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Three Step Inverse Free Kurchatov-Like Methods of Convergence Order Close to Four for Equations

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Abstract. An inverse free Kurchatov-like methods with three steps is introduced of convergence order close to four to generate sequences approximating solutions of equations defined on complete normed spaces. The local analysis shows R-convergence close to four under conditions controlling the divided difference. Numerous experiments demonstrate the performance of the method.

1. Introduction

Let E_1, E_2 represent complete normed spaces [1] and $\Omega \subset E_1$ be open and convex. A plethora of applications from different fields can be formulated as

$$F(x) = 0. \tag{1.1}$$

Here $F : \Omega \to E_2$ is a continuos operator. A solution of equation (1.1), which is denoted by $x^* \in \Omega$ is given in analytical form only in rare cases. That leads to the development of iterative methods generating sequence converging to x^* provided some conditions are satisfied involving the initial information.

One of the most time-consuming part of iterative methods for solving nonlinear problems is finding the inverse operator or solving the corresponding linear problem. To avoid this, methods

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with approximation of the inverse operator were developed. One of the first such methods was proposed by Ulm in [17]

$$\begin{aligned} x_{n+1} &= x_n - T_n F(x_n), \\ T_{n+1} &= 2T_n - T_n F'(x_{n+1}) T_n, \ n = 0, 1, 2, \ldots. \end{aligned} \tag{1.2}$$

Here $x_0 \in \Omega$ and $T_0 \in L(E_2, E_1)$ are given initial approximations for the solution x^* and the inverse operator $F'(x^*)^{-1}$, respectively. The Ulm method (1.2) was studied under conditions of different types and it was shown that it converges with second order [2,3,10,12,13,15,17]. Similar method was prosed by Moser [13,14]

$$\begin{aligned} x_{n+1} &= x_n - T_n F(x_n), \\ T_{n+1} &= 2T_n - T_n F'(x_n) T_n, \ n = 0, 1, 2, \ldots. \end{aligned} \tag{1.3}$$

In (1.3) $F'(x_n)$ appears instead of $F'(x_{n+1})$ in (1.2). The convergence order for (1.3) is equal to $\frac{1+\sqrt{5}}{2}$. The methods with approximation of the inverse operator with higher convergence order were studied in [6,9].

In this paper we propose the three step Kurchatov-like method (TSKLM). This method is defined for $T_0 \in L(E_2, E_1)$ and each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - T_n F(x_n), \\ z_n &= y_n - T_n F(y_n), \\ x_{n+1} &= z_n - T_n F(z_n), \\ K_{n+1} &= [2y_n - x_n, x_n; F], \\ M_n &= 2T_n - T_n K_{n+1} T_n, \\ T_{n+1} &= M_n + M_n(2I - K_{n+1} M_n)(I - K_{n+1} M_n), \end{aligned} \tag{1.4}$$

where $[\cdot, \cdot; F] : \Omega \times \Omega \to E_2$ is a divided difference of order one [1,7] and $L(E_2, E_1)$ is the space of linear operators mapping E_2 into E_1, $x_0 \in \Omega$.

Definition 1.1. *[1,7] Let F be a nonlinear operator defined on a subset Ω of a Banach space E_1 with values in a Banach space E_2, and let x, y, be two different points of Ω. A linear operator from E_1 to E_2 which is denoted by $[x, y; F]$ and satisfies the following conditions*

$$[x, y; F](x - y) = F(x) - F(y)$$

is called a first-order divided difference of F at the points x and y.

If there exists a Fréchet derivative $F'(x)$, then

$$[x, x; F] = F'(x).$$

Notice that there are other selection for the Kurchatov operator K_{n+1} such as $K_{n+1} = [2x_{n+1} - z_n, z_n; F]$ or $K_{n+1} = [2x_{n+1} - x_n, x_n; F]$ or $K_{n+1} = F'(x_{n+1})$ or $K_{n+1} = L(x_{n+1})$, where $L(x_{n+1})$ is an approximation to $F'(x_{n+1})$, or other options [1,4,5,8,16]. Denote the corresponding methods by

$$\begin{aligned} y_n &= x_n - T_nF(x_n),\\ z_n &= y_n - T_nF(y_n),\\ x_{n+1} &= z_n - T_nF(z_n),\\ K_{n+1} &= [2x_{n+1} - z_n, z_n; F],\\ M_n &= 2T_n - T_nK_{n+1}T_n,\\ T_{n+1} &= M_n + M_n(2I - K_{n+1}M_n)(I - K_{n+1}M_n), \end{aligned} \tag{1.5}$$

$$\begin{aligned} y_n &= x_n - T_nF(x_n),\\ z_n &= y_n - T_nF(y_n),\\ x_{n+1} &= z_n - T_nF(z_n),\\ K_{n+1} &= [2x_{n+1} - x_n, x_n; F],\\ M_n &= 2T_n - T_nK_{n+1}T_n,\\ T_{n+1} &= M_n + M_n(2I - K_{n+1}M_n)(I - K_{n+1}M_n), \end{aligned} \tag{1.6}$$

