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Modulation Instability, Dark and Singular Soliton for Weakly Nonlocal Schrodinger Equation

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Abstract. The optical soliton solution of nonlinear complex models holds significant importance in nonlinear optics and communication systems. Considering nonlinear complex models, often described by equations like the nonlinear Schrodinger equation (NLSE), plays a crucial role in defining the balance between dispersive and nonlinear effects, enabling the formation and maintenance of solitons over long distances. This stability is crucial for signal integrity in optical communication systems. The investigation of optical soliton solutions from nonlinear complex models is sometimes complicated. With this in mind, we employed an effective method of extended Tanh function with a Riccatti differential equation to retrieve the dark, singular and periodic wave solutions for the weakly nonlocal nonlinear Schrodinger equation with parabolic law. The obtained solutions were verified by back-substitution in the original equations, with the aid of a Mathematica to affirm the robustness of the chosen approach. Respective 2D and 3D graphs for some of the obtained results was portrayed by choosing suitable values of the parameters that were involved. An analysis of instability that results in the modulation of the steady-state as a result of co-action between the nonlinear and dispersive effects was performed on the proposed model where the condition for stable wave under small perturbation was obtained and presented. The gain spectrum plot for the modulation instability was portrayed.

1. Introduction

Partial Differential Equations (PDEs) serve as powerful mathematical tools for modeling and understanding diverse phenomena in science and engineering. These equations involve multiple variables and their partial derivatives, making them well-suited for describing complex physical processes such as heat conduction, fluid dynamics, and electromagnetic fields. The investigation of PDEs encompasses various aspects, including analytical and numerical methods for solving these equations. Analytically, researchers explore techniques like separation of variables, integral transforms,

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and series solutions to obtain exact solutions or gain insights into the behavior of solutions. On the other hand, numerical methods, such as finite difference, finite element, and spectral methods, provide computational tools to approximate solutions for PDEs in cases where analytical solutions are challenging or impossible to obtain. Additionally, the study of stability, existence, and uniqueness of solutions, as well as the development of new solution methods, continues to be a vibrant area of research within the broader field of partial differential equations. Understanding and harnessing the mathematical intricacies of PDEs play a crucial role in advancing our comprehension of the natural world and technological applications. Numerous scholars have focused their efforts on determining exact solutions for NPDEs, with these equations finding applications across diverse scientific and technological domains, including but not limited to mathematical physics, fluid dynamics, optical fibers, and economics [10, 14, 27].

In recent years, various methodologies have been developed for determining precise solutions to NPDEs. These approaches include the Lie symmetry method [16, 18, 20, 26, 30], the Kudryashov method [21, 22, 33], the sine-Gordon expansion method [6], the invariant subspace method [13, 28], the Sardar subequation method [3, 11], and others [1, 5, 8, 9, 29]. These techniques contribute to the exploration of exact solutions for a wide range of PDEs. The NLSE is a complex PDE that plays a crucial role in various fields of physics and engineering, describing the evolution of complex wave packets. It arises in different contexts, including nonlinear optics, plasma physics, fluid dynamics, and condensed matter physics. The NLSE and its variants have significant importance in understanding and predicting the behavior of wave-like phenomena in diverse real-world applications.

In nonlinear optics, the NLSE governs the propagation of intense laser beams through nonlinear media. This equation describes the interactions among optical waves, considering effects such as self-focusing, self-phase modulation, and optical solitons. Optical solitons, which are stable, localized wave packets that can maintain their shape during propagation, and applications in long-distance communication systems. The NLSE helps optimize and control these phenomena in the design of optical communication systems and laser technologies.

The NLSE also appears in the study of ultra-cold atomic gases, particularly in the context of Bose-Einstein condensates. In this scenario, the NLSE describes the dynamics of the macroscopic wave function of the condensate. Understanding the NLSE for BECs is crucial for investigating phenomena such as matter-wave solitons and vortices, which have applications in precision measurements and quantum information processing.

The NLSE arises in the study of Langmuir waves in plasmas, where it describes the evolution of the electron plasma wave. Nonlinear effects become significant in high-intensity laser-plasma interactions and can lead to the generation of harmonics and other phenomena. This has applications in areas such as controlled nuclear fusion research and the development of high-power particle

