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Uncertainty Principles and Extremal Functions for Bessel Multiplier Operators in Quantum Calculus

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Abstract. Using the q-Jackson integral and some elements of the q-harmonic analysis associated with the q-Bessel operator for fixed $0 < q < 1$, we introduce the q-Bessel multiplier operators and we give some new results related to these operators as Plancherel's, Calderón's reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. Next, using the theory of reproducing kernels we give best estimates and an integral representation of the extremal functions related to these operators on weighted Sobolev spaces.

1. Introduction

The q-theory, called also in some literature quantum calculus began to arise. Interest in this theory is grown at an explosive note by both physicists and mathematicians due to a large number of its application domains, for more information about quantum calculus one can see [20].

Recently, many reasercher have been investigated the behavior of the q-theory to several already studied for the Fourier analysis, for example sampling theorem [2], Paley-Wiener theorem [1], uncertainty principles [31], wavelet transform [15], wavelet packet [6], Ramanujan master theorem [16], Sobolev type spaces [27] and wave equation [29]. In their seminal papers, Hörmander's and Mikhlin's [18,25] initiated the study of boundedness of the translation invariant operators on $\mathbb{R}^d$. The translation invariant operators on $\mathbb{R}^d$ characterized using the classical Euclidean Fourier transform $\mathcal{F}(f)$ therefore they also known as Fourier multipliers. Given a measurable function

$$m : \mathbb{R}^d \longrightarrow \mathbb{C}$$

its Fourier multiplier is the linear map $\mathcal{T}_m$ given for all $\lambda \in \mathbb{R}^d$ by the relation

$$\mathcal{F}(\mathcal{T}_m(f))(\lambda) = m(\lambda)\mathcal{F}(f)(\lambda) \tag{1.1}$$

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The Hörmander-Mikhlin fundamental condition gives a criterion for L^p-boundedness for all $1 < p < \infty$ of Fourier multiplier $\mathcal{T}_m$ in terms of derivatives of the symbol m, more precisely if

$$\left|\partial_\lambda^\gamma m(\lambda)\right| \lesssim |\lambda|^{-|\gamma|} \quad for \quad 0 \leq |\gamma| \leq \left[\frac{d}{2}\right] + 1. \tag{1.2}$$

Then, $\mathcal{T}_m$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^d)$ into itself .

The condition (1.2) imposes m to be a bounded function, smooth over $\mathbf{R}^d \backslash \{0\}$ satisfying certain local and asymptotic behavior. Locally, m admits a singularity at 0 with a mild control of derivatives around it up to order $\left[\frac{d}{2}\right] + 1$. This singularity links to deep concepts in harmonic analysis and justifies the key role of Hörmander-Mikhlin theorem in Fourier multiplier L_p-theory, this condition defines a large class of Fourier multipliers including Riesz transforms and Littelwood-Paley partitions of unity which are crucial in Fourier summability or Pseudo-differential operator.The boundedness of Fourier multipliers is useful to solve problems in the area of mathematical analysis as Probability theory see [24], Stochastic processus see [5], and the study of nonlinear partial differential equations see [22]. For its importance many researcher extend the theory of Fourier multiplier to different setting for example in the Dunkl-Weinstein setting [33], in the Laguerre-Bessel setting [8], in the q-Fourier setting [26, 31, 32] and the q-cosine Fourier setting [3]. The general theory of reproducing kernels is stared with Aronszajn's in [4] in 1950, next the authors in [23, 30] applied this theory to study Tikhonov regularization problem and they obtained approximate solutions for bounded linear operator equations on Hilbert spaces with the viewpoint of numerical solutions by computers. This theory has gained considerable interest in various field of mathematical sciences especially in Engineering and numerical experiments by using computers see [30].

This paper focuses on the generalized Fourier transform associated with the q-Bessel operator called the q-Bessel transform introduced in [11], more precisely we define the following q-differential operator for $0 < q < 1$ by

$$\Delta_{q,\alpha} f(x) = \frac{f\left(q^{-1}x\right) - \left(1 + q^{2\alpha}\right) f(x) + q^{2\alpha} f(qx)}{x^2}, \quad \forall x \neq 0. \tag{1.3}$$

The eigenfunctions of the operator (1.3) are related to the Hahn-Exton q-Bessel function $j_\alpha(x; q^2)$ defined in [15]. The q-Bessel tranform $\mathcal{H}_{q,\alpha}$ is defined on $L^1_\alpha(\mathbb{R}_q^+)$ by

$$\mathcal{H}_{q,\alpha}(f)(\lambda) = \int_0^\infty j_\alpha(\lambda x; q^2) f(x) d\mu_{q,\alpha}(x), \quad \text{for } \lambda \in \mathbb{R}_q^+$$

where $d\mu_{q,\alpha}$ is the measure on $\mathbb{R}_q^+$ given later. Let σ be a function in $L^2_\alpha(\mathbb{R}_q^+)$ and $\beta \in \mathbb{R}_q^+$, the q-Bessel L^2_α-multiplier operators are defined for smooth function f on $\mathbb{R}_q^+$ as

$$\mathcal{M}_{q,\sigma,\beta}(f)(x) := \mathcal{H}_{q,\alpha}^{-1}\left(\sigma_\beta \mathcal{H}_{q,\alpha}(f)\right)(x) \tag{1.4}$$

where the function σ_β is given by

$$\sigma_\beta(\lambda) := \sigma(\lambda\beta). \tag{1.5}$$

These operators are a generalization of all classical multiplier operators introduced in [3, 10, 26, 31, 32]. The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the q-Bessel transform, in section 3, we introduce the q-Bessel L^2_α-multiplier operators $\mathcal{M}_{q,\sigma,\beta}$ and we give for them a Plancherel's, point- wise reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. The last section of this paper is devoted to give an application of the general theory of reproducing kernels to q-Bessel multiplier theory and to give best estimates and an integral representation of the extremal functions related to the q-Bessel L^2_α-multiplier operators $\mathcal{M}_{q,\sigma,\beta}$ on weighted Sobolev spaces.

2. Harmonic Analysis Associated with the q-Bessel Transform

In this section we set some notations and we recall some results in harmonic analysis related to the q-Bessel operator (1.3), all these results can be founded in [11, 17, 19–21, 28].

2.1. **Notations and preliminaries.** In this subsection, we give some notations, definitions and properties of the q-shifted factorial, the Jackson's q-derivatives and the Jackson's q-integrals introduced in [19].

Let $a \in \mathbb{C}$, the q-shifted factorial are defined by:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad (a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right).$$

The Jackson's q-derivative of a function f is given by

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x} \text{ if } x \neq 0.$$

The q-Jackson's integrals from 0 to a and from 0 to ∞ are defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_0^\infty f\left(aq^n\right)q^n,$$

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f\left(q^n\right)q^n.$$

Provided the sums converge absolutely.

