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On the Cumulative Distribution Function of the Difference of Two Dependent Chi Square Random Variables

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Abstract. In this article, we reexamine the derivation of the cumulative distribution function of the difference of two dependent chi-square random variables with the same degrees of freedom. We derive the cdf for this difference for even degrees of freedom and discuss a discrepancy that we have found with a reported cdf of this difference for even degrees of freedom in [6]. For odd degrees of freedom, an expression for the cdf seems to be unknown. In this case, we derive a representation of the cdf in terms of the Meijer G-function. These representations allowed us to compute percentiles for even and odd degrees of freedom.

1. Introduction

In the algebra of random variables, finding the probability density function (pdf) and the cumulative distribution function (cdf) of the difference and the sum of two random variables (rvs) are standard problems. It is well known that such combinations of rvs appear within the theory of statistics and in its applications. It is also clear that when the two rvs are dependent, the analysis of the problem is more technically complicated. In particular, in this note we focus mainly on deriving the cdf for the difference of two dependent central chi-square random variables with the same number of degrees of freedom. Results on this problem seem to have been around for a while and are reported in some detail in [6]. Not aware initially of the results in [6], we have carried out the basic analysis and derived the cdf. The cdf expressions which we have obtained appear in different forms than those reported in [6]. In an attempt to see how these different representations correspond, we discovered a discrepancy between the two forms of the cdfs. In the analysis below we give

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a modification to the cdf expression for even degrees of freedom reported in [6]. The cdf for the difference of dependent central chi square rvs with odd degrees of freedom seems to be unknown, and it is not reprted in [6]. A representation of the cdf in terms of a Meijer G-functions seems to fill in this gap. The article is organized as follows. We first discuss a bivariate chi square distribution of the Kibble-type on which the rest of the analysis is based. Then we present and discuss the pdf of the difference of two dependent chi square rvs, followed by a derivation of the cdf of this difference. We end the article by computing a sampling of percentiles from the cdf expressions for both even and odd degrees of freedom.

2. Density Functions

In this section, we examine a joint pdf for correlated gamma rvs derived by W.F. Kibble in [2], obtain from it the joint pdf of two dependent central chi square rvs, forms an initial reference point for further analysis. This joint pdf turns out to be identical to the joint pdf reported in [6]. We then obtain the pdf for the difference of two dependent central chi square random variables and compare with the piecewise given expressions of the pdf reported in [6].

Given a vector $(X_1, \cdots, X_n)$ of Gaussian rv's with zero mean, then for $n > 1$, $Y = \sum \lim_{i=1}^{n} X_i^2$ is a central chi square rv with n degrees of freedom. For simplicity, we assume that the rvs X_i are standard normal. Take two such vectors Y_1 and $Y_{2.}$, their joint probability density function (pdf) is given by (e.g., see [6], p. 21)

$$p_{Y_1,Y_2}(y_1, y_2) = \frac{(y_1 y_2)^{\frac{1}{2}\left(\frac{n}{2}-1\right)}}{4\Gamma\left(\frac{n}{2}\right)(1-\rho^2)(2|\rho|)^{\frac{n}{2}-1}} \exp\left(-\frac{y_1+y_2}{2\sqrt{1-\rho^2}}\right) I_{\frac{n}{2}-1}\left(\frac{|\rho|\sqrt{y_1 y_2}}{1-\rho^2}\right), \tag{1}$$

where $y_1, y_2 \geq 0$, $-1 < \rho < 1, \rho \neq 0$ (if $n > 2$), and I_ν is the modified Bessel function of the first kind of order ν. As the

$$\lim_{\rho\to 0} \frac{I_{\frac{n}{2}-1}\left(\frac{|\rho|\sqrt{y_1 y_2}}{1-\rho^2}\right)}{|\rho|^{\frac{n}{2}-1}} = \frac{(y_1 y_2)^{\frac{n-2}{4}}}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)},$$

the joint pdf reduces to the product of two univariate chi square pdfs.

Remark 1. *Recall that the pdf of the unscaled univariate gamma distribution is given by* $f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}$, *where* $\alpha > 0, \beta > 0$, *and* $x \geq 0$. *Putting* $\alpha = \frac{n}{2}$, $\beta = 2$, *one obtains the univariate chi square distribution as a special case. It is well known that there are different variants of the bivariate gamma pdf's in the literature (e.g., see [3], Ch. 48). Among the earliest bivariate pdfs for the gamma distribution is that of Kibble [2] derived for the case* $\beta = 1$ *(scaled gamma's). Adjusting Kibble's bivariate gamma for the scale parameter* $\beta = 2$, *with the shape parameter* $\alpha = \frac{n}{2}$, *one obtains the pdf given in (1) - which is a Kibble type bivariate chi square pdf. Adjusting the scale parameter, one would obtain the pdf with* $\sigma_1^2 \neq 1 \neq \sigma_2^2$.