$$\begin{aligned} y_n &= x_n - T_nF(x_n),\\ z_n &= y_n - T_nF(y_n),\\ x_{n+1} &= z_n - T_nF(z_n),\\ K_{n+1} &= F'(x_{n+1}),\\ M_n &= 2T_n - T_nK_{n+1}T_n,\\ T_{n+1} &= M_n + M_n(2I - K_{n+1}M_n)(I - K_{n+1}M_n), \end{aligned} \tag{1.7}$$

$$\begin{aligned} y_n &= x_n - T_nF(x_n),\\ z_n &= y_n - T_nF(y_n),\\ x_{n+1} &= z_n - T_nF(z_n),\\ K_{n+1} &= L(x_{n+1}),\\ M_n &= 2T_n - T_nK_{n+1}T_n,\\ T_{n+1} &= M_n + M_n(2I - K_{n+1}M_n)(I - K_{n+1}M_n), \end{aligned} \tag{1.8}$$

respectively. Notice that method (1.8) specializes to (1.7) if $L = F'$. A possible choice for L is presented in the numerical section.

To test numerically the order of convergence of the iterative methods very often use computational order of convergence (COC) and approximated computational order of convergence (ACOC) [11]. COC is denoted by δ_2, ACOC can be computed by formulas denoted by δ_1 and δ_3:

$$\delta_1 \approx \frac{\ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right)}{\ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right)}, \quad \delta_2 \approx \frac{\ln\left(\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}\right)}{\ln\left(\frac{\|x_n-x^*\|}{\|x_{n-1}-x^*\|}\right)}, \quad \delta_3 \approx \frac{\ln\left(\frac{\|F(x_{n+1})\|}{\|F(x_n)\|}\right)}{\ln\left(\frac{\|F(x_n)\|}{\|F(x_{n-1})\|}\right)}.$$

In this article, we provide the local convergence analysis of the method (1.4) under assumptions that Fréchet derivative and first-order divided differences satisfy classical Lipschitz conditions (see Section 2). Sections 3 and 4 present results of numerical experiments and conclusions, respectively.

2. Convergence

The local analysis of convergence is very important since it provides the degree of difficulty in selecting the initial points x_0 from a ball centered at the solution x^* and of a certain specified radius.

The symbol $S(x, r)$ is denoting an open ball centered at $x \in E_1$ and of a radius $r > 0$.

Define the parameter ρ by

$$\rho = \sup\{t > 0 : S(x^*, t) \subset \Omega\}.$$

Let $a > 0$, $b_0 \geq 0$, $b > 0$, $d_1 > 0$, $d_2 > 0$, $\lambda \geq 0$ and $l \geq 0$ be given parameters. It is convenient to define the parameters

$$\begin{aligned}
q &= \frac{b}{a}, \\
\rho_0 &= \min\left\{1, \rho, \frac{1}{4lb(l+d_1)d_2}\right\}, \\
\bar{\rho} &\in (0, \rho_0), \\
c_1 &= 4lb(l+d_1), \\
c_2 &= 1 + 2bc_1, \\
c_3 &= c_2 + 2lb, \\
c_4 &= c_2 + 4lb(1+c_3), \\
c_5 &= c_4 + 4lb(1+c_3), \\
c_6 &= 1 + 4bc_1 + 8lb, \\
b_0 &= \frac{d_2}{1 - c_1 d_2 \bar{\rho}} \quad for\ \bar{\rho} \in (0, \rho_0), \\
a &= \min\left\{1, \frac{1}{c_3c_4 + 2lbc_3^2}, \frac{1}{c_5 + 2lb}, \frac{1}{c_6^6}\right\}, \\
&\textit{and} \\
b &= \min\{\bar{\rho}, a\}.
\end{aligned} \tag{2.1}$$

Let $x^* \in \Omega$ be a solution of the equation $F(x) = 0$. We assume from now on that for each $u_1, u_2, v_1, v_2 \in S(x^*, \rho)$ there exists $l > 0$ such that

$$\|[u_2, u_1; F] - [v_2, v_1; F]\| \leq l(\|u_2 - v_2\| + \|u_1 - v_1\|). \tag{2.2}$$

It follows by (2.2) that F' exists, $[x, x; F] = F'(x)$ and for each $u, v \in S(x^*, \bar{\rho})$

$$\|F'(u) - F'(v)\| \leq 2l\|u - v\|. \tag{2.3}$$

Moreover, assume $F'(x^*)$ is invertible and set

$$\|F'(x^*)\| \leq d_1, \ \|F'(x^*)^{-1}\| \leq d_2. \tag{2.4}$$

Furthermore, assume T_0, K_0 satisfy

$$\|T_0\| \leq b_0 \leq b,\ \|I - T_0K_0\| \leq \lambda,\ b \geq b_0 \text{ and } \lambda \in [0, b]. \tag{2.5}$$

Finally, assume

$$S(x^*, \rho^*) \subset \Omega,\ \rho^* = \max\{\bar{\rho}, 3b\}. \tag{2.6}$$

The conditions (2.2), (2.4)-(2.6) are called (A) from now on.

Next, the main local analysis of convergence is presented under the conditions (A) and the preceding notation.