The normalized form of the q-Bessel kernel is defined in [14, 17, 28] by

$$j_\alpha(x;q^2) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{n(n+1)}{2}}}{\left(q^{\alpha+1};q\right)_n(q;q)_n}x^{2n}. \tag{2.1}$$

It satisfies the following estimate [11]

$$\forall x \in \mathbb{R}_q^+, \quad \left|j_\alpha(x; q^2)\right| \leq 1. \tag{2.2}$$

2.2. **The q-Bessel transform.** In this section, we define and give some basic properties of q-Bessel transform introduced in [11]. We first introduced the following spaces and norms

- $C_{0,q}(\mathbb{R}_q^+)$ denotes the set of all functions defined on $\mathbb{R}_q^+$ continuous at zero and vanishing at infinity, equiped with the induced topology of uniforme convergence.
- $L_\alpha^p(\mathbb{R}_q^+), 1 \leq p \leq \infty$, denotes the space of measurable functions on $\mathbb{R}_q^+$, satisfying

$$\|f\|_{p,q,\alpha} =: \begin{cases} \left(\int_0^\infty |f(x)|^p d\mu_{q,\alpha}(x)\right)^{1/p} < \infty, & 1 \leq p < \infty, \\ \sup_{x \in \mathbb{R}_q^+} |f(x)| < \infty, & p = \infty. \end{cases}$$

where

$$d\mu_{q,\alpha}(x) = \frac{1}{1-q} \frac{\left(q^{2\alpha+2}; q^2\right)_\infty}{(q^2; q^2)_\infty} x^{2\alpha+1} d_q(x),$$

Definition 2.1. *([11]) The q-Bessel transform $\mathcal{H}_{q,\alpha}$ defined on $L_\alpha^1(\mathbb{R}_q^+)$ by*

$$\mathcal{H}_{q,\alpha}(f)(\lambda) = \int_0^\infty j_\alpha(\lambda x; q^2) f(x) d\mu_{q,\alpha}(x), \quad \text{for } \lambda \in \mathbb{R}_q^+$$

Some basic properties of this transform are as follows, for the proofs, we refer the reader to [11, 13, 14, 25].

Proposition 2.1.
(1) For every $f \in L_\alpha^1(\mathbb{R}_q^+)$ we have $\mathcal{H}_{q,\alpha}(f) \in C_{0,q}(\mathbb{R}_q^+)$ and we have

$$\|\mathcal{H}_{q,\alpha}(f)\|_{\infty,q,\alpha} \leq B_{q,\alpha} \|f\|_{1,q,\alpha}. \tag{2.3}$$

Where

$$B_{q,\alpha} = \frac{1}{1-q} \frac{\left(-q^{2\alpha+2}; q^2\right)_\infty \left(-q^2; q^2\right)_\infty}{(q^2; q^2)_\infty} \tag{2.4}$$

(2)(q-Inversion formula) For $f \in \left(L_\alpha^1 \cap L_\alpha^2\right)(\mathbb{R}_q^+)$ such that $\mathcal{F}_\alpha(f) \in L_\alpha^1(\mathbb{R}_q^+)$ we have

$$f(x) = \int_0^\infty j_\alpha(\lambda x; q^2) \mathcal{H}_{q,\alpha}(f)(\lambda) d\mu_{q,\alpha}(\lambda), \quad a.e \quad x \in \mathbb{R}_q^+. \tag{2.5}$$

(3) (q-Parseval formula) For all $f, g \in L_\alpha^2(\mathbb{R}_q^+)$ we have

$$\langle f, g \rangle_q = \langle \mathcal{H}_{q,\alpha}(f), \mathcal{H}_{q,\alpha}(g) \rangle_q, \tag{2.6}$$

In particular we have

$$\|f\|_{2,q,\alpha} = \|\mathcal{H}_{q,\alpha}(f)\|_{2,q,\alpha}. \tag{2.7}$$

(4) (q-Plancherel theorem) The q-Bessel transform $\mathcal{H}_{q,\alpha}$ can be extended to an isometric isomorphism from $L_\alpha^2(\mathbb{R}_q^+)$ into $L_\alpha^2(\mathbb{R}_q^+)$.

2.3. **The translation operator Associated with the q-Bessel transform.**

Definition 2.2. *([13]) Let $x, y \in \mathbb{R}_q^+$ and f is a measurable function on $\mathbb{R}_q^+$ the translation operator is defined by*

$$\tau_{q,\alpha}^x f(y) = \int_0^\infty j_\alpha(\lambda x; q^2) j_\alpha(\lambda y; q^2) \mathcal{H}_{q,\alpha}(f)(\lambda) d\mu_{q,\alpha}(\lambda),$$

The following proposition summarizes some properties of the q-Bessel translation operator see [13].

Proposition 2.2. *For all $x, y \in \mathbb{R}_q^+$,we have:*

(1)

$$\tau_{q,\alpha}^x f(y) = \tau_{q,\alpha}^y f(x). \tag{2.8}$$

(2)

$$\int_0^\infty \tau_{q,\alpha}^x f(y) d\mu_{q,\alpha}(y) = \int_0^\infty f(y) d\mu_{q,\alpha}(y). \tag{2.9}$$

(3) for $f \in L_\alpha^p(\mathbb{R}_q^+)$ with $p \in [1; +\infty]$ $\quad \tau_{q,\alpha}^x f \in L_\alpha^p(\mathbb{R}_q^+)$ and we have

$$\left\|\tau_{q,\alpha}^x f\right\|_{p,q,\alpha} \leq \|f\|_{p,q,\alpha}, \tag{2.10}$$

(4) For $f \in L_\alpha^1(\mathbb{R}_q^+)$, $\quad \tau_{q,\alpha}^x f \in L_\alpha^1(\mathbb{R}_q^+)$ and we have

$$\mathcal{H}_{q,\alpha}\left(\tau_{q,\alpha}^x f\right)(\lambda) = j_\alpha(\lambda x; q^2) \mathcal{H}_{q,\alpha}(f)(\lambda), \quad \forall \lambda \in \mathbb{R}_q^+. \tag{2.11}$$

The relation (2.11) shows that the translation operator $\tau_{q,\alpha}^x$ is a particular case of the q-Bessel multiplier operator (1.4).

By using the q-Bessel translation operator, we define the generalized convolution product of f, g by

$$(f *_q g)(x) = \int_0^\infty \tau_{q,\alpha}^x(f)(y) g(y) d\mu_{q,\alpha}(y).$$

This convolution is commutative, associative and its satisfies the following properties see [11, 13].

Proposition 2.3.

