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Some Analytical Properties of the Remainder of Binet-Like Expression for the Barnes G-Function

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Abstract. In this paper, we prove some properties such as monotonicity, complete monotonicity, logarithmic convexity, inequalities, subadditivity and starshapedness, involving the Binet-like remainder of the Barnes G-function. The methods of proofs are analytical in nature.

1. Introduction

Special functions are usually encountered in almost every scientific discipline. Particularly, they play important roles in areas such as mathematics, physics and engineering. The gamma function, which is an extension of the factorial function, is arguably the most important special function. This is largely due to its vast areas of applications as well as its connection with other special functions. It is usually defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{1}$$

for $z > 0$. The Binet's formula for logarithm of the gamma function is given as

$$\ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + \ln \sqrt{2\pi} + \theta(z) \tag{2}$$

for $z > 0$, where

$$\theta(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-zt}}{t} dt \tag{3}$$

is known as the remainder of Binet's formula. Due to the important properties exhibited by the function $\theta(z)$, it has been investigated in multiple ways. In [5], the authors proved among other things that, for $p \in (0, 1]$, the function

$$f_p(z) = \theta(pz) - p\theta(z) \tag{4}$$

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is completely monotonic on $(0, \infty)$. In [7], the authors investigated the complete monotonicity of the function

$$f_{p,q,r}(z) = r[\theta(pz) - q\theta(z)] \tag{5}$$

where $p > 0$, q is a real number and $r \neq 0$. They further established that $\theta(z)$ is subadditive on $(0, \infty)$ and $-\theta(z)$ is starshaped on $(0, \infty)$. In [11], the authors considered a generalization of the function $\theta(z)$ which is denoted by $\theta_\alpha(z)$. They investigated the complete monotonicity of the function

$$f_{p,q,\alpha}(z) = \theta_\alpha(pz) - q\theta_\alpha(z) \tag{6}$$

where $p > 0$, q is a real number and $\alpha > 0$. Among other things, they further established that the function $\theta_\alpha(z)$ is subadditive. In [3], the authors considered an inequality for the r-th derivative of $\theta(z)$. Subsequently, they obtain the Turan-type inequality

$$\left(\theta^{(r+1)}(z)\right)^2 \leq \theta^{(r)}(z)\theta^{(r+2)}(z) \tag{7}$$

where $r \in \mathbb{N}$. A generalization of (7) can be found in [2].

The multiple gamma function, which is a generalization of the ordinary gamma function, was defined by Barnes as [1]

$$\begin{aligned} \Gamma_{r+1}(z+1) &= \frac{\Gamma_{r+1}(z+1)}{\Gamma_r(z)}, \quad z \in \mathbb{C}, r \in \mathbb{N}, \\ \Gamma_1(z) &= \Gamma(z), \\ \Gamma_r(1) &= 1. \end{aligned}$$

The particular case $G(z) = \frac{1}{\Gamma_2(z)}$ is referred to as the Barnes G-function or the double gamma function. It satisfies the following basic properties [1].

$$\begin{aligned} G(z+1) &= G(z)\Gamma(z), \quad z \in \mathbb{C}, \\ G(1) &= 1, \\ (\ln G(z))''' &\geq 0, \quad z > 0. \end{aligned}$$

For further properties of the function $G(z)$, one may refer to the papers [10] and [12] and the references in there.

A Binet-like expression for logarithm of the double gamma function is given by Choi [6] as

$$\ln \Gamma_2(z) = \ln A - \frac{z^2}{4} + \left(\frac{z^2}{2} - \frac{z}{2} + \frac{1}{12}\right)\ln z + (1-z)\ln\Gamma(z) + \Theta(z) \tag{8}$$

where $A = 1.282427...$ is the Glaisher-Kinkelin constant and

$$\Theta(z) = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{t}{12}\right)\frac{e^{-zt}}{t^2}dt. \tag{9}$$

is what is referred to as the Binet-like remainder.

