Eur. J. Math. Anal. 5 (2025) 10
doi: 10.28924/ada/ma.5.10

Neumann and Dirichlet Problems for the Cauchy–Riemann and the Poisson Equations in the Partial Eclipse Domain

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Abstract. In this paper, we consider the Neumann boundary value problem and the Dirichlet boundary value problem for complex partial differential equations in the partial eclipse domain. First, By the parqueting–reflection principle and the Cauchy–Pompeiu formula, a modified integral representation formula in the partial eclipse domain is constructed. Then, we explicitly solve the Neumann problem for the homogeneous equation and discuss the solvability conditions. Moreover, we investigate the Dirichlet problem for the Poisson equation in the partial eclipse domain. In other words, with the help of the Green's function, we provide a unique solution for the Dirichlet boundary value problem for the Poisson equation and consider boundary behavior.

1. Introduction

Mathematical analysis is an active branch in mathematics which has grown significantly. It has indeed flourished, playing a pivotal role in advancement of both pure and applied mathematics. Its growth can be seen in the development of new techniques for solving differential equations, advancements in complex analysis, and profound contributions to functional analysis. This expansion has not only deepened our theoretical understanding but has also paved the way for practical applications in fields like mathematical physics, fluid dynamics, engineering, etc [1, 3, 6, 9].

The theory of boundary value problems for partial differential equations is a key area in mathematical analysis and mathematical physics. The theory often focuses on conditions under which solutions exist and are unique. These conditions can depend on the properties of the differential operator, the domain, and the boundary conditions. Boundary value problems are critical in the study of partial differential equations, as they involve finding a solution to a partial differential equation that satisfies certain conditions at the boundaries of the domain. The most common boundary value problems are the Dirichlet, the Neumann and the Schwarz problems. In particular, the Dirichlet

Received: 2 Dec 2024.
Key words and phrases. boundary value problem; Neumann problem; Dirichlet problem; partial eclipse domain.

problem specifies the value of the function on the boundary and the Neumann problem specifies the values of the derivative (normal to the boundary).

Several analytical and computational methods are used to solve boundary value problems, including integral representation formulas, the Green's functions, etc. Integral representation formulas are crucial tools in solving complex boundary value problems, especially for analytic and harmonic functions. The Green's functions are used to construct solutions for complex partial equation with boundary conditions. For the Dirichlet problem, the Green's function represents the influence of a point source on the boundary and is used to build the solution, see [1–9, 16].

In recent years, many mathematician have studied boundary value problems for complex partial differential equations and numerous results have been obtained, can be referenced in [1–16]. In 2024, we introduced a new domain called partial eclipse and also investigated the Schwarz and the Dirichlet boundary value problem for Cauchy–Riemann equations in the partial eclipse domain, (Ali Darya and Nasir Taghizadeh, "Schwarz and Dirichlet problems for complex partial differential equations in the eclipse domain," J Math Sci) see [1].

In the present paper, we consider the Neumann problem for first-order partial differential equation and the Dirichlet problem for second-order partial differential equation in the partial eclipse domain. In other words, we first solve the Neumann problem the homogeneous Cauchy–Riemann equation and discuss the solvability conditions. In the next step, With the help of the Green's function, we construct a unique solution for the Dirichlet problem for the Poisson equation in the partial eclipse domain and investigate the Dirichlet problem. In particular, we study the boundary behavior.

Let M be the partial eclipse domain in the complex plane $\mathbb{C}$ defined by [1]

$$M = \left\{ z \in \mathbb{C} : |z - ai| < \sqrt{2}a,\ |z + ai| > \sqrt{2}a \right\}$$

where $C_1 = \left\{ z : |z - ai| = \sqrt{2}a \right\}$ and $C_2 = \left\{ z : |z + ai| = \sqrt{2}a \right\}$ are two circles with the same radius and the boundary of M is denoted by ∂M. Due to the fact that the real number a has a positive arbitrary amount, the desired partial eclipse can be created by choosing a. So the length of the borders of the partial eclipse changes according to the number.