(1)(q-Young's inequality) for all $p, q, r \in [1; +\infty]$ such that: $\frac{1}{p} + \frac{1}{s} = 1 + \frac{1}{r}$ and for all $f \in L_\alpha^p(\mathbb{R}_q^+)$, $g \in L_\alpha^s(\mathbb{R}_q^+)$ the function $f *_\alpha g$ belongs to the space $L_\alpha^r(\mathbb{R}_q^+)$ and we have

$$\|f *_\alpha g\|_{r,q,\alpha} \leq \|f\|_{p,q,\alpha} \|g\|_{s,q,\alpha} \tag{2.12}$$

(2) For $f, g \in L_\alpha^2(\mathbb{R}_q^+)$ the function $f *_q g$ belongs to $L_\alpha^2(\mathbb{R}_q^+)$ if and only if the function $\mathcal{H}_{q,\alpha}(f)\mathcal{H}_{q,\alpha}(g)$ belongs to $L_\alpha^2(\mathbb{R}_q^+)$ and in this case we have

$$\mathcal{H}_{q,\alpha}(f *_q g) = \mathcal{H}_{q,\alpha}(f)\mathcal{H}_{q,\alpha}(g). \tag{2.13}$$

(3) For all $f, g \in L_\alpha^2(\mathbb{R}_q^+)$ then we have

$$\int_0^\infty |f *_q g(x,t)|^2 d\mu_{q,\alpha}(x) = \int_0^\infty |\mathcal{H}_{q,\alpha}(f)(\lambda)|^2 |\mathcal{H}_{q,\alpha}(g)(\lambda)|^2 d\mu_{q,\alpha}(\lambda), \tag{2.14}$$

where both integrals are simultaneously finite or infinite.

3. The q-Bessel L^2_α-Multiplier Operators

The main purpose of this section is to introduce the q-Bessel L^2_α-multiplier operators on $\mathbb{R}_q^+$ and to establish for them some uncertainty principles and Calderon's reproducing formulas.

3.1. Calderon's Reproducing Formulas for the q-Bessel L^2_α-multiplier operators.

Definition 3.1. *Let $\sigma \in L^2_\alpha(\mathbb{R}_q^+)$ and $\beta \in \mathbb{R}_q^+$, the q-Bessel L^2_α-multiplier operators are defined for smooth function f on $\mathbb{R}_q^+$ as*

$$\mathcal{M}_{q,\sigma,\beta}(f)(x) := \mathcal{H}^{-1}_{q,\alpha}\left(\sigma_\beta \mathcal{H}_{q,\alpha}(f)\right)(x), \tag{3.1}$$

where the function σ_β is given by the relation (1.5) and by a simple change of variable we find that for all $\beta \in \mathbb{R}_q^+, \sigma_\beta \in L^2_\alpha(\mathbb{R}_q^+)$ and

$$\left\|\sigma_\beta\right\|_{2,q,\alpha} = \frac{1}{\beta^{\alpha+1}}\|\sigma\|_{2,q,\alpha}. \tag{3.2}$$

Remark 3.1. *According to the relation (2.13) we find that*

$$\mathcal{M}_{q,\sigma,\beta}(f)(x) = \left(\mathcal{H}^{-1}_{q,\alpha}\left(\sigma_\beta\right) *_\alpha f\right)(x), \tag{3.3}$$

where

$$\mathcal{H}^{-1}_{q,\alpha}\left(\sigma_\beta\right)(x) = \frac{1}{\beta^{2\alpha+2}}\mathcal{H}^{-1}_{q,\alpha}(\sigma)\left(\frac{x}{\beta}\right). \tag{3.4}$$

We give some properties of the q-Bessel L^2_α-multiplier operators.

Proposition 3.1. *(i) For every $\sigma \in L^2_\alpha(\mathbb{R}_q^+)$, and $f \in L^1_\alpha(\mathbb{R}_q^+)$, the function $\mathcal{M}_{q,\sigma,\beta}(f)$ belongs to $L^2_\alpha(\mathbb{R}_q^+)$, and we have*

$$\left\|\mathcal{M}_{q,\sigma,\beta}(f)\right\|_{2,q,\alpha} \leq \frac{1}{\beta^{\alpha+1}}\|\sigma\|_{2,q,\alpha}\|f\|_{1,q,\alpha}.$$

(ii) For every $\sigma \in L^\infty_\alpha(\mathbb{R}_q^+)$, and for every $f \in L^2_\alpha(\mathbb{R}_q^+)$, the function $\mathcal{M}_{q,\sigma,\beta}(f)$ belongs to $L^2_\alpha(\mathbb{R}_q^+)$, and we have

$$\left\|\mathcal{M}_{q,\sigma,\beta}(f)\right\|_{2,q,\alpha} \leq \|\sigma\|_{\infty,q,\alpha}\|f\|_{2,q,\alpha} \tag{3.5}$$

(iii) For every $\sigma \in L^2_\alpha(\mathbb{R}_q^+)$, and for every $f \in L^2_\alpha(\mathbb{R}_q^+)$, $\mathcal{M}_{q,\sigma,\beta}(f) \in L^\infty_\alpha(\mathbb{R}_q^+)$, and we have

$$\mathcal{M}_{q,\sigma,\beta}(f)(x) = \int_0^\infty \sigma(\beta\lambda) j_\alpha(\lambda x; q^2)\mathcal{H}_{q,\alpha}(f)(\lambda)d\mu_{q,\alpha}(\lambda), \qquad a.e \quad x \in \mathbb{R}_q^+ \tag{3.6}$$

and

$$\left\|\mathcal{M}_{q,\sigma,\beta}(f)\right\|_{\infty,q,\alpha} \leq \frac{1}{\beta^{\alpha+1}}\|\sigma\|_{2,q,\alpha}\|f\|_{2,q,\alpha}.$$

Proof. (i) By using the relations (2.12),(3.3) we find that

$$\left\|\mathcal{M}_{q,\sigma,\beta}(f)\right\|^2_{2,q,\alpha} = \left\|\mathcal{H}^{-1}_{q,\alpha}\left(\sigma_\beta\right) *_q f\right\|^2_{2,q,\alpha} \leq \|f\|^2_{1,q,\alpha}\left\|\mathcal{H}^{-1}_{q,\alpha}\left(\sigma_\beta\right)\right\|^2_{2,q,\alpha}$$

Plancherel's formula (2.7) and the relation (3.2) gives the desired result.

(ii) Is a consequence of Plancherel's formula (2.7).

(iii) Is a consequence of the relations (2.7),(2.12),(3.2) and (3.3), on the other hand the relation (3.6) follows from inversion formula (2.5). □

In the following result, we give Plancherel's and pointwise reproducing inversion formula for the q-Bessel L^2_α-multiplier operators.