THEOREM 2.1. *Assume that the conditions (A) hold and $b < a$. Then, the sequence $\{x_n\}$ generated by (TSKLM) converges to the solution of the equation $F(x) = 0$ provided that $x_0 \in S(x^*, b)$. Moreover, the following assertions hold for $e_n = \|x_n - x^*\|$, $h_n = \|I - T_nK_n\|$,*

$$e_n \leq aq^{4^n} \tag{2.7}$$

and

$$h_n \leq aq^{4^{n-1}},\ h_0 \leq b, \tag{2.8}$$

$n = 0, 1, 2, \ldots$, where ρ, a, b, q are given by (2.1).

Proof. The assertions (2.7) and (2.8) are shown by induction. If $n = 0$, assertion (2.7) for $n = 0$ holds, since $e_0 \leq b$. Then, by (2.5), we get $a \in [0, 1]$ and $h_0 \leq \lambda \leq b$, so (2.8) holds if $n = 0$. Assume (2.7) and (2.8) hold if $n = i$. That is

$$e_i \leq aq^{4^i} < b < \bar{\rho} \tag{2.9}$$

and

$$h_i \leq aq^{4^{i-1}}. \tag{2.10}$$

We need the estimate

$$\begin{aligned}
\|K_{i+1} - F'(x_i)\| &= \|[2y_i - x_i, x_i; F] - [x_i, x_i; F]\| \\
&\leq l(\|2y_i - x_i - x_i\| + \|x_i - x_i\|) = 2l\|y_i - x_i\| \\
&\leq 2l\|T_iF(x_i)\| \leq 2l\|T_i\|(l + d_1)\|x_i - x^*\| \\
&\leq c_1 e_i \leq c_1 aq^{4^i},
\end{aligned} \tag{2.11}$$

where we used (2.1), (2.2) (i.e (2.3)), (2.5), the induction hypotheses (2.9), (2.10) and

$$\begin{aligned}
F(x_i) &= F(x_i) - F(x^*) = \int_0^1 F'(x^* + \theta(x_i - x^*))d\theta(x_i - x^*) \\
&= \int_0^1 \Big[F'(x^* + \theta(x_i - x^*)) - F'(x^*) + F'(x^*)\Big]d\theta(x_i - x^*)
\end{aligned}$$

leading to

$$\|F(x_i)\| \leq (l + d_1)\|x_i - x^*\|. \tag{2.12}$$

It follows by the Banach Lemma on invertible operators [1], (2.1) and (2.11) that the operator K_{i+1} is invertible and

$$\|K_{i+1}^{-1}\| \leq \frac{\|F'(x^*)^{-1}\|}{1 - \|F'(x^*)^{-1}\|\|K_{i+1} - F'(x_i)\|} \leq \frac{d_2}{1 - d_2 c_1 e_i} \leq \frac{d_2}{1 - d_2 c_1 \bar{\rho}} = b_0 \leq b, \tag{2.13}$$

so

$$\begin{aligned} \|T_i\| &= \|T_i K_i K_i^{-1}\| = \|(-I + (I - T_i K_i))K_i^{-1}\| \\ &\leq (1 + \|I - T_i K_i\|)\|K_i^{-1}\| \\ &\leq (1 + aq^{4^i})b \leq (1+1)b = 2b, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \xi_i = \|I - T_i F'(x_i)\| &= \|(I - T_i K_i) + (T_i K_i - T_i F'(x_i))\| \\ &\leq \|I - T_i K_i\| + \|T_i\|\|K_i - F'(x_i)\| \\ &\leq aq^{4^i} + 2bc_1 aq^{4^i} = c_2 aq^{4^i}. \end{aligned} \tag{2.15}$$

Then, by the first substep of (TSKLM) we can write in turn

$$\begin{aligned} y_i - x^* &= x_i - x^* - T_i(F(x_i) - F(x^*)) \\ &= \int_0^1 \Big[(I - T_i F'(x_i)) + T_i(F'(x_i) - F'(x^* + \theta(x_i - x^*)))\Big](x_i - x^*)d\theta. \end{aligned} \tag{2.16}$$

It follows by the induction hypotheses (2.9), (2.10), the (2.16) triangle inequality, (2.3), and (2.13)-(2.16), we have in turn that

$$\begin{aligned} \|y_i - x^*\| &= \xi_i e_i + l\|T_i\|e_i^2 \\ &\leq c_2 aq^{4^i} aq^{4^i} + 2lb\left(aq^{4^i}\right)^2 \leq c_3 a^2 q^{2\times 4^i}. \end{aligned} \tag{2.17}$$

and since $a \in [0,1]$

$$\begin{aligned} \|x_i - y_i\| &\leq e_i + \|y_i - x^*\| \\ &\leq aq^{4^i} + c_3 a^2 q^{2\times 4^i} \leq (1 + c_3)aq^{4^i} \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \|I - T_i F'(y_i)\| &= \|(I - T_i F'(x_i)) + (T_i F'(x_i) - T_i F'(y_i))\| \\ &\leq c_2 aq^{4^i} + 4lb(1 + c_3)aq^{4^i} = c_4 aq^{4^i}. \end{aligned} \tag{2.19}$$