It is interesting to observe that the Binet-like remainder, $\Theta(z)$ has some resemblance with the Binet remainder, $\theta(z)$. The natural question that arise is: does the function $\Theta(z)$ satisfy the properties satisfied by the function $\theta(z)$? Motivated by the papers [3,5,7,11], the objective of this paper is to answer this question. We establish some properties of the function $\Theta(z)$ such as monotonicity, complete monotonicity, logarithmic convexity, inequalities, subadditivity and starshapedness, among others. We present our findings in the next section. Before that, we provide the following definitions which shall pave the way for us to prove our results. Throughout this paper, $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

Definition 1.1. A real-valued function $\mathcal{K}$ defined on an interval $I \subseteq \mathbb{R}$ is said to be convex on I iff

$$\mathcal{K}\left(\frac{x}{u} + \frac{y}{v}\right) \leq \frac{\mathcal{K}(x)}{u} + \frac{\mathcal{K}(y)}{v} \tag{10}$$

holds for all $x, y \in I$ and $u > 1$, $v > 1$ such that $\frac{1}{u} + \frac{1}{v} = 1$. Equivalently, $\mathcal{K}$ is said to be convex on $\mathcal{I}$ iff

$$\mathcal{K}''(z) \geq 0 \tag{11}$$

for all $z \in I$. If the inequalities (10) and (11) are reversed, then $\mathcal{K}$ is said to be concave on I.

Definition 1.2. A positive real-valued function $\mathcal{K}$ defined on an interval $I \subseteq \mathbb{R}$ is said to be logarithmically convex on I iff

$$\mathcal{K}\left(\frac{x}{u} + \frac{y}{v}\right) \leq [\mathcal{K}(x)]^{\frac{1}{u}} [\mathcal{K}(y)]^{\frac{1}{v}} \tag{12}$$

holds for all $x, y \in I$ and $u > 1$, $v > 1$ such that $\frac{1}{u} + \frac{1}{v} = 1$. Equivalently, $\mathcal{K}$ is said to be logarithmically convex on I iff

$$[\ln \mathcal{K}(z)]'' \geq 0 \tag{13}$$

for all $z \in I$. If the inequalities (12) and (13) are reversed, then $\mathcal{K}$ is said to be logarithmically concave on I.

Definition 1.3 ([13]). A real-valued function $\mathcal{K}$ defined on an interval $I \subseteq \mathbb{R}$ is said to be completely monotonic on I iff

$$(-1)^n \mathcal{K}^{(n)}(z) \geq 0$$

holds for all $z \in I$ and $n \in \mathbb{N}_0$.

Definition 1.4 ([4]). A real-valued function $\mathcal{K}$ defined on an interval $I \subseteq \mathbb{R}$ is said to be subadditive on I iff

$$\mathcal{K}(x + y) \leq \mathcal{K}(x) + \mathcal{K}(y)$$

holds for all $x, y \in I$. If the inequality is reversed, then $\mathcal{K}$ is said to be superadditive on I.

Definition 1.5 ([4]). A real-valued function $\mathcal{K}$ defined on an interval $I \subseteq \mathbb{R}$ is said to be starshaped on I iff

$$\mathcal{K}(\alpha z) \leq \alpha \mathcal{K}(z),$$

for all $z \in I$ and $\alpha \in [0, 1]$.

2. Results and Discussion

Beginning with the following lemmas, we now present our findings in this section.

Lemma 2.1. *For $t > 0$, the inequality*

$$\frac{1}{t^2} - \frac{1}{12} < \frac{e^{-t}}{(1-e^{-t})^2} < \frac{1}{t^2} \tag{14}$$

holds.

Proof. See Theorem 2 of [8]. □

Lemma 2.2. *For $t > 0$, the function*

$$\mathcal{P}(t) = \frac{1}{t^2} - \frac{e^{-t}}{(1-e^{-t})^2} \tag{15}$$

is strictly decreasing.

Proof. See Theorem 1 of [8] or Theorem 1.1 of [9]. □

Theorem 2.3. *For $t > 0$, let $A(t)$ be defined as*

$$A(t) = \frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{t}{12}. \tag{16}$$

$$= \frac{t}{12} + \frac{1}{t} - \frac{1}{2}\coth\left(\frac{t}{2}\right). \tag{17}$$

Then:

(a) $A(t)$ is strictly increasing.

(b) $A(t)$ is positive.

(c) $A(t)$ is strictly convex.

Proof. By L'hopital's rule and simple computation, we have

$$\lim_{t\to 0} A(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} A(t) = \infty.$$

Then by making use of the left-hand side of (14), we obtain

$$\begin{aligned} A'(t) &= \frac{e^t}{(e^t-1)^2} - \frac{1}{t^2} + \frac{1}{12} \\ &= \frac{e^{-t}}{(e^{-t}-1)^2} - \frac{1}{t^2} + \frac{1}{12} \\ &> 0. \end{aligned}$$

Hence $A(t)$ is strictly increasing and that completes the proof of (a). Next, the increasing property of $A(t)$ implies that for $t>0$,

$$A(t)>\lim_{t\to 0}A(t)=0$$

which completes the proof of (b). Next, as a result of Lemma 2.2, we have

$$A''(t)=-\left(\frac{1}{t^2}-\frac{e^{-t}}{(1-e^{-t})^2}\right)'=-\mathcal{P}'(t)>0$$

which completes the proof of (c). □

Remark 2.4. Theorem 2.3 (c) implies that $\Theta(z)$ is positive.