Theorem 3.1. *Let $\sigma \in L^2_\alpha(\mathbb{R}^+_q)$ satisfying the admissibility condition:*

$$\int_0^\infty \left|\sigma_\beta(\lambda)\right|^2 \frac{d_q(\beta)}{\beta} = 1, \quad \lambda \in \mathbb{R}. \tag{3.7}$$

(i) (Plancherel formula) For all f in $L^2_\alpha(\mathbb{R}^+_q)$, we have

$$\int_0^\infty |f(x)|^2 d\mu_{q,\alpha}(x) = \int_0^\infty \left\|\mathcal{M}_{q,\sigma,\beta}(f)\right\|^2_{2,q,\alpha} \frac{d_q(\beta)}{\beta}. \tag{3.8}$$

(ii) (First calderón's formula) Let $f \in L^1_\alpha(\mathbb{R}^+_q)$ such that $\mathcal{H}_{q,\alpha}(f) \in L^1_\alpha(\mathbb{R}^+_q)$ then we have

$$f(x) = \int_0^\infty \left(\mathcal{M}_{q,\sigma,\beta}(f) *_\alpha \mathcal{H}^{-1}_{q,\alpha}\left(\overline{\sigma_\beta}\right)\right)(x)\frac{d\beta}{\beta}, \qquad a.e.\ x \in \mathbb{R}.$$

Proof. (i) By using the relations (2.14) and (3.3) we get

$$\begin{aligned}\int_0^\infty \left\|\mathcal{M}_{q,\sigma,\beta}(f)\right\|^2_{2,q,\alpha} \frac{d_q(\beta)}{\beta} &= \int_0^\infty \left[\int_0^\infty \left|\mathcal{M}_{q,\sigma,\beta}(f)(x)\right|^2 d\mu_{q,\alpha}(x)\right] \frac{d_q(\beta)}{\beta} \\ &= \int_0^\infty \left[\int_0^\infty \left|\mathcal{H}_{q,\alpha}(f)(\lambda)\right|^2 d\mu_{q,\alpha}(\lambda)\right] \left|\sigma_\beta(\lambda)\right|^2 \frac{d_q(\beta)}{\beta}\end{aligned}$$

the admissibility condition (3.7) and Plancherel's formula (2.7) gives the desired result.
(ii) Let $f \in L^1_\alpha(\mathbb{R}^+_q)$ such that $\mathcal{H}_{q,\alpha}(f) \in L^1_\alpha(\mathbb{R}^+_q)$, by using the relations (2.6),(2.11) we find that

$$\begin{aligned}&\int_0^\infty \left(\mathcal{M}_{q,\sigma,\beta}(f) *_\alpha \mathcal{H}^{-1}_{q,\alpha}\left(\overline{\sigma_\beta}\right)\right)(x)\frac{d\beta}{\beta} \\ &= \int_0^\infty \left[\int_0^\infty \left|\sigma_\beta(\lambda)\right|^2 \mathcal{H}_{q,\alpha}(f)(\lambda) j_\alpha(\lambda x; q^2) d\mu_{q,\alpha}(\lambda)\right] \frac{d_q(\beta)}{\beta} \\ &= \int_0^\infty \left[\int_0^\infty \mathcal{H}_{q,\alpha}(f)(\lambda) j_\alpha(\lambda x; q^2) d\mu_{q,\alpha}(\lambda)\right] \left|\sigma_\beta(\lambda)\right|^2 \frac{d_q(\beta)}{\beta}\end{aligned}$$

the admissibility condition (3.7),inversion formula (2.5) gives the desired result. □

To establish the second Calderon's reproducing formula for the q-Bessel L^2_α-multiplier operators, we need the following technical result.

Proposition 3.2. *Let $\sigma \in L^2_\alpha(\mathbb{R}^+_q) \cap L^\infty_\alpha(\mathbb{R}^+_q)$ satisfy the admissibility condition (3.7) then the function defined by*

$$\Phi_{\gamma,\delta}(\lambda) = \int_\gamma^\delta \left|\sigma_\beta(\lambda)\right|^2 \frac{d_q(\beta)}{\beta}$$

belongs to $L^2_\alpha(\mathbb{R}^+_q) \cap L^\infty_\alpha(\mathbb{R}^+_q)$ for all $0 < \gamma < \delta < \infty$.

Proof. Using Hölder's inequality for the measure $\frac{d_q(\beta)}{\beta}$ and the relation (3.2) we find that

$$\left\|\Phi_{\gamma,\delta}\right\|_{2,q,\alpha}^2 \leq \log(\delta/\gamma)\|\sigma\|_{2,q,\alpha}^2\|\sigma\|_{\infty,q,\alpha}^2 \int_\gamma^\delta \frac{d_q(\beta)}{\beta^{\alpha+2}} < \infty$$

So $\Phi_{\gamma,\delta} \in L^2_\alpha(\mathbb{R}_q^+)$, furthermore by using the relation (3.7) we get $\left\|\Phi_{\gamma,\delta}\right\|_{\infty,q,\alpha} < \infty$ therefore $\Phi_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R}_q^+) \cap L^\infty_\alpha(\mathbb{R}_q^+)$. □

Theorem 3.2. *(Second Calderón's formula). Let $f \in L^2_\alpha(\mathbb{R}_q^+)$ and $\sigma \in L^2_\alpha(\mathbb{R}_q^+) \cap L^\infty_\alpha(\mathbb{R}_q^+)$ satisfy the admissibility condition (3.7) and $0 < \gamma < \delta < \infty$. Then the function*

$$f_{\gamma,\delta}(x) = \int_\gamma^\delta \left(\mathcal{M}_{q,\sigma,\beta}(f) *_\alpha \mathcal{H}^{-1}_{q,\alpha}\left(\overline{\sigma_\beta}\right)\right)(x)\frac{d_q(\beta)}{\beta}, \quad x \in \mathbb{R}_q^+$$

belongs to $L^2_\alpha(\mathbb{R}_q^+)$ and satisfies

$$\lim_{(\gamma,\delta)\to(0,\infty)} \left\|f_{\gamma,\delta} - f\right\|_{2,q,\alpha} = 0 \tag{3.9}$$