The parqueting–reflection principle is a technique used to solve boundary value problems for complex partial differential equations, particularly for domains with complex geometries. The method involves reflecting the domain across its boundaries to simplify the problem. By doing so, the boundary conditions can be transformed into simpler forms, making it easier to find solutions. By using the reflections, one can construct solutions for complex boundary value problems such as the Schwarz, the Dirichlet and Neumann problems. This is particularly useful for domains composed of circular arcs and straight lines [1–4, 6, 8–10].

In this section, using the parqueting–reflection method for the introduced domain M, we achieve coverage for the entire complex plane $\mathbb{C}$. This coverage is obtained from reflections with three repetitions.

The reflection of $z \in M$ at C_1, is

$$|z-ai| = \sqrt{2}a \Rightarrow (z-ai)(\bar{z}+ai) = 2a^2 \Rightarrow z_1^* = \frac{ai\bar{z} - a^2 + 2a^2}{\bar{z}+ai} \Rightarrow z_1^* = \frac{ai\bar{z}+a^2}{\bar{z}+ai}.$$

Similarly, the reflections of z_1^* and z at C_2, are the points,

$$z_2^* = \frac{a^2}{z}, \quad z_3^* = \frac{-ai\bar{z}+a^2}{\bar{z}-ai}.$$

Those reflections produce a parqueting of the entire complex plane and those points will also be needed for constructing the integral representation formula for M. To solve boundary value problems for partial differential equations, the integral formula should be appropriately modified according to the type of problem and partial differential equation, [1–4, 7, 8, 10].

The Neumann boundary value problem is a classical in the field of complex analysis and partial differential equations. It involves finding a function that satisfies complex partial differential equations within a given domain and whose normal derivative on the boundary of the domain matches a specified function. The Neumann boundary value problem is significant, because it is a gateway to understanding the deeper intricacies of complex functions and broader field of complex analysis. The insights gained from solving these problems can lead to advancement in both mathematical theory and practical applications, see [2, 6, 7, 9, 11].

In this section, we investigate the Neumann boundary value problem for the homogeneous Cauchy– Riemann equation. The fundamental tool for complex boundary value problems is the integral representation formula which just has to be properly modified.

Theorem 1.1. *Any $\omega \in C^1(M;\mathbb{C}) \bigcap C(\overline{M};\mathbb{C})$ can be represented as*

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial M} \omega(\zeta)\left[\frac{1}{\zeta - z} + \frac{z}{\zeta z - a^2}\right] d\zeta - \frac{1}{\pi}\int_M \omega_{\bar{\zeta}}(\zeta)\left[\frac{1}{\zeta - z} + \frac{z}{\zeta z - a^2}\right] d\xi d\eta, \tag{1.1}$$

where $\zeta = \xi + i\eta$.

Proof. The Cauchy–Pompieu formula

$$\frac{1}{2\pi i}\int_{\partial M} \omega(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_M \omega_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \begin{cases} \omega(z) & z \in M, \\ 0 & z \notin \overline{M}, \end{cases}$$

applied to $z \in M$ and $z_2^* \notin M$, respectively, gives the following equalities:

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial M} \omega(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_M \omega_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \tag{1.2}$$

$$0=\frac{1}{2\pi i}\int_{\partial M}\omega(\zeta)\frac{zd\zeta}{\zeta z-a^2}-\frac{1}{\pi}\int_M\omega_{\bar{\zeta}}(\zeta)\frac{zd\xi d\eta}{\zeta z-a^2}, \tag{1.3}$$

adding the resulting above relations, leads to claimed the integral representation formula. □

Next, we state the Neumann boundary value problem for Cauchy–Riemann equation in the partial eclipse domain as follows.

Neumann boundary value problem: Find an analytic function in the partial eclipse domain, i.e. a solution to Cauchy–Riemann equation, satisfying,

$$\partial_{v_z}\omega=\gamma,\quad on\ \partial M,\quad \gamma\in C(\partial M;\mathbb{C}).$$

The classical Neumann problem involves finding a function that satisfies Cauchy–Riemann equation in a domain, along with prescribed values of its normal derivative on the boundary. When adjusted for analytic functions, the Neumann condition is often reformulated to align with the concept of analyticity, ensuring the function meets the criteria for complex differentiability [2,6,7,9,11,16].