accelerators.
NLSE is fundamental in understanding the behavior of optical pulses in fiber optic communication systems. Fiber optic channels exhibit nonlinear effects such as self-phase modulation and cross-phase modulation, which can distort transmitted signals. The NLSE is essential for modeling and mitigating these effects, ensuring the reliability and efficiency of long-distance communication networks.
Variants of the NLSE are used to model the propagation of water waves in oceans and other bodies of water. Nonlinear effects, such as wave steepening and wave breaking, can be described using these equations. Understanding these phenomena is crucial for predicting and mitigating the impact of tsunamis, storm surges, and other oceanic events.
NLSE variants are employed in the study of biological systems, such as modeling the propagation of nerve impulses. The NLSE can describe the nonlinear dynamics of excitable media, providing insights into the behavior of electrical signals in biological tissues.
Moreover, the NLSE [4, 12, 23], which is an essential fully-integrated nonlinear dispersive partial differential equation (PDE), has found extensive application in elucidating diverse phenomena like deep water waves, rogue waves, plasmas, and nonlinear optics, including atomic physics. The NLSE serves as a comprehensive description of nonlinear dispersive processes. Zhou et al. [34] considered the weakly nonlocal NLSE having PL nonlinearity with external potential as

$$i\Phi_t + \lambda_1\Phi_{xx} + \left(\lambda_2|\Phi|^2_{xx} + \lambda_3|\Phi|^2 + \lambda_4|\Phi|^4\right)\Phi = 0. \tag{1}$$

The study of Schrodinger equations with nonlinearity is an important area of research in mathematical physics. In recent years, the weakly nonlocal Schrodinger equation with parabolic law nonlinearity has gained significant attention due to its numerous applications in various fields such as quantum mechanics, nonlinear optics, and fluid dynamics.
In this paper, we present Modulation instability analysis and the novel exact solutions for the weakly nonlocal Schrodinger equation with parabolic law nonlinearity, derived using the method extended tanh function. The solutions we obtain, which include dark soliton, bright soliton, and traveling wave solutions, have not been previously reported in the literature and demonstrate the complex dynamics of the considered equation. Dark optical soliton describes the soli- tary waves with lower intensity than the background, bright optical soliton describes the solitary waves whose peak intensity is larger than the background and the sin- gular optical soliton is a solitary wave with discontinuous derivatives; examples of such solitary waves include compactions, which have finite (compact) support, and peakons, whose peaks have a discontinuous first derivative [35, 36]. Dark soli- ton propagates without changing its shape, but it is not made by a normal pulse; rather, it is a lack of energy in a continuous-time beam. The intensity is constant, but for a short time during which it jumps to zero and back again, thus generating a "dark pulse"'. Those solitons can

actually be generated introducing short dark pulses in much longer standard pulses. Dark solitons are more difficult to handle than standard solitons, but they have shown to be more stable and robust to losses. Bright optical soliton causes a temporary increase in an associated wave amplitude [52]. The combined dark-bright optical soliton carries the combine features of the dark and bright optical solitons. Moreover, the modulation instability analysis for the weakly nonlocal NLSE having PL nonlinearity with external potential will also be presented.

Our findings have significant implications for the study of nonlinear phenomena in physical systems. The ability to obtain exact solutions to such complex equations is crucial in understanding the underlying physics and designing new experiments. Additionally, with the aid of mathematica, our results are verified by back-substitution in the original equations, affrming the robustness of our approach. The suggested is not only direct and simple but also suitable for constructing new results, paving the way for future applications on NLSEs with dual power law and perturbed NLSEs with Kerr law. This is particularly useful for identifying solitons in photorefractive and polymer materials. The proposed technique holds potential for further applications in natural science models, aiding in the investigation of other mathematical challenges and characterizing the behavior of nonlinear models.

The remaining sections of this manuscript are organized as follows: In Section 2, we give the description of the approach.In section 3, we apply the method the governing equation. Insection 4, we give the graphical representation of some of the obtained results. In section 5, we present the analysis of modulation instability for the model and its physical description. In section 6, we provide the conclusion of the study.

2. Description of the approach

The method of extended tanh is often employed in obtaining soliton solutions due to is effectiveness in capturing the inherent characteristics of solitons such as their amplitude, width, and velocity. The following are some of the rationale for using the method: Flexibility, asymptotic behavior, Ease of manipulation accuracy, existence of exact solutions.

In order to present the modified extended tanh approach, we need to consider the nonlinear partial differential equation of the form:

$$\rho\left(U, U_x, U_{xx}, U_t U_x, \ldots\right) = 0; \tag{2}$$

In transforming (2); we use the wave transformations:

$$U(x,t) = U(\xi);\ \xi = (x - \mu t) \tag{3}$$

where μ is a nonzero constant. Substitution of the transformation (3) into Eq. (2), it reduces to an ordinary differential equation of the polynomial form:

$$\Xi\left(U(\xi), U'(\xi), U''(\Xi), U'''(\Xi), \ldots\right) = 0. \tag{4}$$

Furthermore, the solution is considered to be a finite series of the form:

$$u(\xi) = a_0 + \sum_{n=1}^{n=N} a_n \psi^n(\xi) + \sum_{n=1}^{n=N} \frac{b_n}{\psi^n}(\xi) \tag{5}$$

where, $a_0, a_n, b_n, n = 1, 2, 3, \ldots N$ are constants which to be computed; and N is a positive integer which is to be determined by balancing the highest order derivative and with the highest nonlinear terms in the equation. Also, $\psi(\xi)$ satisfies the Riccati's differential equation:

$$\psi'(\xi) = \psi^2(\xi) + b \tag{6}$$

where b is a constant. Furthermore, the Riccati differential equation in (6) has solutions of the form:

