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Majorizing Sequences for Newton-Like Method and Their Limit Points

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Abstract. A plethora of problems from diverse disciplines of Mathematics, Mathematical Biology, Chemistry, Medicine, physics and Engineering to mention a few reduce to solving nonlinear equations or systems of equations usually in the finite dimensional Euclidean or more general spaces. The solutions of such equations are numbers or vectors of functions and can be found in closed form only in special cases. That is why researchers and practitioners develop mostly iterative methods which generate sequences approximating the solutions. The least number of iterations to be carried out in order to obtain a pre-decided error tolerance on the distances between consecutive iterates as well as the choice of initial points ensuring the convergence of the methods is very important. These two objectives can be achieved by introducing real majorizing sequences which control the behaviour of the iterates. Moreover, the closed form of the limits of the real sequences determine the radius of the ball that contains the initial points. In this paper we contribute by introducing more precise majorizing sequences and limit points.

1. Introduction

Majorizing sequences have been used extensively to study the semi-local convergence of Newton's method defined for $x_0 \in D$ and each $n = 0, 1, 2, \dots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.1}$$

where $F : D \subset B_1 \to B_2$ is a Fŕechet- differentiable operator between Banach spaces B_1, B_2 and D is an open and convex set [1, 3, 4, 6, 7]. The usually sufficient semi-local convergence conditions differ in general as well as the majorizing sequences and their limit points. We try to relate these conditions, sequences and limit points in a unified way without additional hypotheses.

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2. Lipschitz conditions

The symbols $L(B_1, B_2)$, $U(x, r)$ are used to denote the space of bounded linear operators from B_1 into B_2 and the open ball centered at $x \in B_1$ and of radius $r > 0$, respectively.

We introduce Lipscits conditions used to control F'. Then, we copare them to each other.

Definition 2.1. *Suppose $M \in \mathscr{L}(B_1, B_2)$ is an invertible operator and $x_0 \in D$. We say that F' is center-Lipschitz continuous if there exists $L_0 > 0$ such that*

$$M^{-1}(F'(x) - M)\| \leq L_0\|x - x_0\| \text{ for each } x \in D_0. \tag{2.1}$$

Define the region

$$D_0 = D \cap U(x_0, \frac{1}{L_0}). \tag{2.2}$$

Definition 2.2. *Suppose $M \in \mathscr{L}(B_1, B_2)$ is an invertible operator. We say that F' is restricted Lipschitz continuous if there exists $L > 0$ such that*

$$\|M^{-1}(F'(y) - F'(x))\| \leq L\|y - x\| \text{ for each } x, y \in D_0. \tag{2.3}$$

Definition 2.3. *Suppose $M \in \mathscr{L}(B_1, B_2)$ is an invertible operator. We say that F' is Lipschitz continuous if there exists $L_1 > 0$ such that*

$$\|M^{-1}(F'(y) - F'(x))\| \leq L_1\|y - x\| \text{ for each } x, y \in D. \tag{2.4}$$

REMARK 2.4. *It follows by these definitions that since $D_0 \subseteq D$, we have*

$$L_0 \leq L_1 \tag{2.5}$$

and

$$L \leq L_1. \tag{2.6}$$

It is worth noting that L_0 and L_1 depend on x_0, F' and D. But L depends on x_0, F' and D_0. Moreover, in practice the computation of L_1 requires that of L_0 and L as special cases. These constants are related to majorizing sequences in Section 3.

3. Convergence of Majorizing sequences.

Let $\Omega \geq 0, L_1 > 0$ and $\lambda \geq 1$ be parameters. Define μ and β by

$$\mu = \lambda\Omega \text{ and } \beta = \frac{\lambda}{1 + (\lambda - 1)L_1\mu}. \tag{3.1}$$

Moreover, define the quadratic majorizing function f_1 by

$$f_1(t) = \frac{\beta L_1 t^2}{2} - t + \mu. \tag{3.2}$$

Further more, define the scalar sequence $\{v_n\}$ for $v_0 = 0$ and each $n = 0, 1, 2, \ldots$ by

$$v_{n+1} = v_n - \frac{f_1(v_n)}{f_1'(v_n)}. \tag{3.3}$$

An auxiliary result is needed for the convergence of the sequence $\{v_n\}$.