Proof. By a simple computation we find that

$$f_{\gamma,\delta}(x) = \int_0^\infty \Phi_{\gamma,\delta}(\lambda) j_\alpha(\lambda x; q^2)\mathcal{H}_{q,\alpha}(f)(\lambda)d\mu_{q,\alpha}(\lambda) = \mathcal{H}^{-1}_{q,\alpha}\left(\Phi_{\gamma,\delta}\mathcal{H}_{q,\alpha}(f)\right)(x),$$

by using proposition 3.2 we find that $\Phi_{\gamma,\delta} \in L^\infty_\alpha(\mathbb{R}_q^+)$ then we have $f_{\gamma,\delta} \in L^2_\alpha(\mathbb{R}_q^+)$ and

$$\mathcal{H}_{q,\alpha}\left(f_{\gamma,\delta}\right)(\lambda) = \Phi_{\gamma,\delta}(\lambda, m)\mathcal{H}_{q,\alpha}(f)(\lambda)$$

on the other hand by using Plancherel's formula (2.7) we find that

$$\lim_{(\gamma,\delta)\to(0,\infty)} \left\|f_{\gamma,\delta} - f\right\|_{2,q,\alpha}^2 = \lim_{(\gamma,\delta)\to(0,\infty)} \int_0^\infty |\mathcal{H}_{q,\alpha}(f)(\lambda)|^2 \left(1 - \Phi_{\gamma,\delta}(\lambda)\right)^2 d\mu_{q,\alpha}(\lambda)$$

by using the admissibility condition (3.7), the relation (3.9) follows from the dominated convergence theorem. □

3.2. **Uncerainty principles for the q-Bessel L^2_α-multiplier operators.** *The main purpose of this subsection is to establish Heisenberg's and Donoho-Stark's uncertainty principles for the q-Bessel L^2_α-multiplier operators $\mathcal{M}_{q,\sigma,\beta}$.*

3.2.1. *Heisenberg's uncertainty principle for $\mathcal{M}_{q,\sigma,\beta}$. Heisenberg's uncertainty principle for the q-Bessel Fourier transform $\mathcal{H}_{q,\alpha}$ has been established in [9, 11] as follows, for all $f \in L^2_\alpha(\mathbb{R}_q^+)$ we have*

$$\||x|f\|_{2,q,\alpha}\||\lambda|\mathcal{H}_{q,\alpha}(f)\|_{2,q,\alpha} \geq \mathbf{k}_{q,v}\|f\|_{2,q,\alpha}^2, \tag{3.10}$$

where $\mathbf{k}_{q,\alpha} = \frac{\left[1+\sqrt{q}\times q^{\alpha+1}\right]}{1-q^{2(\alpha+1)}}$.

The inequality (3.10) says that if f is highly localized, then $\mathcal{H}_{q,\alpha}(f)$ cannot be concentrated near a single point. We will generalize this inequality for $\mathcal{M}_{q,\sigma,\beta}$, we have the following result

Theorem 3.3. *For all $f \in L^2_\alpha(\mathbb{R}_q^+)$ we have*

$$\|f\|^2_{2,q,\alpha} \leq \frac{\left\||\lambda|^2\mathcal{H}_{q,\alpha}(f)\right\|_{2,q,\alpha}}{\mathbf{k}_{q,\alpha}} \left[\int_0^\infty \left\||x|\mathcal{M}_{q,\sigma,\beta}(f)\right\|^2_{2,q,\alpha} \frac{d_q(\beta)}{\beta}\right]^{\frac{1}{2}}$$

Proof. Let us suppose that $\left\||\lambda|^2\mathcal{H}_{q,\alpha}(f)\right\|_{2,q,\alpha} + \left[\int_0^\infty \left\||x|^2\mathcal{M}_{q,\sigma,\beta}(f)\right\|^2_{2,q,\alpha} \frac{d_q(\beta)}{\beta}\right] < \infty$, by using the relation (3.10) we find that

$$\mathbf{k}_{q,\alpha}\int_0^\infty |\mathcal{M}_{q,\sigma,\beta}(f)(x)|^2 d\mu_{q,\alpha}(x) \leq \left\||x|\mathcal{M}_{q,\sigma,\beta}(f)\right\|_{2,q,\alpha} \left\||\lambda|\sigma_\beta\mathcal{H}_{q,\alpha}(f)\right\|_{2,q,\alpha},$$

integrating over $]0,+\infty[$ with respect to measure $\frac{d_q(\beta)}{\beta}$ and using Plancherel's formula (3.8) and Schwartz's inequality we get

$$\mathbf{k}_{q,\alpha}\|f\|^2_{2,q,\alpha}$$
$$\leq \left[\int_0^\infty \||x|\mathcal{M}_{q,\sigma,\beta}(f)\|^2_{2,q,\alpha}\frac{d_q(\beta)}{\beta}\right]^{\frac{1}{2}} \left[\int_0^\infty \left[\int_0^\infty ||\lambda\sigma_\beta(\lambda)|^2 \,|\mathcal{H}_{q,\alpha}(f)(\lambda)|^2|(\lambda)|d\mu_{q,\alpha}(\lambda)\right]\frac{d_q(\beta)}{\beta}\right]^{\frac{1}{2}}$$

the admissibility condition (3.7) gives the desired result. □

3.2.2. *Donoho-Stark's uncertainty principle for $\mathcal{M}_{q,\sigma,\beta}$. Building on the ideas of Donoho and Stark In [3], the main purpose of this subsection is to give an uncertainty inequality of concentration type in $L^2_\theta(\mathbb{R}_q^+)$ where $L^2_\theta(\mathbb{R}_q^+)$ the space of measurables functions on $\mathbb{R}_q^+ \times \mathbb{R}_q^+$ such that*

$$\|f\|_{2,\theta_\alpha} = \left[\int_0^\infty \|f(\beta,.)\|^2_{2,q,\alpha}\frac{d_q(\beta)}{\beta}\right]^{\frac{1}{2}}.$$

We denote by θ_α the measure defined on $\mathbb{R}_q^+ \times \mathbb{R}_q^+$ by

$$d\theta_\alpha(\beta,x) = d\mu_{q,\alpha}(x) \otimes \frac{d_q(\beta)}{\beta},$$

Definition 3.2. *[12]*

(i) Let E be a measurable subset of $\mathbb{R}_q^+$, we say that the function $f \in L^2_\alpha(\mathbb{R}_q^+)$ is ϵ-concentrated on E if

$$\|f - \mathbb{1}_E f\|_{2,q,\alpha} \leq \epsilon\|f\|_{2,q,\alpha}, \tag{3.11}$$

where $\mathbb{1}_E$ is the indicator function of the set E.
(ii) Let F be a measurable subset of $\mathbb{R}_q^+ \times \mathbb{R}_q^+$, we say that the function $\mathcal{T}_{\sigma,\beta}(f)$ is ρ-concentrated on F if

$$\|\mathcal{M}_{q,\sigma,\beta}(f) - \mathbb{1}_F\mathcal{M}_{q,\sigma,\beta}(f)\|_{2,\theta_\alpha} \leq \rho\|\mathcal{M}_{q,\sigma,\beta}(f)\|_{2,\theta_\alpha}. \tag{3.12}$$