Consider now the difference W of two dependent central chi square rvs, X, Y, each having a $n = 2m$, $m > 1$, degrees of freedom (using the notation in [6]). Then it is reported in [6] that the rv $W = X - Y$

has the following piecewise defined pdf

$$p_W(w) = \frac{|w|^{m-1}}{(m-1)!2^{2m}(1-\rho^2)^{\frac{m}{2}}} \exp\left(\frac{w}{2\sqrt{1-\rho^2}}\right) \sum_{i=0}^{m-1} \frac{(m+i-1)!}{i!(m-i-1)!} \left(\frac{\sqrt{1-\rho^2}}{|w|}\right)^i, \ w < 0$$

$$p_W(w) = \frac{|w|^{m-1}}{(m-1)!2^{2m}(1-\rho^2)^{\frac{m}{2}}} \exp\left(-\frac{w}{2\sqrt{1-\rho^2}}\right) \sum_{i=0}^{m-1} \frac{(m+i-1)!}{i!(m-i-1)!} \left(\frac{\sqrt{1-\rho^2}}{|w|}\right)^i, \ w \geq 0$$

Using the following expansion of the Macdonald function [8]

$$K_{m-\frac{1}{2}}(w) = \left(\frac{\pi}{2w}\right)^{\frac{1}{2}} e^{-w} \sum_{i=0}^{m-1} \frac{(m+i-1)!}{i!(m-i-1)!(2w)^i},$$

and $K_{-\nu}(w) = K_\nu(w)$, the pdf given in [6] for W can be written as

$$p_W(w) = \frac{|w|^{m-\frac{1}{2}}}{2^{2m}\sqrt{\pi}(1-\rho^2)^{\frac{2m+1}{4}}(m-1)!} K_{m-\frac{1}{2}}\left(\frac{|w|}{2\sqrt{1-\rho^2}}\right), \ w \neq 0,$$

and, $p_W(0) := \lim_{w\to 0} p_W(w)$, which evaluates to $\frac{\Gamma(m-\frac{1}{2})}{4\sqrt{\pi}\sqrt{1-\rho^2}(m-1)!}$. The simplest definition of the Macdonald function is $K_{m-\frac{1}{2}}(w) := \frac{\pi(-1)^{k-1}}{2}\left(I_{-m+\frac{1}{2}}(w) - I_{m-\frac{1}{2}}(w)\right)$, m integer, (among other names of K_ν, it is often also called the modified Bessel function of the second kind (like in Mathematica)). Replacing the m used in ([6], p. 29) by $\frac{n}{2}$, where n now is the number of degrees of freedom of the rvs X and Y, then followed by replacing n by m, we obtain

$$f_W(w) = \frac{|w|^{\frac{m-1}{2}}}{2^m\sqrt{\pi}(1-\rho^2)^{\frac{m+1}{4}}\Gamma\left(\frac{m}{2}\right)} K_{\frac{m-1}{2}}\left(\frac{|w|}{2\sqrt{1-\rho^2}}\right), \ w \in \mathbf{R}\backslash\{0\}. \tag{2}$$

and $f_W(0) := \frac{\Gamma\left(\frac{m-1}{2}\right)}{4\sqrt{\pi}\sqrt{1-\rho^2}\Gamma\left(\frac{m}{2}\right)}$. Clearly $f_W(w) = f_W(-w)$. For $m = 1$, $f_W(w) = \frac{1}{2\pi\sqrt{1-\rho^2}} K_0\left(\frac{|w|}{2\sqrt{1-\rho^2}}\right)$. For $m = 2$, as $K_{\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}$, then $f_W(w) = \frac{1}{4\sqrt{1-\rho^2}} e^{-\frac{|w|}{2\sqrt{1-\rho^2}}}$. These cases match with equations (4.20) and (4.23) given in ([6], p. 29). For odd degrees of freedom, it is reported in ([6], p. 30), using his notation for $n = n_1 = n_2 = 2m+1$, and for $\sigma_1 = \sigma_2 = 1$, that the pdf is given by

$$p_W(w) = \frac{|w|^m}{\sqrt{\pi}\Gamma\left(m+\frac{1}{2}\right)(1-\rho^2)^{\frac{m+1}{2}}} K_m\left(\frac{|w|}{2\sqrt{1-\rho^2}}\right). \tag{3}$$

Writing this in terms of n and setting $n = m$ to adjust back to our notation, we see that (3) matches with (2). Figure 1 shows a plot of $f_W(w)$.