LEMMA 3.1. *Suppose*

$$h_1 = 2\beta L_1 \mu \leq 1. \tag{3.4}$$

Then, the following assertions hold

(i) *The zeros of the function f_1 are real and given by*

$$v^* = \frac{1-\sqrt{1-2\beta L_1\mu}}{\beta L_1} \text{ and } v^{**} = \frac{1+\sqrt{1-2\beta L_1\mu}}{\beta L_1}. \tag{3.5}$$

(ii)

$$v_{n+1} - v_n = \frac{\beta L_1 (v_n - v_{n-1})^2}{2(1-\beta L_1 v_n)} = -\frac{f_1(v_n)}{f_1'(v_n)}. \tag{3.6}$$

(iii) *The sequence $\{v_n\}$ is increasingly convergent to v^* and can also be written in closed for as*

$$v_n = \frac{\sum_{j=0}^{2^n-2} q_1^j}{\sum_{j=0}^{2^n-1} q_1^j} v^*, n = 1, 2, \ldots, \tag{3.7}$$

where,

$$q_1 = \frac{v^*}{v^{**}} = \frac{1-\sqrt{1-2\beta L_1\mu}}{1+\sqrt{1-2\beta L_1\mu}}. \tag{3.8}$$

Proof. (i) The zeros of the function f are real by (3.4).

By setting $f(t) = 0$ and using the quadratic formula we obtain v^* and v^{**}.

(ii) Let

$$v_{n+1} = g_1(v_n), v_0 = 0, n = 0, 1, \ldots \tag{3.9}$$

where,

$$g_1(t) = \frac{\frac{1}{2}\beta L_1 t^2 - \mu}{\beta L_1 t - 1}. \tag{3.10}$$

Multiply (3.9) by $(1-\beta L_1 v_n)$ and simplify to get

$$\begin{aligned} (1-\beta L_1 v_n)v_{n+1} &= \mu - \frac{1}{2}\beta L_1 v_n^2, \\ &\text{or} \\ \frac{1}{2}\beta L_1 v_n^2 - \beta L_1 v_n v_{n+1} + \frac{1}{2}\beta L_1 v_{n+1}^2 &= \frac{1}{2}\beta L_1 v_{n+1}^2 - v_{n+1} + \mu, \end{aligned} \tag{3.11}$$

so

$$\frac{1}{2}\beta L_1 (v_{n+1} - v_n)^2 = \frac{1}{2}\beta L_1 v_{n+1}^2 - v_{n+1} + \mu.$$

Thus, we can write

$$v_{n+1} - v_n = \frac{\frac{1}{2}\beta L_1 (v_n - v_{n-1})^2}{1-\beta L_1 v_n} = -\frac{f_1(v_n)}{f_1'(v_n)}. \tag{3.12}$$

(iii) The proof can be found in [5].

□

REMARK 3.2. (i) *In view of (3.1) the results of the Lemma 3.1 can be given without* β*. For example (3.4) becomes*

$$\mu L_1(\lambda+1) \le 1. \tag{3.13}$$

(ii) *Let* $L > 0$*. Define the quadratic majorizing function* f *by*

$$f(t) = \frac{\beta L t^2}{2} - t + \mu, \tag{3.14}$$

and the scalar sequence $\{u_n\} for u_0 = 0$ *and each* $n = 0, 1, 2, \ldots$ *by*

$$u_{n+1} = u_n - \frac{f(u_n)}{f'(u_n)}. \tag{3.15}$$

Denote the corresponding zeros of $f(t) = 0$ *by* v^* *and* v^{**}*, respectvely provided that*

$$h_2 = 2\beta L\mu \le 1 \tag{3.16}$$

Clearly, the results of the Lemma 3.1 hold, if L *replaces* L_1 *and*

$$u_{n+1} - u_n = \frac{\beta L(u_n - u_{n-1})^2}{2(1-\beta L u_n)}. \tag{3.17}$$