Corollary 2.5. *For $t>0$, the inequality*

$$\frac{e^{-t}}{(1-e^{-t})^2}<\frac{2}{t^3} \tag{18}$$

holds.

Proof. The convexity of $A(t)$ implies that

$$A''(t)=\frac{2}{t^3}+\frac{e^t}{(e^t-1)^2}-\frac{2e^{2t}}{(e^t-1)^3}>0.$$

This simplifies to

$$\frac{2}{t^3}(e^t-1)^3-e^t(e^t-1)>0$$

which further simplifies to

$$\frac{2}{t^3}>\frac{e^t}{(e^t-1)^2}=\frac{e^{-t}}{(1-e^{-t})^2}.$$

This completes the proof. □

Remark 2.6. We note that $\frac{1}{t^2}-\frac{2}{t^3}<0$ if $0<t<2$ and $\frac{1}{t^2}-\frac{2}{t^3}>0$ if $t>2$. Thus, the upper bound in (14) is better that upper bound in (18) if $0<t<2$ and the upper bound in (18) is better than the upper bound in (14) if $t>2$.

Theorem 2.7. *The function $\Theta(z)$ is completely monotonic on $(0,\infty)$.*

Proof. Differentiating r number of times of (9) gives

$$\Theta^{(r)}(z)=(-1)^r\int_0^\infty A(t)t^{r-2}e^{-zt}dt \tag{19}$$

where $r\in\mathbb{N}_0$ and $\Theta^{(0)}(z)=\Theta(z)$. This implies that

$$\begin{aligned}(-1)^r\Theta^{(r)}(z)&=(-1)^{2r}\int_0^\infty A(t)t^{r-2}e^{-zt}dt\\&=\int_0^\infty A(t)t^{r-2}e^{-zt}dt>0.\end{aligned}$$

This completes the proof. □

Remark 2.8. The representation (19) implies that the function $\Theta(z)$ is decreasing and convex on $(0,\infty)$.

Theorem 2.9. *Let $r \in \mathbb{N}_0$ be even. Then the function $\Theta^{(r)}(z)$ is logarithmically convex on $(0,\infty)$. That is, the inequality*

$$\Theta^{(r)}\left(\frac{x}{u}+\frac{y}{v}\right) \leq \left[\Theta^{(r)}(x)\right]^{\frac{1}{u}}\left[\Theta^{(r)}(y)\right]^{\frac{1}{v}} \tag{20}$$

holds for $x>0$, $y>0$, $u>1$, $v>1$ and $\frac{1}{u}+\frac{1}{v}=1$.

Proof. Let $r \in \mathbb{N}_0$ be an even number. Then, by using (19) and Holder's inequality for integrals, we obtain

$$\begin{aligned}
\Theta^{(r)}\left(\frac{x}{u}+\frac{y}{v}\right) &= \int_0^\infty A(t)t^{r-2}e^{-(\frac{x}{u}+\frac{y}{v})t}dt \\
&= \int_0^\infty (A(t)t^{r-2})^{\frac{1}{u}+\frac{1}{v}}e^{-(\frac{x}{u}+\frac{y}{v})t}dt \\
&= \int_0^\infty A(t)^{\frac{1}{u}}t^{\frac{r-2}{u}}e^{\frac{-xt}{u}}A(t)^{\frac{1}{v}}t^{\frac{r-2}{v}}e^{\frac{-yt}{v}}dt \\
&\leq \left(\int_0^\infty A(t)t^{r-2}e^{-xt}dt\right)^{\frac{1}{u}}\left(\int_0^\infty A(t)t^{r-2}e^{-yt}dt\right)^{\frac{1}{v}} \\
&= \left[\Theta^{(r)}(x)\right]^{\frac{1}{u}}\left[\Theta^{(r)}(y)\right]^{\frac{1}{v}}
\end{aligned}$$

which completes the proof. □

Remark 2.10. The particular case where $r=0$ in Theorem 2.9 proves that $\Theta(z)$ is logarithmically convex on $(0,\infty)$.