- For Hyperbolic solution, if $b < 0$, then
  $\Phi(\xi) = -\sqrt{-b}\; Tanh\sqrt{-b}\; \xi$, $\Phi(\xi) = -\sqrt{-b}\; Coth\sqrt{-b}\; \xi$
- For rational solution, if $b = 0$, then $\frac{1}{\xi}$
- For Periodic solution, if $b > 0$, then
  $\Phi(\xi) = \sqrt{b}\; Tan\sqrt{b}\; \xi$, $\Phi(\xi) = \sqrt{b}\; Cot - \sqrt{-b}\; \xi$

Now, after we substitute Eq. (5) and its derivatives together with the Riccati Eq. (6), into Eq. (4), it gives a polynomial in $\Phi(\xi)$. Collecting the coefficients of the same power of $\Phi(\xi)$ in the polynomial and setting each of them to zero, we shall get a set of algebraic equations. Solving these system of algebraic equation with the aid of symbolic computation using Mathematica to get the values of $a_0, a_n, b_n, (n = 1, 2, 3, \ldots)$ *and* b. Finally, substituting these values into Eq. (5) from which we obtain the solution of Eq. (4).

3. Application

In this section, the application of the modified extended tanh expansion method with Riccati differential equation [49] to our equation (1) shall be presented.

Consider the transformation:

$$\phi(x, t) = \Phi(\xi) e^{i\varpi};\ \xi = x - \nu t \text{ and } \varpi = \omega x - rt + \delta \tag{7}$$

Substituting equation (7) into equation (1), gives the following nonlinear ODE:

$$2\lambda_2 \Phi^2 \Phi'' + \lambda_1 \Phi'' + \lambda_4 \Phi^5 + \lambda_3 \Phi^3 + \Phi \left(2\lambda_2 \Phi'^2 - \lambda_1 \omega^2 + r\right) = 0. \tag{8}$$

From the real part and

$$\left(v - 2\lambda_1 \omega\right) \Phi' = 0. \tag{9}$$

From the imaginary part. Thus, we have the constraint condition as

$$v = 2\lambda_1 \omega. \tag{10}$$

Balancing between $\Phi^2\Phi''$ and Φ^5 , that is:

$$2N + N + 2 = 5N \implies N = 1. \tag{11}$$

With equation (11), our equation (5) takes the form:

$$\phi(\Omega) = a_1\Phi(\Omega) + a_0 + \frac{b_1}{\Phi(\Omega)} \tag{12}$$

Substituting Eq. (12), its first and second derivative along with equation (6) into equation (8), we get a polynomial in powers of $\Phi(\Omega)$. Summing the coefficients of $\Phi(\Omega)$ with the same power and equating each summation to zero, gives a set of algebraic equations. Solving these set of algebraic equations, yields the following cases of solutions:

case one

When $a_0 = 0$; $a_1 = -\frac{i\sqrt{6}\sqrt{\lambda_2}}{\sqrt{\lambda_4}}$; $b_1 = 0$; $r \to \frac{1}{16}\left(\lambda_1\left(\frac{2\lambda_3}{\lambda_2} + 16\omega^2\right) - \frac{\lambda_4\lambda_1^2}{\lambda_2^2} + \frac{3\lambda_3^2}{\lambda_4}\right)$;
$w = \frac{\lambda_1\lambda_4 - 3\lambda_2\lambda_3}{24\lambda_2^2}$, we obtain the following solutions to equation (3):

If $W < 0$, then we have

$$\Phi_1(\xi) = \pm\frac{i\sqrt{3\lambda_2\lambda_3 - \lambda_1\lambda_4}\tanh\left(\frac{\sqrt{3\lambda_2\lambda_3 - \lambda_1\lambda_4}\Omega}{2\sqrt{6}\lambda_2}\right)}{2\sqrt{\lambda_2}\sqrt{\lambda_4}}e^{i(\omega x - rt + \delta)}. \tag{13}$$

$$\Phi_2(\xi) = \pm\frac{i\sqrt{3\lambda_2\lambda_3 - \lambda_1\lambda_4}\coth\left(\frac{\sqrt{3\lambda_2\lambda_3 - \lambda_1\lambda_4}\Omega}{2\sqrt{6}\lambda_2}\right)}{2\sqrt{\lambda_2}\sqrt{\lambda_4}}e^{i(\omega x - rt + \delta)}. \tag{14}$$