We have the following result

Theorem 3.4. *Let $f \in L^2_\alpha(\mathbb{R}_q^+)$ and $\sigma \in \sigma \in L^2_\alpha(\mathbb{R}_q^+)) \cap L^\infty_\alpha(\mathbb{R}_q^+)$ satisfying the admissibility condition (3.7), if f is ϵ-concentrated on E and $\mathcal{M}_{q,\sigma,\beta}(f)$ is ρ-concentrated on F then we have*

$$\|\sigma\|_{2,q,\alpha}(\mu_\alpha(E))^{\frac{1}{2}}\left[\int_F \frac{d\theta_\alpha(\beta,x)}{\beta^{4\alpha+2}}\right]^{\frac{1}{2}} \geq 1-(\epsilon+\rho).$$

Proof. Let $f \in L^2_\alpha(\mathbb{R}^+_q)$ and $\sigma \in L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R}^+_q)$ satisfying (3.7) and assume that $\mu_\alpha(E) < \infty$ and $\left[\int_F \frac{d\theta_\alpha(\beta,x)}{\beta^{4\alpha+2}}\right]^{\frac{1}{2}} < \infty$. According to the relations (3.11),(3.12) we have

$$\|\mathcal{M}_{q,\sigma,\beta}(f) - \mathbb{1}_F\mathcal{M}_{q,\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha} \leq \|\mathcal{M}_{q,\sigma,\beta}(f) - \mathbb{1}_F\mathcal{M}_{q,\sigma,\beta}(f)\|_{2,\theta_\alpha} + \|\mathbb{1}_F\mathcal{M}_{q,\sigma,\beta}(f - \mathbb{1}_E f)\|_{2,\theta_\alpha}$$
$$\leq \rho\|\mathcal{M}_{q,\sigma,\beta}(f)\|_{2,\theta_\alpha} + \|\mathcal{M}_{q,\sigma,\beta}(f - \mathbb{1}_E f)\|_{2,\theta_\alpha},$$

by using Plancherel's relation (3.8) we get

$$\|\mathcal{M}_{q,\sigma,\beta}(f)\|_{2,\theta_\alpha} \leq \|\mathcal{M}_{q,\sigma,\beta}(f) - \mathbb{1}_F\mathcal{M}_{q,\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha} + \|\mathbb{1}_F\mathcal{M}_{q,\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha}$$
$$\leq (\epsilon + \rho)\|f\|_{2,q,\alpha} + \|\mathbb{1}_F\mathcal{M}_{q,\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha}, \tag{3.13}$$

on the other hand by using the relation (3.6) and Hölder's inequality we find that

$$\|\mathbb{1}_F\mathcal{M}_{q,\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha} \leq \|f\|_{2,q,\alpha}\|\sigma\|_{1,q,\alpha}(\mu(E))^{\frac{1}{2}}\left[\int_F \frac{d\theta_\alpha(\beta,x)}{\beta^{4\alpha+2}}\right]^{\frac{1}{2}}, \tag{3.14}$$

by the relations (3.13),(3.14) we deduce that

$$\|\mathcal{M}_{q,\sigma,\beta}(f)\|_{2,\theta_\alpha} \leq \|f\|_{2,q,\alpha}\left[(\epsilon + \rho) + \|\sigma\|_{1,q,\alpha}(\mu_\alpha(E))^{\frac{1}{2}}\left[\int_F \frac{d\theta_\alpha(\beta,x}{\beta^{4\alpha+2}}\right]^{\frac{1}{2}}\right]$$

Plancherel's formula (3.8) for $\mathcal{M}_{\sigma,\beta}$ gives the desired result. □

4. Extremal Functions Associated with the q-Bessel L^2_α-Multiplier Operators

In the following, we study the extremal functions associated with the the q-Bessel L^2_α-multiplier operators.

Definition 4.1. *Let ψ be a positive function on $\mathbb{R}^+_q$ satisfying the following conditions*

$$\frac{1}{\psi} \in L^1_\alpha(\mathbb{R}^+_q) \tag{4.1}$$

and

$$\psi(\lambda) \geq 1, \quad \lambda \in \mathbb{R}^+_q. \tag{4.2}$$

We define the Sobolev-type space $\mathcal{S}_\psi(\mathbb{R}^+_q)$ by

$$\mathcal{S}_\psi(\mathbb{R}^+_q) = \left\{f \in L^2_\alpha(\mathbb{R}^+_q) : \sqrt{\psi}\mathcal{H}_{q,\alpha}(f) \in L^2_\alpha(\mathbb{R}^+_q)\right\}$$

provided with inner product

$$\langle f, g\rangle_\psi = \int_0^\infty \psi(\lambda, m)\mathcal{H}_{q,\alpha}(f)(\lambda)\overline{\mathcal{H}_{q,\alpha}(g)(\lambda)}d\mu_{q,\alpha}(\lambda),$$

and the norm

$$\|f\|_\psi = \sqrt{\langle f, f\rangle_\psi}.$$

Proposition 4.1. *Let σ be a function in $L^\infty_\alpha(\mathbb{R}^+_q)$. Then the q-Bessel L^2_α-multiplier operators $\mathcal{M}_{q,\sigma,\beta}$ are bounded and linear from $\mathcal{S}_\psi(\mathbb{R}^+_q)$ into $L^2_\alpha(\mathbb{R}^+_q)$ and we have for all $f \in \mathcal{S}_\psi(\mathbb{R}^+_q)$*

$$\left\|\mathcal{M}_{q,\sigma,\beta}(f)\right\|_{2,q,\alpha} \leq \|\sigma\|_{\infty,q,\alpha}\|f\|_\psi. \tag{4.3}$$

Proof. By using the relations (2.8),(3.5),(4.2) we get the result □

Definition 4.2. *Let $\eta > 0$ and let σ be a function in $L^\infty_\alpha(\mathbb{R}^+_q)$. We denote by $\langle f, g\rangle_{\psi,\eta}$ the inner product defined on the space $\mathcal{S}_\psi(\mathbb{R}^+_q)$ by*

$$\langle f, g\rangle_{\psi,\eta} = \int_0^\infty \left(\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2\right) \mathcal{H}_{q,\alpha}(f)(\lambda)\overline{\mathcal{H}_{q,\alpha}(g)(\lambda)}d\mu_{q,\alpha}(\lambda),$$

and the norm

$$\|f\|_{\psi,\eta} = \sqrt{\langle f, f\rangle_{\psi,\eta}}$$

Theorem 4.1. *Let $\sigma \in L^\infty_\alpha(\mathbb{R}^+_q)$ the Sobolev-type space $\left(\mathcal{S}_\psi(\mathbb{R}^+_q), \langle\cdot,\cdot\rangle_{\psi,\eta}\right)$ is a reproducing kernel Hilbert space with kernel*

$$\mathcal{K}_{q,\psi,\eta}(x, y) = \int_0^\infty \frac{j_\alpha(\lambda x; q^2)j_\alpha(\lambda y; q^2)}{\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2} d\mu_{q,\alpha}(\lambda),$$

that is

(i) For all $y \in \mathbb{R}^+_q$, the function $x \mapsto \mathcal{K}_{q,\psi,\eta}(x, y)$ belongs to $\mathcal{S}_\psi(\mathbb{R}^+_q)$.