Let $L > 0$*. Define the sequence* $\{s_n\}$ *for* $_0 = 0$,

$$\begin{aligned} s_1 = \mu, s_2 = s_1 + \frac{\beta L_0(s_1 - s_0)^2}{2(1 - L_0\beta s_1)} \\ \text{and} \\ s_{n+1} = s_n + \frac{\beta L(s_n - s_{n-1})^2}{2(1 - L_0\beta s_n)}. \end{aligned} \tag{3.18}$$

Next, we compare the sequences $\{v_n\}, \{u_n\}$*, and* $\{s_n\}$.

LEMMA 3.3. *Suppose (2.5),(2.6) and (3.4) hold.*
Then, the following assertions hold

$$\begin{aligned} 0 &\le s_n \le s_{n+1}, \\ 0 &\le u_n \le u_{n+1} \\ 0 &\le v_n \le v_{n+1}, \\ 0 &\le s_n \le u_n \le v_n \end{aligned}$$

and

$$0 \le s^* = \lim_{n\to+\infty} \le u^* = \lim_{n\to+\infty} = \frac{1-\sqrt{1-2\beta L\mu}}{\beta L} \le v^*.$$

Proof. It follows by simple induction (2.5),(2.6) and the definition of these sequences. □

In the next section, we relate sequences $\{v_n\}, \{u_n\}$ and $\{s_n\}$ to $\{x_n\}$.

4. Convergence of Newton's method

The celebrated Newton-Kantorovich Theorem for solving nonlinear equations using Newton's method is stated next. The proof can be found in [3, 6] for $M = F'(x_0)$. Moreover, the proof for general M follows by simply using M instead of $F'(x_0)$ in the Newton-Kantorovich Theorem.

THEOREM 4.1. *Suppose that (2.4) and (3.4) hold for $\lambda = 1, \mu = \Omega$ and $\Omega \geq \|F'(x_0)^{-1}F(x_0)\|$. Then, the sequence $\{x_n\}$ generated by Newton's method (1.1) is well defined in $U(x_0, v^*)$, remains in $U(x_0, v^*)$ for each $n = 0, 1, 2, \ldots$ and converges to a unique solution $x^* \in U(x_0, r^*)$ of the equation $F(x) = 0$. Moreover, the sequence $\{v_n\}$ majorizes $\{x_n\}$,*

$$\|x_{n+1} - x_n\| \leq v_{n+1} - v_n \tag{4.1}$$

and

$$\|x^* - x_n\| \leq v^* - v_n. \tag{4.2}$$

Furthermore, if there exists $\bar{v} \geq v^*$ such that

$$\frac{L_1}{2}(v^* + \bar{v}) < 1, \tag{4.3}$$

then the solution x^* is more unique in $U[x_0, \frac{2}{L_1} - v^*]$, where $U[x_0, r]$ is the closure of $U(x_0, r)$.

REMARK 4.2. *In view of (2.6) Theorem (4.1) holds provided that $L, \{u_n\}$ replace $L_1, \{v_n\}$, respectively. By Lemma 3.1 and 3.3 the sequence $\{u_n\}$ is tighter than $\{v_n\}$ and the limit point u^* is atleast as small as v^*. Moreover, they are given in closed form. This is not however the case for s^*. The convergence condition for $\{s_n\}$ given in [2] for*

$$M = F'(x_0), \lambda = 1 \quad h_3 = 2\bar{L}\mu \leq 1, \tag{4.4}$$

where

$$\bar{L} = \frac{1}{8}(4L_0 + \sqrt{L_0L + 8L_0^2} + \sqrt{L_0L}).$$

Notice that

$$h_1 \leq 1 \implies h_2 \leq 1 \text{ and } h_3 \leq 1 \tag{4.5}$$

but not necessarily vice versa unless if $L_0 = L = L_1$. Moreover,

$$\frac{h_3}{h_1} \to 0 \text{ as } \frac{L_0}{L_1} \to 0. \tag{4.6}$$

$$\frac{h_3}{h_1} \to 0 \text{ as } \frac{L_0}{L} \to 0. \tag{4.7}$$

In view of (4.5)-(4.7) and the Lemma 3.3 the results using (4.4) improve the ones by Theorem 4.1 infinitely many times. However, s^* is not given in closed form. But we have