In an analogous way, we get in turn

$$\begin{aligned} \|z_i - x^*\| &\leq \|(I - T_i F'(y_i))(y_i - x^*)\| + l\|T_i\|\|y_i - x^*\|^2 \\ &\leq c_4 a q^{4^i} c_3 a^2 q^{2\times 4^i} + l(2bc_3^2 a^4 q^{4\times 4^i}) \\ &\leq (c_3 c_4 + 2lbc_3^2) a^2 q^{3\times 4^i} = a^2 q^{3\times 4^i}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \|y_i - z_i\| &\leq \|y_i - x^*\| + \|z_i - x^*\| \\ &\leq c_3 a^2 q^{2\times 4^i} + a^2 q^{3\times 4^i} = (1 + c_3) a^2 q^{2\times 4^i} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \|I - T_i F'(z_i)\| &\leq \|I - T_i F'(y_i)\| + \|T_i\|\|F'(y_i) - F'(z_i))\| \\ &\leq c_4 a q^{4^i} + 4lb(1 + c_3) a^2 q^{2\times 4^i} = c_5 a q^{4^i} \end{aligned} \tag{2.22}$$

leading to

$$\begin{aligned} e_{i+1} &\leq \|I - T_i F'(z_i)\|\|z_i - x^*\| + l\|T_i\|\|z_i - x^*\|^2 \\ &\leq c_5 a q^{4^i} a^2 q^{3\times 4^i} + l(2ba^4 q^{6\times 4^i}) \\ &\leq (c_5 + 2lb) q^{4^{i+1}} \leq a q^{4^{i+1}} \end{aligned} \tag{2.23}$$

showing (2.7) for $n = i + 1$. Moreover, we have

$$\begin{aligned} \|x_{i+1} - x_i\| &\leq \|x_{i+1} - x^*\| + \|x_i - x^*\| \\ &\leq a q^{4^{i+1}} + a q^{4^i} \leq 2a q^{4^i} \end{aligned} \tag{2.24}$$

Notice that $2y_i - x_i \in S(x^*, 3b)$ by (2.6) and $e_{i+1} \leq q^{4^{i+1}} < b < \bar{\rho}$.

Hence, we can have

$$\begin{aligned} \|K_{i+1} - K_i\| &\leq \|(K_{i+1} - F'(x_i)) + (F'(x_i) - F'(x_{i-1})) + (F'(x_{i-1}) - K_i)\| \\ &\leq c_1 a q^{4^i} + 4la q^{4^{i-1}} + c_1 a q^{4^{i-1}} \\ &\leq (2c_1 + 4l) a q^{4^{i-1}} \end{aligned} \tag{2.25}$$

$$\begin{aligned} \|I - T_i K_{i+1}\| &= \|(I - T_i K_i) + (T_i K_{i+1} - T_i K_i)\| \\ &\leq a q^{4^i} + 2b(2c_1 + 4l) a q^{4^{i-1}} = c_6 a q^{4^{i-1}}. \end{aligned} \tag{2.26}$$

Next, by the fourth equation of (TSKLM), we can write $I - M_i K_{i+1} = (I - T_i K_{i+1})^2$, so

$$\|I - M_i K_{i+1}\| \leq \|I - T_i K_{i+1}\|^2 \leq c_6^2 a^2 q^{2\times 4^{i-1}}. \tag{2.27}$$

But, we can also write

$$T_{i+1} = M_i + M_i(2I - K_{i+1} M_i)(I - K_{i+1} M_i). \tag{2.28}$$

Thus, we get

$$I - T_{i+1}K_{i+1} \quad = \quad I - (M_i + M_i(2I - K_{i+1}M_i)(I - K_{i+1}M_i))K_{i+1} = (I - M_iK_{i+1})^3. \quad (2.29)$$

Therefore, since $a \in [0, 1]$, we get by the inductions hypotheses and (2.29) that

$$\|I - T_{i+1}K_{i+1}\| \quad \leq \quad \|I - M_iK_{i+1}\|^3 \leq c_6^6 a^6 q^{6\times 4^{i-1}} \leq aq^{4^i}$$

showing (2.8) for $n = i + 1$.

Consequently the induction for (2.7) and (2.8) is completed.

Finally, by letting $n \to \infty$ in (2.7), we deduce that $\lim_{n\to\infty} x_n = x^*$, since $q \in [0, 1)$.

REMARK 2.2. *It turns out that the proof of Theorem 2.1 can be repeated in the case of the method (1.8) as long as we add an additional condition of the form for each* $n = 0, 1, 2, \ldots$

$$\|L_n - F'(x_n)\| \leq \sigma_n\|F(x_n)\|, \quad (2.30)$$

where $\{\sigma_n\}$ *is a nonnegative sequence such that* $\sup_{n\geq 0} \sigma_n \leq \sigma$, *where* $\sigma \geq 0$

3. Numerical Examples

In this section, we present the results of numerical investigation of the three-step Kurchatov-like method for solving the nonlinear equation (1.1). We give errors at each iteration and ACOC and COC for considered methods (1.4), (1.5), (1.6), (1.7), (1.8). The computations were carried out on a PC with 1.00 GHz processor and 8 GB of memory with use of software GNU Octave 7.3.0. The Euclidean norm was used. The initial approximation T_0 was computed by formulas $T_0 = [2x_0 - x_{-1}, x_{-1}; F]^{-1}$ for methods (1.4), (1.5), (1.6), $T_0 = F'(x_0)^{-1}$ – for method (1.7) and $T_0 = L(x_0)^{-1}$ – for method (1.8), where (a) $L(x) = [x, x + \alpha; F]$ and (b) $L(x) = [x + \alpha_1 F(x), x + \alpha_2 F(x); F]$ with $\alpha \in \mathbb{R}$.