(ii) For all $f \in \mathcal{S}_\psi(\mathbb{R}^+_q)$ and $y \in \mathbb{R}^+_q$, we have the reproducing property

$$f(y) = \left\langle f, \mathcal{K}_{q,\psi,\eta}(\cdot,(y))\right\rangle_{\psi,\eta}.$$

Furthermore the kernel $\mathcal{K}_{q,\psi,\eta}$ is a positive definite function.

Proof. (i) Let $y \in \mathbb{R}^+_q$, from the relations (2.2),(4.1) we have the function

$$g_y : \lambda \longrightarrow \frac{j_\alpha(\lambda y; q^2)}{\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2}$$

belongs to $L^1_\alpha(\mathbb{R}^+_q) \cap L^2_\alpha(\mathbb{R}^+_q)$. Hence the function $\mathcal{K}_{q,\psi,\eta}$ is well defined and by the inversion formula (2.5), we get

$$\mathcal{K}_{q,\psi,\eta}(x, y) = \mathcal{H}^{-1}_{q,\alpha}(g_y)(x)$$

by using Plancherel's theorem for $\mathcal{H}_{q,\alpha}$ we find that $\mathcal{K}_{q,\psi,\eta}(\cdot, y)$ belongs to $L^2_\alpha(\mathbb{R}^+_q)$ and we have

$$\mathcal{H}_{q,\alpha}(\mathcal{K}_{q,\psi,\eta}(\cdot, y))(\lambda) = \frac{j_\alpha(\lambda y; q^2)}{\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2} \tag{4.4}$$

by using the relations (2.2),(4.1) and (4.4) we find that

$$\|\sqrt{\psi}\mathcal{H}_{q,\alpha}(\mathcal{K}_{q,\psi,\eta}(\cdot, y))\|_{2,q,\alpha} \leq \frac{1}{\eta^2}\left\|\frac{1}{\psi}\right\|_{1,q,\alpha} < \infty,$$

this prove that for every $y \in \mathbb{R}_q^+$ the function $x \mapsto \mathcal{K}_{q,\psi,\eta}(x,y)$ belongs to $\mathcal{S}_\psi(\mathbb{R}_q^+)$.

(ii) By using the relation (4.4) we find that for all $f \in \mathcal{H}_\psi(\mathbb{R})$,

$$\langle f, \mathcal{K}_{q,\psi,\eta}(\cdot,y)\rangle_{\psi,\eta} = \int_0^\infty \left(\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2\right) \mathcal{H}_{q,\alpha}(f)(\lambda)\overline{\mathcal{H}_{q,\alpha}(\mathcal{K}_{q,\psi,\eta)}(\cdot,y))(\lambda)}d\mu_{q,\alpha}(\lambda)$$
$$= \int_0^\infty j_\alpha(\lambda y; q^2)\mathcal{H}_{q,\alpha}(f)(\lambda)d\mu_{q,\alpha}(\lambda),$$

inversion formula (2.5) gives the desired result. On the other hand since $\frac{1}{\psi}$ is positive function then for all $z_1,\ldots,z_n$ complex numbers and $x_1,\ldots\ldots,x_n$ in $\mathbb{R}_q^+$, we obtain

$$\sum_{r=1}^n\sum_{l=1}^n z_r\overline{z_l}\mathcal{K}_{q,\psi,\eta}(x_r,x_l) = \int_0^{+\infty}\left[\sum_{r=1}^n\sum_{l=1}^n z_r\overline{z_l}j_\alpha\left(x_r\lambda;q^2\right)j_\alpha\left(x_l\lambda;q^2\right)\right]\frac{1}{\psi}(\lambda)d\mu_{q,\alpha}(\lambda)$$
$$= \int_0^{+\infty}\left|\sum_{r=1}^n z_r j\left(x_r\lambda;q^2\right)\right|^2\frac{1}{\psi}(\lambda)d\mu_{q,\alpha}(\lambda) \geq 0$$

Which proves that the kernel $\mathcal{K}_{q,\psi,\eta}$ is positive definite. □

The main result of this section can be stated as follows

Theorem 4.2. *Let $\sigma \in L_\alpha^\infty(\mathbb{R}_q^+)$ and $\beta \in \mathbb{R}_q^+$, for any $h \in L_\alpha^2\left(\mathbb{R}_q^+\right)$ and for any $\eta > 0$, there exist a unique function $f^*_{q,\eta,\beta,h}$ where the infimum*

$$\inf_{f\in\mathcal{S}_\psi(\mathbb{R}_q^+)}\left\{\eta\|f\|_\psi^2 + \left\|h - \mathcal{M}_{q,\sigma,\beta}(f)\right\|_{2,q,\alpha}^2\right\} \tag{4.5}$$

is attained. Moreover the extremal function $f^*_{q,\eta,\beta,h}$ is given by

$$f^*_{q,\eta,\beta,h}(y) = \int_0^\infty h(x)\overline{\Theta_{q,\eta,\beta}(x,y)}d\mu_{q,\alpha}(x),$$

where $\Theta_{q,\eta,\beta}$ is given by

$$\Theta_{q,\eta,\beta}(x,y) = \int_0^\infty \frac{\sigma_\beta(\lambda)j_\alpha(\lambda x;q^2)j_\alpha(\lambda y;q^2)}{\eta\psi(\lambda)+|\sigma_\beta(\lambda)|^2}d\mu_{q,\alpha}(\lambda)$$

Proof. The existence and the unicity of the extremal function $f^*_{q,\eta,\beta,h}$ satisfying (4.5) is given in [23,30], furthermore $f^*_{q,\eta,\beta,h}$ is given by

$$f^*_{q,\eta,\beta,h}(y) = \langle h, \mathcal{M}_{q,\sigma,\beta}(\mathcal{K}_{q,\psi,\eta}(\cdot,y))\rangle_q$$