To formulate the Neumann boundary value problem, we need to define the outward normal derivative at the boundary of M. The normal derivative on the boundary of is given by the formulas,

$$\partial_{v_z}\omega=\begin{cases}(\frac{z-ai}{\sqrt{2}a})\omega_z, & on\ C_1,\\ (\frac{z+ai}{\sqrt{2}a})\omega_z, & on\ C_2.\end{cases}$$

Now, according to the above definition, we state and prove the following theorem.

Theorem 1.2. *The Neumann boundary value problem*

$$\omega_{\bar{z}}=0,\quad z\in M,$$

$$\partial_{v_z}\omega=\gamma,\quad z\in\partial M, \tag{1.4}$$

$$\omega(t)=c,$$

where $t\in M$, $c\in\mathbb{C}$ and

$$\partial_{v_z}\omega=\begin{cases}(\frac{z-ai}{\sqrt{2}a})\omega_z, & on\ C_1,\\ (\frac{z+ai}{\sqrt{2}a})\omega_z, & on\ C_2,\end{cases}$$

is solvable, if and only if, for $z\in M$,

$$\frac{1}{2\pi i}\int_{\partial M}\gamma(\zeta)\left[\frac{\bar{z}-ai}{\zeta(\bar{z}-ai)+ai\bar{z}-a^2}+\frac{\bar{z}+ai}{\zeta(\bar{z}+ai)-ai\bar{z}-a^2}\right]d\zeta=0,$$

and its solution is

$$\omega(z)=\frac{1}{2\pi i}\int_{\partial M}\gamma(\zeta)\left[\frac{z-t}{\zeta}-\log(\frac{\zeta-z}{\zeta-t})+\frac{a^2}{\zeta^2}\log(\frac{\zeta z-a^2}{\zeta t-a^2})\right]d\zeta,$$

where $\zeta=\xi+i\eta$.

Proof. Suppose ω is a solution to the Neumann problem. Introducing a new function $\phi = \omega_z$, since ϕ is an analytic function, $\phi = \omega_z$ is a solution to the following problem,

$$\phi_{\bar{z}} = 0, \quad in \quad M,$$

$$\phi = \omega_z, \quad on \;\; \partial M, \tag{1.5}$$

where ω_z on ∂M is represented by,

$$\omega_z(z) = \begin{cases} (\frac{\bar{z}-ai}{\sqrt{2}a})\gamma, & on \quad C_1, \\ (\frac{\bar{z}+ai}{\sqrt{2}a})\gamma, & on \quad C_2. \end{cases}$$

Equation (5) is equivalent to the Dirichlet boundary value problem for the homogeneous Cauchy–Riemann equation. By Theorem 3.2 in [1], the above Dirichlet problem is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial M}\gamma(\zeta)\left[\frac{\bar{z}-ai}{\zeta(\bar{z}-ai)+ai\bar{z}-a^2}+\frac{\bar{z}+ai}{\zeta(\bar{z}+ai)-ai\bar{z}-a^2}\right]d\zeta = 0,$$

then, the unique solution is given by,

$$\omega_z(z) = \frac{1}{2\pi i}\int_{\partial M}\gamma(\zeta)\left[\frac{1}{\zeta - z}+\frac{z}{\zeta z - a^2}\right]d\zeta. \tag{1.6}$$

The primitive of the function in (6) is

$$\omega(z) = \frac{1}{2\pi i}\int_{\partial M}\gamma(\zeta)\left[\frac{z}{\zeta} - \log(\zeta - z) + \frac{a^2}{\zeta^2}\log(\zeta z - a^2)\right]d\zeta + c.$$

Define c as

$$c = -\frac{1}{2\pi i}\int_{\partial M}\gamma(\zeta)\left[\frac{t}{\zeta} - \log(\zeta - t) + \frac{a^2}{\zeta^2}\log(\zeta t - a^2)\right]d\zeta.$$