$$s^* \leq \bar{s}, \tag{4.8}$$

where

$$\bar{s} = \mu + \frac{L_0\mu^2}{2(1-\alpha)(1-L_0\mu)}, \tag{4.9}$$

where

$$\alpha = \frac{2L}{L+\sqrt{L^2+8L_0L}}. \tag{4.10}$$

Next, we shall find an upper bound on s^* which is given in closed form and may be tighter than $\bar{s}$.

Let

$$a = \frac{1}{8L}(4L_0 + \sqrt{L_0L + 8L_0^2} + \sqrt{L_0L}) \tag{4.11}$$

Then, the condition (4.4) is equivalent to

$$h = 2aL\mu \leq 1 \tag{4.12}$$

Then, the corresponding Theorem in [2] can be written as

THEOREM 4.3. *Suppose for $\mu \geq \|F'(x_0)^{-1}F(x_0)\|$ conditions (2.1), (2.2), (4.12) and $\bar{U}[x_0, s^*] \subset D$. Then, the sequences $\{x_n\}$ generated by Newton's method (1.1) is well defined in $U(x_0, s^*)$, remains in $U(x_0, s^*)$ for each $n = 0, 1, 2, \dots$ and is convergent to a solution $x^* \in U[x_0, s^*]$ of the equation $F(x) = 0$. Moreover, the following error estimates hold*

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n \tag{4.13}$$

and

$$\|x^* - x_n\| \leq s^* - s_n. \tag{4.14}$$

Additionally, if for some $b \geq s^*$

$$L_0(s^* + b) < 1 \tag{4.15}$$

then, the solution x^* is unique in the region $D \cap U[x_0, b]$.

Proof. Simply notice that (4.12) is equivalent to (4.4) used in [2]. □

REMARK 4.4. *By the definition of a*
It follows that

$$0 < a \leq 1 \quad \text{if} \quad L_0 \leq L \tag{4.16}$$

and

$$a \geq 1 \quad \text{if} \quad L \leq L_0. \tag{4.17}$$

Define the function f by

$$f(t) = \frac{aLt^2}{2} - t + \mu \tag{4.18}$$

and the sequence $\{\bar{s}_n\}$ *for* $\bar{s}_0 = 0, \bar{s}_1 = \mu$,

$$\bar{s}_2 = \bar{s}_1 + \frac{L_0(\bar{s}_1 - \bar{s}_0)^2}{2(1 - L_0\bar{s}_1)}, \tag{4.19}$$

$$\bar{s}_{n+1} = \bar{s}_{n+1} - \frac{f(\bar{s}_{n+1})}{f'(\bar{s}_{n+1})}, n = 1, 2, \ldots \tag{4.20}$$

PROPOSITION 4.5. *Suppose that the conditions of Theorem 4.3 hold. Then, we have that smallest solution denoted by* $\bar{s}^*$ *of the equation* $f(t) = 0$, *i.e.*

$$\bar{s}^* = \frac{1 - \sqrt{1 - 2aL\mu}}{aL} \tag{4.21}$$

is an upper bound in closed form of the sequence $\{\bar{s}_n\}$ *and*

$$0 \leq s_n \leq \bar{s}_n, \tag{4.22}$$

$$0 \leq s_{n+1} - s_n \leq \bar{s}_{n+1} - s_n \tag{4.23}$$

and

$$s^* \leq \bar{s}^*. \tag{4.24}$$

Proof. Indeed, this is clear under (4.16), whereas if (4.17) holds the, we have from

$$\|M^{-1}(F(x_{n+1} - F(x_n) - F'(x_n)(x_{n+1} - x_n)\|$$
$$\leq \frac{\tilde{L}\|x_{n+1} - x_n\|^2}{2} \frac{\tilde{L}}{2}(\bar{s}_{n+1} - \bar{s}_n)^2 = \frac{f(\bar{s}_{n+1}}{a},$$