3. Cumulative Distribution Functions

In this section, we derive representations of the cdf of W for even and odd degrees of freedom. In the case of even degrees of freedom, we compare our result with the cdf expression reported in [6]. Let $c = 16\left(1-\rho^2\right)$.

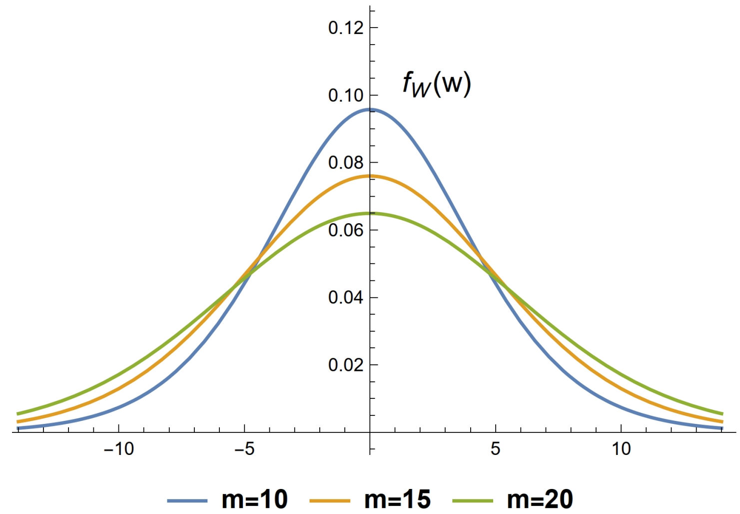

FIGURE 1. The pdf $f_W(w)$ for $m = 10, 15, 20$, and $\rho = 0.7$.

Theorem 1. *If the symmetric pdf of W is given by (2), then the cdf of W for $w > 0$ is given by*

$$F_W(w) = \int_{-\infty}^{0} f_W(w)\, dw + \int_0^w f_W(s)\, ds = \frac{1}{2} + J. \tag{4}$$

For m an even positive integer > 3, J is given by

$$\begin{aligned} J &= \frac{\Gamma\left(\frac{m-1}{2}\right) w}{4\Gamma\left(\frac{m}{2}\right)\sqrt{\pi}\sqrt{1-\rho^2}}\, {}_1F_2\left(\frac{1}{2}; \frac{3-m}{2}, \frac{3}{2}; \frac{w^2}{c}\right) \\ &\quad + \frac{\Gamma\left(\frac{1-m}{2}\right) w^m}{2^{2m} m \Gamma\left(\frac{m}{2}\right)\sqrt{\pi}\left(1-\rho^2\right)^{\frac{m}{2}}}\, {}_1F_2\left(\frac{m}{2}; \frac{m+1}{2}, \frac{m+2}{2}; \frac{w^2}{c}\right), \end{aligned}$$

where $-1 < \rho < 1$, ${}_1F_2(w) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b_1)_k (b_2)_k} \frac{w^k}{k!}$, and $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$, $b_1, b_2 \neq 0, -1, -2, \cdots$.

Proof. First, observe that by the symmetry of the pdf in (2), $I = \frac{1}{2}$. To evaluate J, we use formula 03.04.21.0014.01 in [9], namely,

$$\int z^{\nu} K_{\nu}(z)\, dz = \frac{\pi z \csc(\nu\pi)}{2^{\nu+2}} \begin{pmatrix} 4^{\nu}\sqrt{\pi}\, {}_1\widetilde{F}_2\left(\frac{1}{2}; 1-\nu, \frac{3}{2}; \frac{z^2}{4}\right) \\ -z^{2\nu}\Gamma\left(\nu+\frac{1}{2}\right)\, {}_1\widetilde{F}_2\left(\nu+\frac{1}{2}; \nu+1, \nu+\frac{3}{2}; \frac{z^2}{4}\right) \end{pmatrix},$$

where ${}_1\widetilde{F}_2(a; b_1, b_2; z) := {}_1F_2(a; b_1, b_2; z) / (\Gamma(b_1)\Gamma(b_2))$ is the regularized generalized hypergeometric function (rhgf). Then, it is straight forward to show that

$$\begin{aligned} J &= \frac{\pi \csc\left(\frac{m-1}{2}\pi\right) w}{2^3 \Gamma\left(\frac{m}{2}\right)\sqrt{1-\rho^2}}\, {}_1\widetilde{F}_2\left(\frac{1}{2}; \frac{3-m}{2}, \frac{3}{2}; \frac{w^2}{c}\right) \\ &\quad - \frac{\sqrt{\pi}\csc\left(\frac{m-1}{2}\pi\right) w^m}{2^{2m+1}\left(1-\rho^2\right)^{\frac{m}{2}}}\, {}_1\widetilde{F}_2\left(\frac{m}{2}; \frac{m+1}{2}, \frac{m+2}{2}; \frac{w^2}{c}\right), \end{aligned}$$