If $W > 0$, then we have

$$\Phi_3(\xi) = \mp - \frac{i\sqrt{\lambda_1\lambda_4 - 3\lambda_2\lambda_3}\tan\left(\frac{\sqrt{\lambda_1\lambda_4 - 3\lambda_2\lambda_3}\Omega}{2\sqrt{6}\lambda_2}\right)}{2\sqrt{\lambda_2}\sqrt{\lambda_4}}e^{i(\omega x - rt + \delta)}. \tag{15}$$

$$\Phi_4(\xi) = \pm\frac{i\sqrt{\lambda_1\lambda_4 - 3\lambda_2\lambda_3}\cot\left(\frac{\sqrt{\lambda_1\lambda_4 - 3\lambda_2\lambda_3}\Omega}{2\sqrt{6}\lambda_2}\right)}{2\sqrt{\lambda_2}\sqrt{\lambda_4}}e^{i(\omega x - rt + \delta)}. \tag{16}$$

case two

When $a_0 = 0$; $a_1 = \frac{i\sqrt{6}\sqrt{\lambda_2}}{\sqrt{\lambda_4}}$; $b_1 = \frac{i(3\lambda_2\lambda_3 - \lambda_1\lambda_4)}{8\sqrt{6}\lambda_2^{3/2}\sqrt{\lambda_4}}$; $r = \frac{\lambda_1\left(3\lambda_2\left(4\lambda_2\omega^2 + \lambda_3\right) - \lambda_1\lambda_4\right)}{12\lambda_2^2}$;
$w = \frac{3\lambda_2\lambda_3 - \lambda_1\lambda_4}{48\lambda_2^2}$; we obtain the following solutions to equation (1):

If $W < 0$, then we have

$$\Phi_5(\xi) = \pm\frac{i\sqrt{\lambda_1\lambda_4 - 3\lambda_2\lambda_3}\operatorname{csch}\left(\frac{\sqrt{\frac{\lambda_1\lambda_4}{3} - \lambda_2\lambda_3}\Omega}{2\lambda_2}\right)}{\sqrt{2}\sqrt{\lambda_2}\sqrt{\lambda_4}}e^{i(\omega x - rt + \delta)}. \tag{17}$$

If $W > 0$, then we have

$$\Phi_6(\xi) = \mp\frac{i\sqrt{3\lambda_2\lambda_3 - \lambda_1\lambda_4}\csc\left(\frac{\sqrt{\lambda_2\lambda_3 - \frac{\lambda_1\lambda_4}{3}}\Omega}{2\lambda_2}\right)}{\sqrt{2}\sqrt{\lambda_2}\sqrt{\lambda_4}}e^{i(\omega x - rt + \delta)}. \tag{18}$$

4. Graphical Representations of Results

In this section, the 2D and 3D graphical representation for some of the obtained results for the weakly nonlocal NLSE having PL nonlinearity, which is given by equations (1) shall be presented by choosing different values of parameters that are involved.

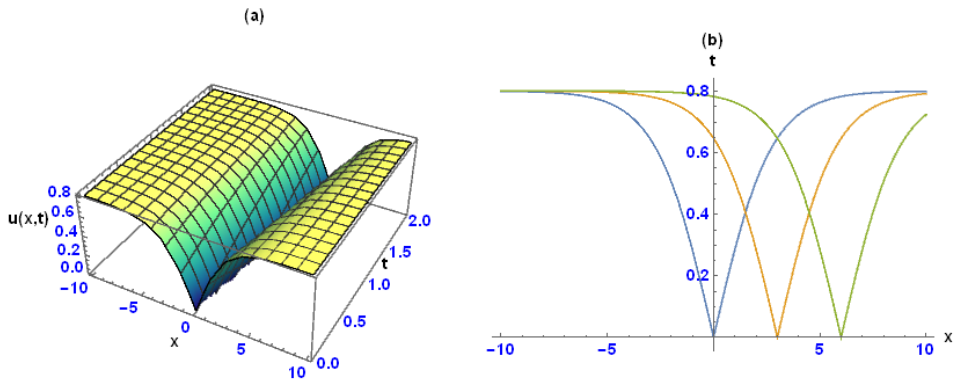

Figure 1. 3D and 2D Abs plot for $\Phi_1(\xi)$.

In figure (1) above, we have the surface profile of 2D and 3D absolute plot representation of equation (13) (that is, $\Phi_1(\xi)$) by taking the following parameter values $(\delta = -1.5;)(\lambda_4 = 1.5;)(\lambda_1 = 0.5;)(\lambda_3 = 1.5;)(\lambda_2 = 1.15;)(\omega = 1.5;)$. The range of values of x for which the graphs were plotted is [-10,10]. The 2D graphs was plotted by taking values of t at 0, 2, 4, .