where

$$\tilde{L} = \begin{cases} L_0, & n = 0 \\ L, & n = 1, 2, \ldots \end{cases},$$

$$\|F'(x_{n+1})^{-1}M\| \leq \frac{1}{1 - L_0\|x_{n+1} - x_0\|} \leq \frac{1}{1 - L_0\bar{s}_{n+1}}$$

so

$$\|x_{n+1} - x_{n+1}\| \leq \|F'(x_{n+1}^{-1}M\|\|M^{-1}F(x_{n+1})\|$$
$$\leq \frac{f(\bar{s}_{n+1})}{a(1 - L_0\bar{s}_{n+1})} \leq -\frac{f(\bar{s}_{n+1})}{f'(\bar{s}_{n+1}},$$
$$\text{since } \frac{1}{a(1 - L_0\bar{s}_{n+1})} \leq \frac{1}{1 - aL\bar{s}_{n+1}} = -\frac{1}{f'(\bar{s}_{n+1})}.$$

□

5. A Numerical Example

The convergence conditions, majorizing sequences and limit points are compared with each other.

EXAMPLE 5.1. *Let* $B_1 = B_2 = \mathbb{R}, D = U(x_0, 1 - p), p \in (0, 1)$ *and* $x_0 = 1$. *Define the function* $\Psi : D \to \mathbb{R}$ *by*

$$\Psi(t) = x^3 - p \tag{5.1}$$

Then, for $\lambda = 1, \mu = \frac{1}{3}(1-p)$ *and* $\beta = 1$. *Moreover, the definitions(2.1)-(2.3) hold if* $L_0 = 3 - p, L = 2(1 + \frac{1}{3-p})$ *and* $L_1 = 2(2-p)$.
Notice that

$$L_0 < L_1, L < L_1 \text{ for each } p \in (0, 1).$$

We also have that

$$L \leq L_0 \text{ if } p \in (0, 2 - \sqrt{3}]$$

and

$$L_0 \leq L \text{ if } p \in [2 - \sqrt{3}, 1).$$

Let us restrict $p \in (0, \frac{1}{2})$ *Then, the Newton-Kantorovich condition (3.4) [3, 6] does not hold, since (3.4) is not satisfied for any* $p \in (0, \frac{1}{2})$. *However, our condition (4.12) hold provided that* $p \in (.46, \frac{1}{2})$. *Thus, the old results [6] cannot guarantee the convergence of Newton's method for any* $p \in (0, \frac{1}{2})$. *However, Newton's method converges to* $x^* = \sqrt[3]{p}$ *if we say* $p = 0.48$. *In order to compare sequences and limit points. Let* $p = 0.7$. *Then, both (3.4) and (4.12) hold.*

Thus, the old results [3, 5–7] cannot guarantee the convergence of Newton's method for any $p \in (0, \frac{1}{2})$. *However, Newton's method converges to* $x^* = \sqrt[3]{p}$. *If say* $p = 0.48$. *In order to compare sequences and limit points. Let* $p = 0.7$. *Then, both (3.4) and (4.12) hold.*

Then, we have $\bar{s} = 0.3965$, $\bar{v} = 0.6511$, $b = 0.3194$. *Therefore, the new error bounds and limit points are tighter than the ones given before [3, 5–7] and under weaker sufficient semi-local convergence criteria.*

Table 1. Comparison Between Majorizing Sequences and their Limit Points

n	v_n	$v_{n+1} - v_n$	s_n	$s_{n+1} - s_n$	$\bar{s}_n$	$\bar{s}_{n+1} - \bar{s}_n$
0	0	0	0	0	0	0
1	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
2	0.1176	0.0176	0.1149	0.4350e-03	0.1149	0.7619e-03
3	$0.1181 = v^*$	0.0006	$0.1154 = s^*$	0.0004e-03	$0.1157 = \bar{s}^*$	0.0009e-03

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