, by using inversion formula (2.5) and the relation (4.4) we get

$$\mathcal{M}_{q,\sigma,\beta}(\mathcal{K}_{q,\psi,\eta}(\cdot,y)(x) = \int_0^\infty \frac{\sigma_\beta(\lambda)j_\alpha(\lambda x;q^2)j_\alpha(\lambda y;q^2)}{\eta\psi(\lambda)+|\sigma_\beta(\lambda)|^2}d\mu_{q,\alpha}(\lambda)$$
$$= \Theta_{q,\eta,\beta}(x,y)$$

and the proof is complete. □

Theorem 4.3. *$\sigma \in L^\infty_\alpha(\mathbb{R}^+_q)$ and $h \in L^2_\alpha\left(\mathbb{R}^+_q\right)$ then the function $f^*_{q,\eta,\beta,h}$ satisfies the following properties*

$$\mathcal{H}_{q,\alpha}(f^*_{q,\eta,\beta,h})(\lambda) = \frac{\overline{\sigma_\beta(\lambda)}}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}\mathcal{H}_{q,\alpha}(h)(\lambda) \tag{4.6}$$

and

$$\|f^*_{q,\eta,\beta,h}\|_\psi \leq \frac{1}{\sqrt{2\eta}}\|h\|_{2,q,\alpha}.$$

Proof. Let $y \in \mathbb{R}^+_q$ then the function

$$k_y : \lambda \longrightarrow \frac{\sigma_\beta(\lambda) j_\alpha(\lambda y; q^2)}{\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2}$$

belongs to $L^2_\alpha(\mathbb{R}^+_q) \cap L^1_\alpha(\mathbb{R}^+_q)$ and by using inversion formula (2.5) we get

$$\Theta_{q,\eta,\beta}(x,y) = \mathcal{H}^{-1}_{q,\alpha}(k_y)(x)$$

using Plancherel's theorem and Parseval's relation (2.6) we find that $\Theta_{q,\eta,\beta}(\cdot,y) \in L^2_\alpha(\mathbb{R}^+_q)$ and

$$f^*_{q,\eta,\beta,h}(y) = \int_0^\infty \mathcal{H}_{q,\alpha}(\lambda)\overline{k_y(\lambda)}d\mu_{q,\alpha}(\lambda) = \int_0^\infty \frac{\overline{\sigma_\beta(\lambda)}}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}\mathcal{H}_{q,\alpha}(h)(\lambda)d\mu_{q,\alpha}(\lambda)$$

on the other hand the function

$$F : \lambda \longrightarrow \frac{\overline{\sigma_\beta(\lambda)}\mathcal{H}_{q,\alpha}(h)(\lambda)}{\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2}$$

belongs to $L^1_\alpha(\mathbb{R}^+_q) \cap L^2_\alpha(\mathbb{R}^+_q)$, by using inversion formula (2.5), Plancherel's theorem we find that $f^*_{q,\eta,\beta,h}$ belongs to $L^2_\alpha(\mathbb{R}^+_q)$ and

$$\mathcal{H}_{q,\alpha}(f^*_{q,\eta,\beta,h})(\lambda) = F(\lambda)$$

on the other hand we have

$$|\mathcal{H}_{q,\alpha}(f^*_{q,\eta,\beta,h})(\lambda)|^2 = \frac{\left|\sigma_\beta(\lambda)\right|^2}{\left(\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2\right)^2}|\mathcal{H}_{q,\alpha}(h)(\lambda)|^2 \leq \frac{1}{2\eta\psi(\lambda)}|\mathcal{H}_{q,\alpha}(h)(\lambda)|^2$$

by Plancherel's formula (2.7) we find that

$$\|f^*_{q,\eta,\beta,h}\|_\psi \leq \frac{1}{\sqrt{2\eta}}\|h\|_{2,q,\alpha}.$$

□

Theorem 4.4. *(Third Calderón's formula) Let $\sigma \in L^\infty_\alpha(\mathbb{R}^+_q)$ and $f \in \mathcal{S}_\psi(\mathbb{R}^+_q)$ then the extremal function given by*

$$f^*_{q,\eta,\beta}(y) = \int_0^\infty \mathcal{M}_{q,\sigma,\beta}(f)(x)\overline{\Theta_{q,\eta,\beta}(x,y)}d\mu_{q,\alpha}(x),$$

satisfies

$$\lim_{\eta\to 0^+} \left\|f^*_{q,\eta,\beta} - f\right\|_{2,q,\alpha} = 0 \tag{4.7}$$

moreover we have $f^*_{q,\eta,\beta} \longrightarrow f$ uniformly when $\eta \longrightarrow 0^+$.

Proof. $f \in \mathcal{S}_\psi(\mathbb{R}_q^+)$, we put $h = \mathcal{M}_{q,\sigma,\beta}(f)$ and $f^*_{q,\eta,\beta,h} = f^*_{q,\eta,\beta}$ in the relation (4.6) we find that

$$\mathcal{H}_{q,\alpha}(f^*_{q,\eta,\beta} - f)(\lambda) = \frac{-\eta\psi(\lambda)\mathcal{H}_{q,\alpha}(f)(\lambda)}{\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2} \tag{4.8}$$

therefore

$$\left\|f^*_{q,\eta,\beta} - f\right\|_\psi^2 = \int_0^\infty \frac{\eta^2 (\psi(\lambda))^3}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} \left|\mathcal{H}_{q,\alpha}(f)(\lambda)\right|^2 d\mu_{q,\alpha}(\lambda)$$

On the other hand we have

$$\frac{\eta^2 (\psi(\lambda))^3}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} \left|\mathcal{H}_{q,\alpha}(f)(\lambda)\right|^2 \leq \psi(\lambda) \left|\mathcal{H}_{q,\alpha}(f)(\lambda)\right|^2 \tag{4.9}$$

the result (4.7) follows from (4.9) and the dominated convergence theorem. Now, for all $f \in \mathcal{S}_\psi(\mathbb{R}_q^+)$ we have $\mathcal{H}_{q,\alpha}(f) \in L^2_\alpha(\mathbb{R}_q^+) \cap L^1_\alpha(\mathbb{R}_q^+)$ and by using the relations (2.5), (4.8) we find that

$$f^*_{q,\eta,\beta}(y) - f(y) = \int_0^\infty \frac{-\eta\psi(\lambda)\mathcal{H}_{q,\alpha}(f)(\lambda)}{\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2} j_\alpha(\lambda y; q^2) d\mu_{q,\alpha}(\lambda)$$

and

$$\left|\frac{-\eta\psi(\lambda)\mathcal{H}_{q,\alpha}(f)(\lambda)}{\eta\psi(\lambda) + \left|\sigma_\beta(\lambda)\right|^2} j_\alpha(\lambda y; q^2)\right| \leq \left|\mathcal{H}_{q,\alpha}(f)(\lambda)\right| \tag{4.10}$$

By using the relation (4.10) and the dominated convergence theorem we deduce that

$$\lim_{\eta \to 0^+} \left|f^*_{q,\eta,\beta}(y) - f(y)\right| = 0$$

which complete the proof of the theorem. □

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