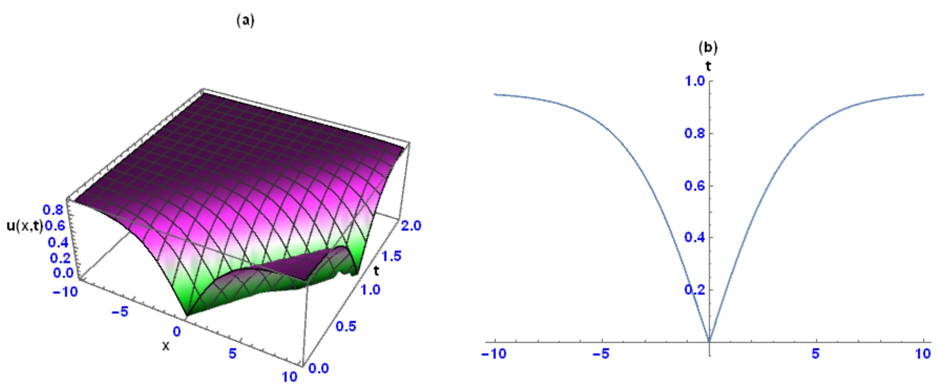

Figure 2. 3D and 2D Abs plot for $\Phi_3(\xi)$.

In figure (2) below, we have the surface profile of 2D and 3D absolute plot representation of equation (15) (that is, $\Phi_3(\xi)$) by taking the following parameter values $(\delta = -1.5;)(\lambda_4 = 1.5;)(\lambda_1 = 0.5;)(\lambda_3 = 1.5;)(\lambda_2 = 1.15;)(\omega = 1.5;)$. The range of values of x for which the graphs were plotted is [-10,10]. The 2D graphs was plotted by taking values of t at

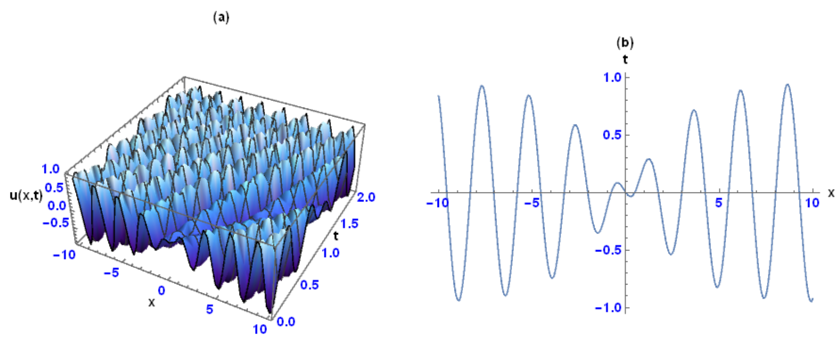

FIGURE 3. 3D and 2D IM plot for $\Phi_3(\xi)$.

the origin.

In figure (3) above, we have the surface profile of 2D and 3D imaginary plot representation of equation (15) (that is, $\Phi_3(\xi)$) by taking the following parameter values ($\delta = -1.5;$) ($\lambda_4 = 1.5;$) ($\lambda_1 = 0.5;$) ($\lambda_3 = 1.5;$) ($\lambda_2 = 1.15;$) ($\omega = 1.5;$). The range of values of x for which the graphs were plotted is [-10,10]. The 2D graphs was plotted by taking values of t at the origin.

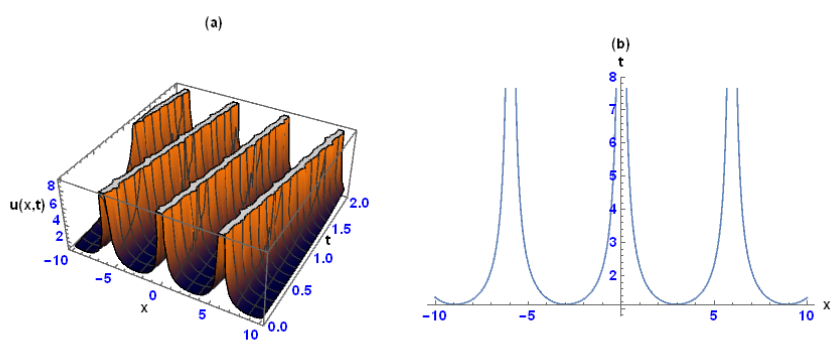

FIGURE 4. 3D and 2D Abs plot for $\Phi_5(\xi)$.

