] \\ &\leq (x - 1)\prod_{k=m}^{n} \frac{2}{1 + x^{2^{-k}}},\end{aligned}$$

  with the convention $\prod_{k=1}^{0}[2/(1 + x^{2^{-k}})] = 1$.
- For $x \in (0, 1)$ and any integer $k \geq 1$, we have $x^{2^{-k}} \leq 1$, implying that $2/(1 + x^{2^{-k}}) \geq 1$ and $x - 1 \leq 0$. The exact same inequality as above is obtained.

The proof is therefore finished. □

This lemma generalizes the famous inequality $\log(x) \leq x-1$ for $x>0$. It also shows how the Seidel formula, i.e., Equation (3), can be applied to refine it.

A similar inequality involving the Einstein function is examined below.

Lemma 3.6. *For $x>0$ and any integers m and n such that $n \geq m \geq 1$, we have*

$$E_2(x) \leq \prod_{k=m}^{n} \frac{2}{1+e^{2^{-k}x}}.$$

For $x<0$, the reversed inequality holds.

Proof. The proof follows from the infinite product expansion in Equation (4). Indeed, for $x>0$ and any integer $k \geq 1$, we have $e^{2^{-k}x} > 1$, implying that $2/(1+e^{2^{-k}x}) < 1$. As a result, we have

$$\begin{aligned} E_2(x) &= \left[\prod_{k=1}^{m-1} \frac{2}{1+e^{2^{-k}x}}\right]\left[\prod_{k=m}^{n} \frac{2}{1+e^{2^{-k}x}}\right]\left[\prod_{k=n+1}^{+\infty} \frac{2}{1+e^{2^{-k}x}}\right] \\ &\leq \prod_{k=m}^{n} \frac{2}{1+e^{2^{-k}x}}, \end{aligned}$$

with the convention $\prod_{k=1}^{0}[2/(1+e^{2^{-k}x})] = 1$. For $x<0$, for $x>0$ and any integer $k \geq 1$, we have $e^{2^{-k}x} < 1$, implying that $2/(1+e^{2^{-k}x}) > 1$. Using this, the reversed inequality above is immediately established. The desired results are obtained. □

The inequalities above are just a sample of what can be derived from our results; other explorations are left for future studies.

4. Conclusion

In conclusion, the logarithmic function $\log(x)$ is central to several mathematical disciplines. In particular, its series expansions play a crucial role in mathematical analysis. While the classical series expansion initially presented limitations for the natural domain, i.e., $x \in (0,+\infty)$, a refined telescoping technique elaborated in [4] allows to relax the constraint. This allows a broader understanding of the logarithmic function. In our first investigations, based on this result and a well-known series expansion established by Srinivasa Ramanujan (see [2] and [5]), we discovered a unified functional pattern. This connection can be expressed in the form "$\phi(x-1) - \phi[\log(x)]$". In light of this, using telescoping techniques and thorough factorization developments, we generate new series extensions for such functions. Several examples are given and discussed. As illustrated, these results lead to new product expansions, including one for the Einstein function, and to inequalities involving the logarithmic function. This article thus contributes to a better understanding of the logarithmic function and series expansions in general, and lays some foundations for future work.

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