Remark 2.11. Theorem 2.9 shows that, for even $r \in \mathbb{N}_0$, the function

$$T(z) = \frac{\Theta^{(r+1)}(z)}{\Theta^{(r)}(z)} \tag{21}$$

is increasing on $(0,\infty)$.

Corollary 2.12. *Let $r \in \mathbb{N}_0$ be even and $p \in (0,1]$. Then the function*

$$\mathcal{B}(z) = \frac{\Theta^{(r)}(pz)}{[\Theta^{(r)}(z)]^p}$$

decreasing on $(0,\infty)$. Consequently, for $0<x\leq y$, the inequality

$$\left(\frac{\Theta^{(r)}(y)}{\Theta^{(r)}(x)}\right)^p \geq \frac{\Theta^{(r)}(py)}{\Theta^{(r)}(px)} \tag{22}$$

holds.

Proof. Logarithmic differentiation of $\mathcal{B}(z)$ and applying the increasing property of $T(z)$ gives

$$\begin{aligned}\frac{\mathcal{B}'(z)}{\mathcal{B}(z)} &= p\frac{\Theta^{(r+1)}(pz)}{\Theta^{(r)}(pz)} - p\frac{\Theta^{(r+1)}(z)}{\Theta^{(r)}(z)} \\ &= p\left[\frac{\Theta^{(r+1)}(pz)}{\Theta^{(r)}(pz)} - \frac{\Theta^{(r+1)}(z)}{\Theta^{(r)}(z)}\right] \\ &\leq 0.\end{aligned}$$

Hence $\mathcal{B}(z)$ is decreasing. Consequently, for $0 < x \leq y$, we have $\mathcal{B}(x) \geq \mathcal{B}(y)$ which when rearranged gives (22). □

Remark 2.13. If $p \geq 1$ in Corollary 2.12, then the reverse cases of the conclusions are obtained.

Theorem 2.14. *Let $r \in \mathbb{N}_0$ and $s \in \mathbb{N}_0$. Then the inequality*

$$\left|\Theta^{(\frac{r}{u}+\frac{s}{v})}\left(\frac{x}{u}+\frac{y}{v}\right)\right| \leq \left|\Theta^{(r)}(x)\right|^{\frac{1}{u}} \left|\Theta^{(s)}(y)\right|^{\frac{1}{v}} \tag{23}$$

holds for $x > 0$, $y > 0$, $u > 1$, $v > 1$ and $\frac{1}{u}+\frac{1}{v} = 1$.

Proof. By using (19) and Holder's inequality for integrals, we obtain

$$\begin{aligned}\left|\Theta^{(\frac{r}{u}+\frac{s}{v})}\left(\frac{x}{u}+\frac{y}{v}\right)\right| &= \int_0^\infty A(t)t^{(\frac{r}{u}+\frac{s}{v})-2}e^{-(\frac{x}{u}+\frac{y}{v})t}dt \\ &= \int_0^\infty A(t)^{(\frac{1}{u}+\frac{1}{v})}t^{(\frac{r}{u}+\frac{s}{v})-2(\frac{1}{u}+\frac{1}{v})}e^{-(\frac{x}{u}+\frac{y}{v})t}dt \\ &= \int_0^\infty A(t)^{\frac{1}{u}}t^{\frac{r-2}{u}}e^{\frac{-xt}{u}}A(t)^{\frac{1}{v}}t^{\frac{s-2}{v}}e^{\frac{-yt}{v}}dt \\ &\leq \left(\int_0^\infty A(t)t^{r-2}e^{-xt}dt\right)^{\frac{1}{u}}\left(\int_0^\infty A(t)t^{s-2}e^{-yt}dt\right)^{\frac{1}{v}} \\ &= \left[\Theta^{(r)}(x)\right]^{\frac{1}{u}}\left[\Theta^{(s)}(y)\right]^{\frac{1}{v}}\end{aligned}$$

which completes the proof. □

Remark 2.15. If $r = s$ in Theorem 2.14, then we obtain

$$\left|\Theta^{(r)}\left(\frac{x}{u}+\frac{y}{v}\right)\right| \leq \left|\Theta^{(r)}(x)\right|^{\frac{1}{u}} \left|\Theta^{(r)}(y)\right|^{\frac{1}{v}} \tag{24}$$

which shows that, for $r \in \mathbb{N}_0$, the function $\left|\Theta^{(r)}(z)\right|$ is logarithmically convex on $(0, \infty)$.