In figure (4) above, we have the surface profile of 2D and 3D absolute plot representation of equation (17) (that is, $\Phi_5(\xi)$) by taking the following parameter values ($\delta = -1.5;$) ($\lambda_4 = 1.5;$) ($\lambda_1 = 0.5;$) ($\lambda_3 = 1.5;$) ($\lambda_2 = 1.15;$) ($\omega = 1.5;$). The range of values of x for which the graphs were plotted is [-10,10]. The 2D graphs was plotted by taking values of t at the origin.

In figure (5) above, we have the surface profile of 2D and 3D real plot representation of equation (17) (that is, $\Phi_5(\xi)$) by taking the following parameter values ($\delta = -1.5;$) ($\lambda_4 = 1.5;$) ($\lambda_1 = 0.5;$) ($\lambda_3 = 1.5;$) ($\lambda_2 = 1.15;$) ($\omega = 1.5;$). The range of values of x

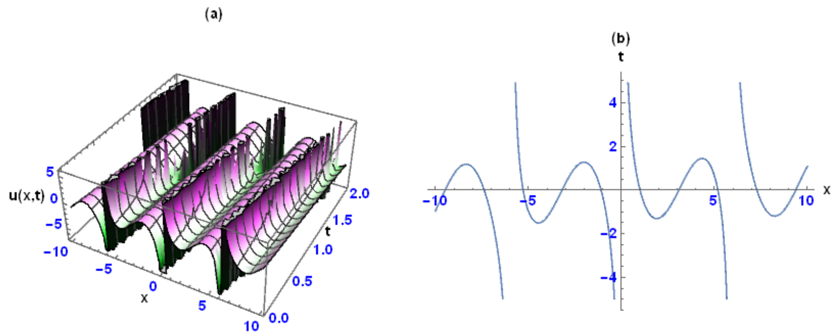

FIGURE 5. 3D and 2D real plot for $\Phi_5(\xi)$.

for which the graphs were plotted is [-10,10]. The 2D graphs was plotted by taking values of t at the origin.

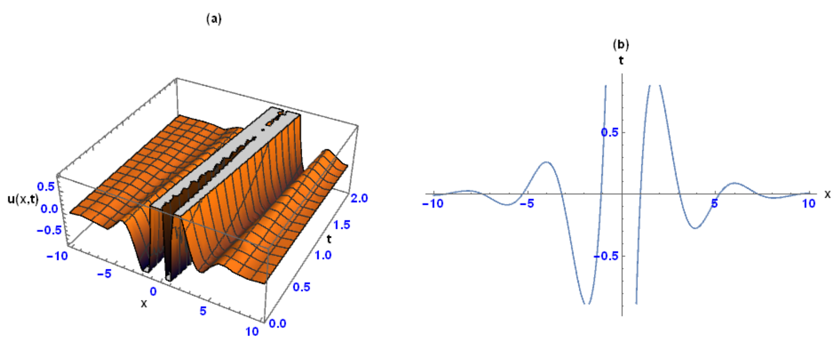

FIGURE 6. 3D and 2D real plot for $\Phi_6(\xi)$.

In figure (6) above, we have the surface profile of 2D and 3D real plot representation of equation (18) (that is, $\Phi_6(\xi)$) by taking the following parameter values $(\delta = -1.5;)\,(\lambda_4 = 1.5;)\,(\lambda_1 = 0.5;)\,(\lambda_3 = 1.5;)\,(\lambda_2 = 1.15;)\,(\omega = 1.5;)$. The range of values of x for which the graphs were plotted is [-10,10]. The 2D graphs was plotted by taking values of t at the origin.

5. Modulation Instability (MI) Analysis

Various nonlinear phenomena display an instability that results in the modulation of the steady-state as a result of co-action between the nonlinear and dispersive effects [37]. In this section, we derive modulation instability for the weakly nonlocal Schrodinger equation with parabolic law.

Theorem 1: Suppose that equation (1) characterizes a wave system featuring a non-trivial dispersion relation, alongside nonlinear components. Given that $\phi(x,t)$ represents a solution to equation, it is possible, under appropriate circumstances, for a range of parameters to exist where MI takes place. Over time, the amplitude and configuration of the wave undergo significant alterations due to instability, resulting in a rapid growth of small perturbations.