EXAMPLE 3.1. *Let* $F : \mathbb{R} \to \mathbb{R}$ *and consider the nonlinear equation*

$$F(x) = e^{x-0.1} - 10x|x - 1| - 0.1 = 0$$

with the exact solution $x^* = 0.1$.

EXAMPLE 3.2. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *and consider the system of two equations with* $x = (\xi; \eta)$

$$\begin{cases} F_1(x) = 3\xi^2\eta + \eta^2 + |\xi - 1| - 0.75 = 0, \\ F_2(x) = \xi^4 + \xi\eta^3 + |\eta| - 0.5625 = 0, \end{cases}$$

and the exact solution $x^* = (0.5; -1)$.

TABLE 1. Error's value at each iteration for Example 3.1.

n	Method (1.4)		Method (1.5)		Method (1.6)	
	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$
0	6.0000e-01	7.9488e+00	6.0000e-01	7.9488e+00	6.0000e-01	7.9488e+00
1	6.0089e-02	4.5850e-01	6.0089e-02	4.5850e-01	6.0089e-02	4.5850e-01
2	2.3413e-03	1.6447e-02	2.1002e-04	1.4706e-03	1.9390e-04	1.3577e-03
3	3.4615e-07	2.4231e-06	3.2876e-14	2.3012e-13	1.2934e-14	9.0566e-14
4	1.3878e-17	8.3267e-17				

TABLE 2. Error's value at each iteration for Example 3.1 (Method (1.8)).

n	(a), $\alpha = 10^{-6}$		(b), $\alpha_1 = 0$, $\alpha_2 = 0.01$	
	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$
0	6.0000e-01	7.9488e+00	6.0000e-01	7.9488e+00
1	6.0098e-02	4.5857e-01	5.2350e-02	3.9520e-01
2	2.1022e-04	1.4720e-03	1.2109e-04	8.4780e-04
3	3.2724e-14	2.2890e-13	3.0254e-15	2.1178e-14

TABLE 3. Error's value at each iteration for Example 3.1 (Method (1.8)).

n	(b), $\alpha_1 = -1$, $\alpha_2 = 1$		(b), $\alpha_1 = 0$, $\alpha_2 = 1$		(b), $\alpha_1 = -1$, $\alpha_2 = 0$	
	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$
0	2.2500e-01	2.1048e+00	2.2500e-01	2.1048e+00	2.2500e-01	2.1048e+00
1	4.3916e-02	3.2765e-01	2.8627e-04	2.0031e-03	8.9687e-02	7.1215e-01
2	1.1100e-04	7.7714e-04	3.3741e-12	2.3618e-11	1.4334e-02	1.0249e-01
3	2.4702e-15	1.7292e-14			6.9350e-05	4.8550e-04
4					5.5539e-14	3.8877e-13

TABLE 4. COC, ACOC for Example 3.1.

Method	δ_1	δ_2	δ_3
Method (1.4)	2.7545	2.7145	2.7309
Method (1.5)	3.1543	3.9915	3.9319
Method (1.6)	4.0870	4.0870	4.0244
Method (1.8) (a), $\alpha = 10^{-6}$	3.9956	3.9931	3.9935
Method (1.8) (b), $\alpha_1 = 0$, $\alpha_2 = 0.01$	3.2266	4.0224	3.9731

Tables 1, 2, 3, 6 and 7 contain the values $\|x_n - x^*\|$ and $\|F(x_n)\|$ at each iteration. The iterative process was stopped if $\|F(x_n)\| \le 10^{-10}$.

Table 5. COC, ACOC for Example 3.1.

Method	δ_1	δ_2	δ_3
Method (1.8) (b), $\alpha_1 = -1$, $\alpha_2 = 1$	3.2944	4.1014	4.0583
Method (1.8) (b), $\alpha_1 = 0$, $\alpha_2 = 1$	2.0169	2.7384	2.6240
Method (1.8) (b), $\alpha_1 = -1$, $\alpha_2 = 0$	3.0250	3.9288	3.9133

Table 6. Error's value at each iteration for Example 3.2.

n	Method (1.4)		Method (1.5)		Method (1.6)	
	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$
0	2.9069e-01	4.9986e-01	2.9069e-01	4.9986e-01	2.9069e-01	4.9986e-01
1	1.3484e-02	8.8111e-03	1.3484e-02	8.8111e-03	1.3484e-02	8.8111e-03
2	1.1530e-04	9.6933e-05	1.5071e-05	9.5531e-06	7.9188e-05	5.5281e-05
3	1.5872e-10	9.9579e-11	1.1102e-16	1.1102e-16	4.7092e-11	4.4613e-11

Table 7. Error's value at each iteration for Example 3.2 (Method (1.8)).