Remark 2.16. If $r = k-1$, $s = k+1$, $u = v = 2$ and $x = y = z$ in Theorem 2.14, then we obtain the Turan-type inequality

$$\left|\Theta^{(k)}(z)\right|^2 \leq \left|\Theta^{(k-1)}(z)\right| \left|\Theta^{(k+1)}(z)\right|. \tag{25}$$

where $k \in \mathbb{N}_0$. This is equivalent to

$$\left|\Theta^{(k+1)}(z)\right|^2 \leq \left|\Theta^{(k)}(z)\right| \left|\Theta^{(k+2)}(z)\right| \tag{26}$$

where $k \in \mathbb{N}_0$.

Lemma 2.17. *The function $\Theta'(z)$ is increasing on $(0, \infty)$.*

Proof. This follows directly from (19) since $(\Theta'(z))' = \Theta''(z) > 0$. □

Theorem 2.18. *The function $-\Theta(z)$ is starshaped on $(0, \infty)$. That is, the inequality*

$$-\Theta(\alpha z) \leq -\alpha\Theta(z) \tag{27}$$

holds for $\alpha \in [0, 1]$ and $z \in (0, \infty)$.

Proof. Let $\mathcal{K}(z) = \Theta(\alpha z) - \alpha\Theta(z)$ for $\alpha \in [0, 1]$ and $z \in (0, \infty)$. Then

$$\begin{aligned}\mathcal{K}'(z) &= \alpha\Theta'(\alpha z) - \alpha\Theta'(z) \\ &= \alpha\left[\Theta'(\alpha z) - \Theta'(z)\right] \\ &< 0\end{aligned}$$

since $\Theta'(z)$ is increasing. Hence $\mathcal{K}(z)$ is decreasing. Then for $z \in (0, \infty)$, we have

$$\mathcal{K}(z) \geq \lim_{z\to\infty} \mathcal{K}(z) = 0$$

which implies that

$$\Theta(\alpha z) \geq \alpha\Theta(z).$$

This then gives rise to the inequality (27) and that completes the proof. □

Theorem 2.19. *Let $r \in \mathbb{N}_0$. Then $\Theta^{(r)}(z)$ is strictly subadditive if r is even and $\Theta^{(r)}(z)$ is strictly superadditive if r is odd. That is, for $x, y \in (0, \infty)$, it holds that*

$$\Theta^{(r)}(x + y) < \Theta^{(r)}(x) + \Theta^{(r)}(y) \tag{28}$$

if r is even, and

$$\Theta^{(r)}(x + y) > \Theta^{(r)}(x) + \Theta^{(r)}(y) \tag{29}$$

if r is odd.

Proof. Let $\mathcal{U}(x, y) = \Theta^{(r)}(x + y) - \Theta^{(r)}(x) - \Theta^{(r)}(y)$. With no loss of generality, let y be fixed. Then by differentiating with respect to x, and using (19), we have

$$\begin{aligned}\mathcal{U}'(x, y) &= \Theta^{(r+1)}(x + y) - \Theta^{(r+1)}(x) \\ &(-1)^{r+1}\int_0^\infty A(t)t^{r-1}e^{-(x+y)t}dt - (-1)^{r+1}\int_0^\infty A(t)t^{r-1}e^{-xt}dt \\ &= (-1)^{r+1}\int_0^\infty A(t)t^{r-1}\left[e^{-(x+y)t} - e^{-xt}\right]dt \\ &:= \mathcal{V}(x, y)\end{aligned}$$

Suppose that r is even. Then $\mathcal{V}(x, y) > 0$. This implies that $\mathcal{U}(x, y)$ is increasing in terms of x. Hence, for $x \in (0, \infty)$, we have

$$\mathcal{U}(x, y) < \lim_{x \to \infty} \mathcal{U}(x, y) = -\Theta^{(r)}(y) < 0$$

which gives rise to the inequality (28). Likewise, suppose that r is odd. Then $\mathcal{V}(x, y) < 0$. This implies that $\mathcal{U}(x, y)$ is decreasing in terms of x. Hence, for $x \in (0, \infty)$, we have

$$\mathcal{U}(x, y) > \lim_{x \to \infty} \mathcal{U}(x, y) = -\Theta^{(r)}(y) > 0$$

which gives rise to the inequality (29). This completes the proof. □

Remark 2.20. The particular case where $r = 0$ in Theorem 2.19, shows that the function $\Theta(z)$ is strictly subadditive on $(0, \infty)$.

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