Proof: To explore the MI of equation (1), let take the initial assumption that equation (1) is subjected to a small perturbation in the following manner:

$$\phi(x,t) = \left(\sqrt{m} + \Phi(x,t)\right) e^{imx}; \tag{19}$$

where m is the normalized optical power,$\phi(x,t)$ shows real valued amplitude of perturbation with relatively dispersion. Substituting equation (19) into equation (1) and linearizing, gives

$$i\Phi_t + (\lambda_1 + \lambda_2 m)\,\Phi_{xx} + i\lambda_1 m\,(\Phi_x + \Phi_x^*) + \left((\lambda_4 - \lambda_1)\,m^2 + \lambda_3 m\right)(\Phi + \Phi^*) = 0, \tag{20}$$

where Φ^* denotes complex conjugate.
Assume the solutions of equation (20) to be of the form

$$\Phi(x,t) = A_1 e^{(i(x\varpi - \xi t))} + A_2 e^{(-i(x\varpi - \xi t))} \tag{21}$$

$$\Phi(x,t)^* = A_1 e^{(-i(x\varpi - \xi t))} + A_2 e^{(i(x\varpi - \xi t))} \tag{22}$$

Where ϖ and ξ represent normalize wave number and frequency of perturbation respectively. Substituting (21) and (22) into (20) and collect the coefficients of $e^{(-i(x\varpi-\xi t))}, and$ $e^{(i(x\varpi-\xi t))}$, and solve the determinant of the resulting matrix of coefficients we obtain the following dispersion relation:

$$-\lambda_1^2\varpi^4 - 2\lambda_1\lambda_2 m^3\varpi^2 + 2\lambda_2\lambda_4 m^3\varpi^2 - \lambda_2^2 m^2\varpi^4 - 2\lambda_1^2 m^2\varpi^2 + 2\lambda_2\lambda_3 m^2\varpi^2$$

$$2\lambda_1\lambda_4 m^2\varpi^2 - 2\lambda_1 m\xi\varpi - 2\lambda_1\lambda_2 m\varpi^4 + 2\lambda_1\lambda_3 m\varpi^2 + \xi^2 = 0 \tag{23}$$

Solving the dispersion relation(23) for ξ, we obtain:

$$\xi = \lambda_1 m\varpi \mp \begin{pmatrix} \lambda_1^2\varpi^4 + 2\lambda_1\lambda_2 m^3\varpi^2 + \lambda_2^2 m^2\varpi^4 + 3\lambda_1^2 m^2\varpi^2 + 2\lambda_1\lambda_2 m\varpi^4 \\ -2\lambda_2\lambda_4 m^3\varpi^2 - 2\lambda_2\lambda_3 m^2\varpi^2 - 2\lambda_1\lambda_4 m^2\varpi^2 - 2\lambda_1\lambda_3 m\varpi^2 \end{pmatrix}^{\frac{1}{2}}. \tag{24}$$

In a situation whereby
$\begin{pmatrix} \lambda_1^2\varpi^4 + 2\lambda_1\lambda_2 m^3\varpi^2 + \lambda_2^2 m^2\varpi^4 + 3\lambda_1^2 m^2\varpi^2 + 2\lambda_1\lambda_2 m\varpi^4 \\ -2\lambda_2\lambda_4 m^3\varpi^2 - 2\lambda_2\lambda_3 m^2\varpi^2 - 2\lambda_1\lambda_4 m^2\varpi^2 - 2\lambda_1\lambda_3 m\varpi^2 \end{pmatrix} \geqslant 0$,the wave number is real for any real value of m and the steady-state is stable against small perturbations. However, in contrary to the above condition, the steady-state solution turns to be unstable, that is, the wave number becomes imaginary, when $\begin{pmatrix} \lambda_1^2\varpi^4 + 2\lambda_1\lambda_2 m^3\varpi^2 + \lambda_2^2 m^2\varpi^4 + 3\lambda_1^2 m^2\varpi^2 + 2\lambda_1\lambda_2 m\varpi^4 \\ -2\lambda_2\lambda_4 m^3\varpi^2 - 2\lambda_2\lambda_3 m^2\varpi^2 - 2\lambda_1\lambda_4 m^2\varpi^2 - 2\lambda_1\lambda_3 m\varpi^2 \end{pmatrix} < 0$ and the perturbation grows exponentially. Under this condition, the growth rate of modulation stability gain spectrum G(m) may be given as

$G(r) = 2Im(\xi)$=

$$2Im\left(\lambda_1 m\varpi \mp \begin{pmatrix} \lambda_1^2\varpi^4 + 2\lambda_1\lambda_2 m^3\varpi^2 + \lambda_2^2 m^2\varpi^4 + 3\lambda_1^2 m^2\varpi^2 + 2\lambda_1\lambda_2 m\varpi^4 \\ -2\lambda_2\lambda_4 m^3\varpi^2 - 2\lambda_2\lambda_3 m^2\varpi^2 - 2\lambda_1\lambda_4 m^2\varpi^2 - 2\lambda_1\lambda_3 m\varpi^2 \end{pmatrix}\right). \tag{25}$$