n	(a), $\alpha = 10^{-6}$		(b), $\alpha_1 = -1$, $\alpha_2 = 1$		(b), $\alpha_1 = 0$, $\alpha_2 = 1$		(b), $\alpha_1 = -1$, $\alpha_2 = 0$	
	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$	$\|x_n - x^*\|$	$\|F(x_n)\|$
0	2.9069e-01	4.9986e-01	2.9069e-01	4.9986e-01	2.9069e-01	4.9986e-01	2.9069e-01	4.9986e-01
1	4.2177e-02	3.3550e-02	1.2130e-01	1.3882e-01	1.2302e-01	1.1058e-01	2.6651e-02	9.4616e-02
2	4.8296e-04	3.1889e-04	1.9815e-02	1.6501e-02	1.7140e-02	1.1420e-02	7.4376e-04	8.7777e-04
3	1.2251e-11	7.9157e-12	9.5643e-05	7.1351e-05	5.0435e-05	3.1888e-05	7.0982e-11	9.6911e-11
4			5.0469e-14	3.6791e-14	3.5108e-15	2.2453e-15		

Table 8. COC, ACOC for Example 3.2.

Method	δ_1	δ_2	δ_3
Method (1.4)	2.8293	2.8343	3.0575
Method (1.5)	3.2733	3.7717	3.6881
Method (1.6)	2.7872	2.7904	2.7665
Method (1.8) (a), $\alpha = 10^{-6}$	3.9231	3.9130	3.7611
Method (1.8) (b), $\alpha_1 = -1$, $\alpha_2 = 1$	3.1016	4.0053	3.9286
Method (1.8) (b), $\alpha_1 = 0$, $\alpha_2 = 1$	3.1851	4.0127	3.9750
Method (1.8) (b), $\alpha_1 = -1$, $\alpha_2 = 0$	3.1562	4.5167	3.4227

Tables 4, 5 and 8 show COC and ACOC with δ_1 was calculated if the condition $\|x_{n+1} - x_n\| \le 10^{-10}$ was fulfilled δ_2 and δ_3 were calculated if the condition $\|F(x_n)\| \le 10^{-10}$ was fulfilled.

The initial approximations x_0 and x_{-1} for Example 3.1 are $x_0 = -0.5$, $x_{-1} = -0.6$ (Tables 1, 2, 4), $x_0 = -0.15$, $x_{-1} = -0.25$ (Tables 3, 5) and for Example 3.2 – $x_0 = (0.63; -1.26)$, $x_{-1} = (0.73; -1.16)$.

EXAMPLE 3.3. *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ and consider the system of equations with $x = (\xi_1; \ldots; \xi_m)$*

$$F_i(x) = \sum_{j=1}^{m} \xi_j + e^{\xi_i} - 1 = 0, \quad i = 1, \ldots, m.$$

Here the exact solution $x^* = (0; \ldots; 0)$.

TABLE 9. COC, ACOC for Example 3.3.

Method	δ_1	δ_2	δ_3
Method (1.4)	2.6007	2.5860	2.6621
Method (1.5)	3.9096	3.9035	3.7967
Method (1.6)	2.0176	2.0059	2.0663
Method (1.7)	3.9118	3.9059	3.7993
Method (1.8) (a), $\alpha = 10^{-6}$	3.9117	3.9058	3.7992
Method (1.8) (b), $\alpha_1 = 0$, $\alpha_2 = -0.1$	3.9097	3.9064	3.9064
Method (1.8) (c)	2.8165	2.8012	2.8803

TABLE 10. COC, ACOC for Example 3.3.

Method	δ_1	δ_2	δ_3
Method (1.8) (b), $\alpha_1 = -1$, $\alpha_2 = 1$	4.3139	4.3096	4.2617
Method (1.8) (b), $\alpha_1 = 0$, $\alpha_2 = 1$	2.7472	2.7326	2.8050
Method (1.8) (b), $\alpha_1 = -1$, $\alpha_2 = 0$	3.9118	3.8946	3.8502

For solving nonlinear equations with differentiable operator can be also used method (1.8) with (c) $L(x_{n+1}) = \frac{1}{2}(F'(x_{n+1}) + [2x_{n+1} - x_n, x_{n+1}; F])$. For this method, the errors decrease faster than for (1.4) and (1.6).

Tables 9 and 10 show COC and ACOC for Example 3.3 with $m = 1$, the initial approximation $x_0 = 0.5$ and $x_0 = 0.9$, respectively.

Figures 1 and 2 show changing of $\|x_n - x_{n-1}\|$, $\|x_n - x^*\|$ and $\|F(x_n)\|$ for $m = 20$, $x_0 = (5; \ldots; 5)$, $x_{-1} = (5.1; \ldots; 5.1)$, Figure 3 – for $x_0 = (0.05; \ldots; 0.05)$. The iterative process was stopped under the condition $\|x_{n+1} - x_n\| \leq 10^{-10}$.

From the obtained results, we see that among methods (1.4)–(1.6), for method (1.5) the error decreases faster and it has highest computational order of convergence.