This completes the proof. □

2. The Dirichlet Problem for M

In this section, we consider the Dirichlet problem for the Poisson equation in the partial eclipse domain. In order to treat the Dirichlet boundary value problem for second order complex partial differential equations some special kernel functions, the Green functions, have to be constructed. It is essential to construct the Green's functions tailored to the specific domain. These Green's functions serve as fundamental tools, transforming the differential equation into an integral form that can be more easily analyzed and solved, see [5–9]. The harmonic Green function for the partial eclipse domain M is

$$G_1(z,\zeta) = \log\left|\frac{\bar{\zeta}(z+ai) - aiz - a^2}{\zeta - z}\frac{\bar{\zeta}(z-aiz) + aiz - a^2}{\zeta z - a^2}\right|^2.$$

The outward normal derivative of the boundary ∂M is given by

For $z \in \partial M \cap C_1$, that is, $|z - ai| = \sqrt{2}a$, we have

$$\partial_{\nu_z} G_1(z,\zeta) = \left((\frac{z - ai}{\sqrt{2}a})\partial_z + (\frac{\bar{z} + ai}{\sqrt{2}a})\partial_{\bar{z}} \right) G_1(z,\zeta),$$

and for $z \in \partial M \cap C_2$, that is, $|z + ai| = \sqrt{2}a$, we have

$$\partial_{\nu_z} G_1(z,\zeta) = \left((\frac{z + ai}{\sqrt{2}a})\partial_z + (\frac{\bar{z} - ai}{\sqrt{2}a})\partial_{\bar{z}} \right) G_1(z,\zeta).$$

The Harmonic Green functions play an essential role in solving the Dirichlet boundary value problem for second order complex partial differential equations. The next theorem contains a representation formula for a class of functions via the Green function, which is used to solve the Dirichlet problem for Poisson equation (see [5,7,9]).

Theorem 2.1. *Let $\Omega \subset \mathbb{C}$ be a regular domain, and let G_1 be the harmonic Green function for Ω Then any $\omega \in C^2(\Omega;\mathbb{C}) \cap C^1(\overline{\Omega};\mathbb{C})$ can be represented as follows:*

$$\omega(z) = -\frac{1}{4\pi}\int_{\partial\Omega} \omega(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)\mathrm{d}t_\zeta - \frac{1}{\pi}\int_{\Omega} \omega_{\zeta\bar{\zeta}}(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

where ν is the outward normal derivative on $\partial\Omega$ and t is the arc length parameter [5,7,9].

Therefore, based on Theorem 3, the explicit form of the Green representation formula for the partial eclipse domain is as following:

$$\begin{aligned}
\omega(z) = &\frac{1}{2\pi i}\int_{\partial M \bigcap C_1} \omega(\zeta)\left(\frac{\zeta + ai}{\zeta - z} + \frac{\bar{\zeta} - ai}{\bar{\zeta} - \bar{z}} - 1\right.\\
&\left. + \frac{z(\zeta - ai)}{\zeta z - a^2} + \frac{\bar{z}(\bar{\zeta} + ai)}{\bar{\zeta}\bar{z} - a^2} - 1\right)\frac{d\zeta}{\zeta - ai}\\
&+\frac{1}{2\pi i}\int_{\partial M \bigcap C_2} \omega(\zeta)\left(\frac{\zeta + ai}{\zeta - z} + \frac{\bar{\zeta} - ai}{\bar{\zeta} - \bar{z}} - 1\right.\\
&\left. + \frac{z(\zeta + ai)}{\zeta z - a^2} + \frac{\bar{z}(\bar{\zeta} - ai)}{\bar{\zeta}\bar{z} - a^2} - 1\right)\frac{d\zeta}{\zeta + ai}\\
&-\frac{1}{\pi}\int_M \omega_{\zeta\bar{\zeta}}(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta. \qquad (2.1)
\end{aligned}$$

In fact formula (7) provides a solution to the Dirichlet problem for the Poisson equation in M.