The figure below shows the gained dispersion relation to investigate the steady-state stability by taking the following parameter values $\lambda_1 \to 2, \lambda_2 \to 2.4, \lambda_3 \to 1.5, \lambda_4 \to 2.3$ under distinct

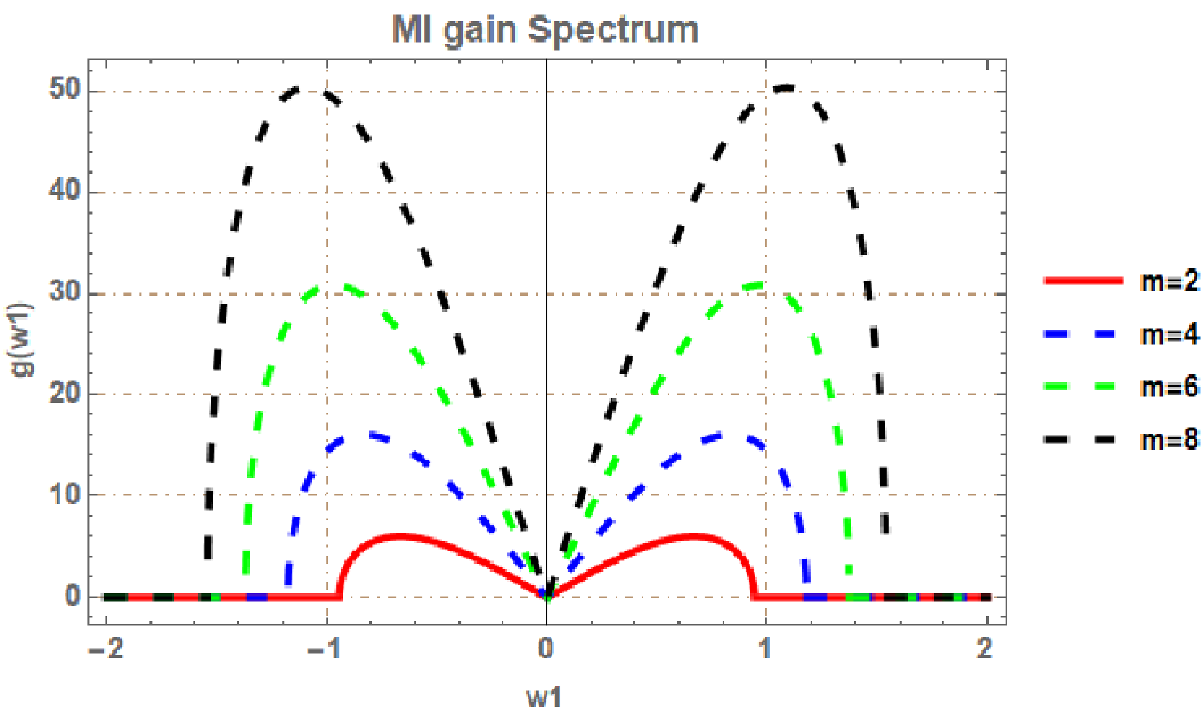

Figure 7. Gain spectrum of MI under different values of m

values of m.

6. Conclusions

Variants of the NLSE are used to model the propagation of water waves in oceans and other bodies of water. Nonlinear effects, such as wave steepening and wave breaking, can be described using these equations. Understanding these phenomena is crucial for predicting and mitigating the impact of tsunamis, storm surges, and other oceanic events. Furthermore, NLSE variant are employed in the study of biological systems, such as modeling the propagation of nerve impulses. The NLSE can describe the nonlinear dynamics of excitable media, providing insights into the behavior of electrical signals in biological tissues. Also, NLSE which are essential in fully-integrated nonlinear dispersive partial differential equation (PDE), have found extensive application in elucidating diverse phenomena like deep water waves, rogue waves, plasmas, and nonlinear optics, including atomic physics. The NLSE serves as a comprehensive description of nonlinear dispersive processes. With all of these in mind, we found the courage and motivation in this study to provide the distinct types of exact soliton solutions for the weakly nonlocal Schrodinger equation. We obtain dark and periodic singular soliton solutions via the reliable approach of the modified extended tanh function method. The obtained solutions were verified by back-substitution in the original equations, with the aid of a Mathematica to affirm the robustness of the chosen approach. The obtained results were portrayed using two-dimensional and three-dimensional graphs. Additionally, modulation instability was performed to study the stationary state of the governing model. Results are helpful in the progress of the concerned system. The gained results will be of high importance in the interaction of quantum-mechanical fluctuations, granular matters, and other fields of weakly nonlocal Schrodinger with parabolic law applications. The achieved results are also useful in various naturally occurring phenomena, industries, geophysics, civil engineering, pharmaceutical, and many others. It is suggested that the method used is also useful for the other nonlinear models of different fields of science and engineering.

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