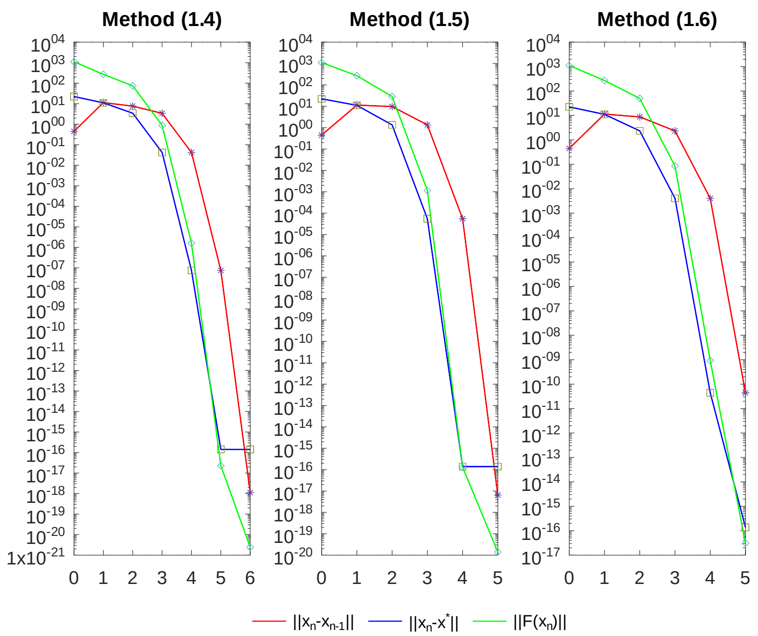

FIGURE 1. Example 3.3: error's value at each iteration.

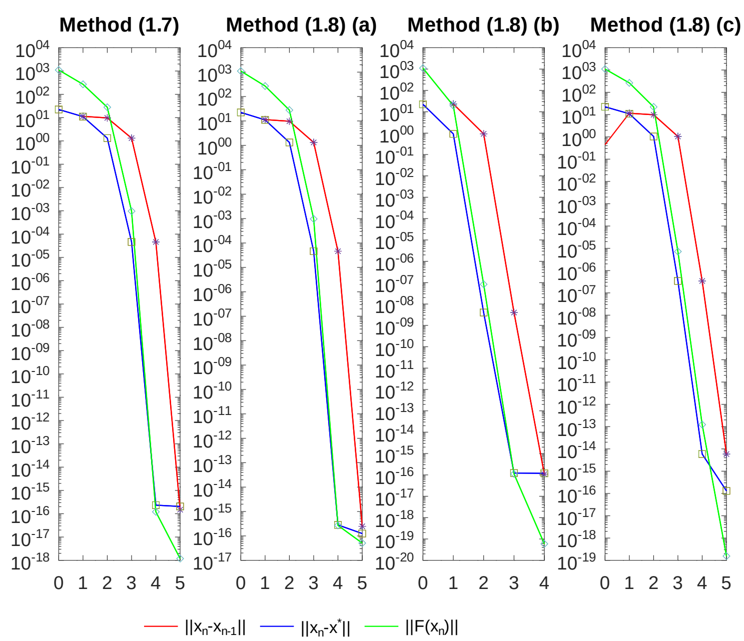

FIGURE 2. Example 3.3: error's value at each iteration; (b), $\alpha_1 = 0$, $\alpha_2 = -0.1$.

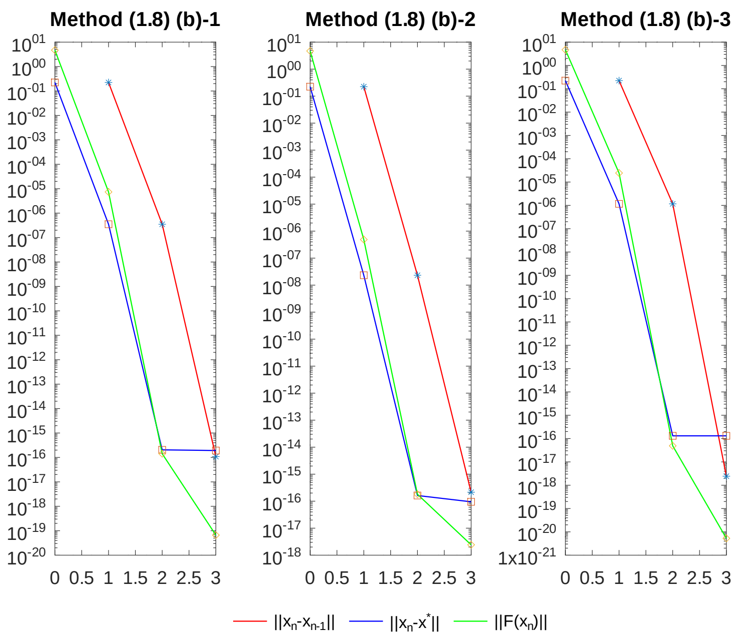

Figure 3. Example 3.3: error's value at each iteration ((b)-1, $\alpha_1 = -1$, $\alpha_2 = 1$; (b)-2, $\alpha_1 = 0$, $\alpha_2 = 1$; (b)-3, $\alpha_1 = -1$, $\alpha_2 = 0$).