Theorem 2.2. *The Dirichlet problem for the Poisson equation in M*

$$\omega_{z\bar{z}} = f, \quad z \in M, \quad f \in C(M;\mathbb{C}),$$

$$\omega = \gamma, \ \ on\ \partial M, \ \ \gamma \in C(\partial M;\mathbb{C}), \qquad (2.2)$$

is uniquely solvable and the solution is given by

$$\begin{aligned}\omega(z) = \frac{1}{2\pi i}\int_{\partial M\bigcap C_1}\gamma(\zeta)\left(\frac{\zeta-ai}{\zeta-z}+\frac{\bar\zeta+ai}{\bar\zeta-\bar z}-1\right.\\ \left.+\frac{z(\zeta-ai)}{\zeta z-a^2}+\frac{\bar z(\bar\zeta+ai)}{\bar\zeta\bar z-a^2}-1\right)\frac{d\zeta}{\zeta-ai}\\ +\frac{1}{2\pi i}\int_{\partial M\bigcap C_2}\gamma(\zeta)\left(\frac{\zeta+ai}{\zeta-z}+\frac{\bar\zeta-ai}{\bar\zeta-\bar z}-1\right.\\ \left.+\frac{z(\zeta+ai)}{\zeta z-a^2}+\frac{\bar z(\bar\zeta-ai)}{\bar\zeta\bar z-a^2}-1\right)\frac{d\zeta}{\zeta+ai}\\ -\frac{1}{\pi}\int_M f(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{2.3}$$

where $\zeta=\xi+i\eta$.

Proof. By the properties of the Green function and the harmonicity of the boundary integrals ω is seen to be a solution to the Poisson equation (see [7]). So, it remain remains to check the boundary relation. The study of integral boundary behavior requires calculations in different parts of the boundary.

Since, for $z\in\partial M\bigcap C_1$,

Case1: $\zeta\in C_1$,

$$\frac{\bar z(\bar\zeta+ai)}{\bar\zeta\bar z-a^2}=\frac{aiz-a^2}{\zeta z-a^2}.$$

Case2: $\zeta\in C_2$,

$$\frac{\bar\zeta-ai}{\bar\zeta-\bar z}=\frac{-aiz-a^2}{\zeta z-a^2},$$

$$\frac{\bar z(\bar\zeta-ai)}{\bar\zeta\bar z-a^2}=\frac{-z-ai}{\zeta-z}.$$

Thus, on $\partial M\bigcap C_1$,

$$\begin{aligned}\lim_{z\to\zeta}\omega(z) &= \lim_{z\to\zeta}\frac{1}{2\pi i}\int_{\partial M\bigcap C_1}\gamma(\zeta)\left[\frac{\zeta-ai}{\zeta-z}+\frac{\bar\zeta+ai}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta-ai}\\ &=\lim_{z\to\zeta}\frac{1}{2\pi i}\int_{C_1}\Upsilon(\zeta)\left[\frac{\zeta-ai}{\zeta-z}+\frac{\bar\zeta+ai}{\bar\zeta-\bar z}-1\right]\frac{d\zeta}{\zeta-ai},\end{aligned}$$

where

$$\Upsilon(\zeta)=\begin{cases}\gamma(\zeta) & \zeta\in\partial M\bigcap C_1,\\ 0 & \zeta\in C_1\backslash(\partial M).\end{cases}$$

So based on the properties of the Poisson kernel for C_1, [6]

$$\lim_{z\to\zeta}\omega(z)=\gamma(\zeta),$$

follows for $\zeta \in \partial M\bigcap C_1$ up to the corner points $\pm a$ of the domain M, because Υ fails to be continuous there if γ not accidentally vanishes at these points.