4. Conclusion

In this article a three-step Kurchatov-like method with approximation of inverse operator is introduced and convergence analysis is provided. The R-convergence four is shown theoretically under Lipschitz conditions for Fréchet derivative and first-order divided differences. Numerous experiments demonstrate the performance of the method for different cases of K_{n+1}. Among the methods with divided differences, the best results are demonstrated by the methods (1.5), namely the highest computational order of convergence. The operator L was chosen in the form $L(x_{n+1}) = [x_{n+1}, x_{n+1} + \alpha; F]$, where α is a small number, and $L(x_{n+1}) = [x_{n+1} + \alpha_1 F(x_{n+1}), x_{n+1} + \alpha_2 F(x_{n+1}); F]$ with α_1, $\alpha_2 \in \mathbb{R}$. These approximating of the derivative give quite good results, in particular if the nonlinear operator is not differentiable. For some values α_1 and α_2 these methods require a good initial approximation. In the case of the differentiable operator, the following choice $L(x_{n+1}) = \frac{1}{2}(F'(x_{n+1}) + [2x_{n+1} - x_n, x_{n+1}; F])$ is also possible. It demonstrates advantages over some methods with divided differences.

References

[1] I. K. Argyros, Convergence and Applications of Newton-type Iterations, New York: Springer-Verlag, 2008. https://doi.org/10.1007/978-0-387-72743-1.

[2] I. K. Argyros, On Ulm's Method for Frechet Differentiable Operators, J. Appl. Math. Comput. 31 (2009) 97-111. https://doi.org/10.1007/s12190-008-0194-5.

[3] I. K. Argyros, On Ulm's Method Using Divided Differences of Order One, Numer. Algorithms 52 (2009) 295-320. https://doi.org/10.1007/s11075-009-9274-3.

[4] I.K. Argyros, S. Shakhno, S. Regmi, H. Yarmola, On the Complexity of a Unified Convergence Analysis for Iterative Methods, J. Complex. 79 (2023) 101781. https://doi.org/10.1016/j.jco.2023.101781.

[5] I.K. Argyros, S. Shakhno, H. Yarmola, Improving Convergence Analysis of the Newton–Kurchatov Method under Weak Conditions Two-Step Solver for Nonlinear Equations, Computation 8 (2020) 8. https://doi.org/10.3390/computation8010008.

[6] I.K. Argyros, S.M. Shakhno, H.P. Yarmola, Method of Third-Order Convergence with Approximation of Inverse Operator for Large Scale Systems, Symmetry 12 (2020), 978. https://doi.org/10.3390/sym12060978.

[7] M. Balázs, G. Goldner, On Existence of Divided Differences in Linear Spaces, Rev. Anal. Numér. Théorie Approx. 2 (1973) 5-9. https://doi.org/10.33993/jnaat21-6.

[8] J.E. Dennis, R.B. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear Equations, Prentice-Hall, Englewoods Cliffs, 1983.

[9] J. A. Ezquerro, M. A. Hernández, An Ulm-type Method with R-order of Convergence Three, Nonlinear Anal.: Real World Appl. 13 (2012) 14-26. https://doi.org/10.1016/j.nonrwa.2011.07.039.

[10] J. A. Ezquerro, M. A. Hernández, The Ulm Method Under Mild Differentiability Conditions, Numer. Math. 109 (2008) 193-207. https://doi.org/10.1007/s00211-008-0144-z.

[11] M. Grau-Sánchez, M. Noguera, J.M. Gutiérrez, On Some Computational Orders of Convergence, Appl. Math. Lett. 23 (2010) 472-478. https://doi.org/10.1016/j.aml.2009.12.006

[12] J. M. Gutiérrez, M. A. Hernández, N. Romero, A Note on a Modification of Moser's Method, J. Complex. 24 (2008), 185-197. https://doi.org/10.1016/j.jco.2007.04.003.

[13] O. H. Hald, On a Newton-Moser Type Method, Numer. Math. 23 (1975), 411-426. https://doi.org/10.1007/BF01437039.

[14] J. Moser, Stable and Random Motions in Dynamical Systems: with Special Emphasis on Celestial Mechanics. Herman Weil Lectures, Annals of Mathematics Studies, vol. 77, Princeton University Press, Princeton, NJ, 1973. https://www.jstor.org/stable/j.ctt1bd6kg5.

[15] H. Petzeltova, Remark on Newton-Moser Type Method, Commentat. Math. Univ. Carol. 21 (1980), 719-725. https://zbmath.org/0455.65042.

[16] S. M. Shakhno, Nonlinear Majorants for Investigation of Methods of Linear Interpolation for the Solution of Nonlinear Equations, European Congress on Computational Methods in Applied Sciences and Engineering ECCOMAS 2004 – P. Neittaanmäki, T. Rossi, K. Majava and O. Pironneau (eds.) O. Nevanlinna and R. Rannacher (assoc. eds.) Yuväskylä, 24-28 July 2004, 11 p. https://www.researchgate.net/publication/238701776.

[17] S. Ulm, On Iterative Methods With Successive Approximation of the Inverse Operator, Izv. Akad. Nauk Est. SSR, 16 (1967), 403-411. (in Russian).