By the same way, for $z \in \partial M\bigcap C_2$,

Case 1: $\zeta \in C_1$,

$$\frac{\bar{\zeta}+ai}{\bar{\zeta}-\bar{z}} = \frac{aiz-a^2}{\zeta z-a^2},$$

$$\frac{\bar{z}(\bar{\zeta}+ai)}{\bar{\zeta}\bar{z}-a^2} = \frac{-z+ai}{\zeta-z}.$$

Case 2: $\zeta \in C_2$,

$$\frac{\bar{z}(\bar{\zeta}+a)}{\bar{\zeta}\bar{z}-a^2} = \frac{-aiz-a^2}{\zeta z-a^2}.$$

Thus, on $\partial M\bigcap C_2$,

$$\lim_{z\to\zeta}\omega(z) = \lim_{z\to\zeta}\frac{1}{2\pi i}\int_{\partial M\bigcap C_2}\gamma(\zeta)\left[\frac{\zeta+ai}{\zeta-z}+\frac{\bar{\zeta}-ai}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta+ai}$$

$$= \lim_{z\to\zeta}\frac{1}{2\pi i}\int_{C_2}\Upsilon(\zeta)\left[\frac{\zeta+ai}{\zeta-z}+\frac{\bar{\zeta}-ai}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta+ai},$$

where

$$\Upsilon(\zeta) = \begin{cases} \gamma(\zeta) & \zeta \in \partial M\bigcap C_2, \\ 0 & \zeta \in C_2\backslash(\partial M). \end{cases}$$

So based on the properties of the Poisson kernel for C_2,

$$\lim_{z\to\zeta}\omega(z) = \gamma(\zeta),$$

follows for $\zeta \in \partial M\bigcap C_2$ up to the corner points $\pm a$ of the domain M, because Υ fails to be continuous there if γ not accidentally vanishes at these points.

Now, we consider the boundary behaviors at the tips $\pm a$. We represent the constant function 1 as

$$1 = \frac{1}{2\pi i}\int_{\partial M\bigcap C_1}\left[\frac{\zeta-ai}{\zeta-z}+\frac{\bar{\zeta}+ai}{\bar{\zeta}-\bar{z}}-1+\frac{z(\zeta-ai)}{\zeta z-a^2}+\frac{\bar{z}(\bar{\zeta}+ai)}{\bar{\zeta}\bar{z}-a^2}-1\right]\frac{d\zeta}{\zeta-ai}$$

$$+\frac{1}{2\pi i}\int_{\partial M\bigcap C_2}\left[\frac{\zeta+ai}{\zeta-z}+\frac{\bar{\zeta}-ai}{\bar{\zeta}-\bar{z}}-1+\frac{z(\zeta+ai)}{\zeta z-a^2}+\frac{\bar{z}(\bar{\zeta}-ai)}{\bar{\zeta}\bar{z}-a^2}-1\right]\frac{d\zeta}{\zeta+ai}.$$

Multiplying this relation with $\gamma(\pm a)$ and subtracting the resulting equation from $\omega(z)$ shows for $z \in \partial M\bigcap C_1$,

$$\lim_{z\to\zeta}(\omega(z)-\gamma(\pm a)) = \lim_{z\to\zeta}\frac{1}{2\pi i}\int_{\partial M\bigcap C_1}\tilde{\gamma}(\zeta)\left[\frac{\zeta-ai}{\zeta-z}+\frac{\bar{\zeta}+ai}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta-ai},$$

where $\tilde{\gamma}(\zeta)=\gamma(\zeta)-\gamma(\pm a)$ and $\tilde{\gamma}(\pm a)=0$,

$$\lim_{z\to\pm a}\omega(z)=\gamma(\pm a).$$

Similarly, for $z\in\partial M\bigcap C_2$,

$$\lim_{z\to\zeta}(\omega(z)-\gamma(\pm a))=\lim_{z\to\zeta}\frac{1}{2\pi i}\int_{\partial M\bigcap C_2}\hat{\gamma}(\zeta)\left[\frac{\zeta+ai}{\zeta-z}+\frac{\bar{\zeta}-ai}{\bar{\zeta}-\bar{z}}-1\right]\frac{d\zeta}{\zeta+ai},$$

where $\hat{\gamma}(\zeta)=\gamma(\zeta)-\gamma(\pm a)$ and $\hat{\gamma}(\pm a)=0$,
therefore,

$$\lim_{z\to\pm a}\omega(z)=\gamma(\pm a).$$

Therefore, the proof is finished. □

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