where,

$$
\begin{aligned}
{}_1\widetilde{F}_2\left(\frac{1}{2};\frac{3-m}{2},\frac{3}{2};z\right) &= \sum_{k=0}^{\infty}\frac{1}{\sqrt{\pi}\left(\frac{1}{2}+k\right)\Gamma\left(\frac{3-m}{2}+k\right)}\frac{z^k}{k!},\\
{}_1\widetilde{F}_2\left(\frac{m}{2};\frac{m+1}{2},\frac{m+2}{2};z\right) &= \sum_{k=0}^{\infty}\frac{1}{\left(\frac{m}{2}+k\right)\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m+1}{2}+k\right)}\frac{z^k}{k!};
\end{aligned}
$$

are the rhgfs defined for all $z\in\mathbb{C}$. Note that for m even, the $\lim_{z\to 0} J = 0 = J(0)$.
Using Euler's reflection formula ($\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin\pi z}$, $z = \frac{m-1}{2}$, which is not an integer when m is even), we obtain

$$
\begin{aligned}
J &= \frac{\Gamma\left(\frac{3-m}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}{2^3\Gamma\left(\frac{m}{2}\right)\sqrt{1-\rho^2}}w\,{}_1\widetilde{F}_2\left(\frac{1}{2};\frac{3-m}{2},\frac{3}{2};\frac{w^2}{c}\right)\\
&\quad-\frac{\Gamma\left(\frac{3-m}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}{2^{2m+1}\sqrt{\pi}\left(1-\rho^2\right)^{\frac{m}{2}}}w^m\,{}_1\widetilde{F}_2\left(\frac{m}{2};\frac{m+1}{2},\frac{m+2}{2};\frac{w^2}{c}\right).
\end{aligned}
$$

Then using $\Gamma(1+z) = z\Gamma(z)$, the J given in the theorem follows. □

Note that the integral $I = \int_{-\infty}^{w} f_W(w)\,dw$ for $Re(w) < 0$, and even $m > 0$, evaluates to

$$
\begin{aligned}
I = &\ \frac{1}{2}+\frac{\Gamma\left(\frac{m-1}{2}\right)w}{4\Gamma\left(\frac{m}{2}\right)\sqrt{\pi}\sqrt{1-\rho^2}}\,{}_1F_2\left(\frac{1}{2};\frac{3-m}{2},\frac{3}{2};\frac{w^2}{c}\right)\\
&-\frac{\Gamma\left(\frac{1-m}{2}\right)(-w)^m}{2^{2m}m\Gamma\left(\frac{m}{2}\right)\sqrt{\pi}\left(1-\rho^2\right)^{\frac{m}{2}}}\,{}_1F_2\left(\frac{m}{2};\frac{m+1}{2},\frac{m+2}{2};\frac{w^2}{c}\right).
\end{aligned} \tag{5}
$$

Figure 2 shows plots for the cdf $F_W(w)$ by Eq.(4).

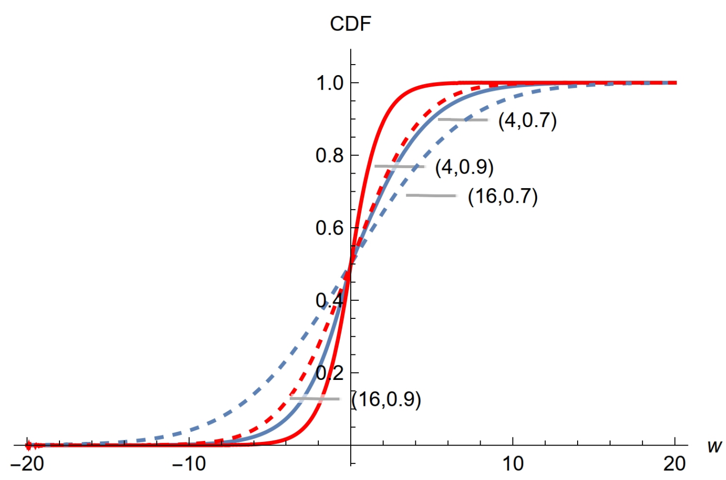

Figure 2. The cdf $F_W(w)$ for $m = 4, 16$, and $\rho = 0.7, 0.9$.

The cdf for the difference W of two dependent central chi square random variables with $2m$ degrees of freedom is reported in ([6], p. 30) (for $\sigma_1^2 = \sigma_2^2 = 1$) as

$$P_W(w) = \left\{ \begin{array}{ll} \frac{(1-\rho^2)^{\frac{m}{2}}}{2^{2m}(m-1)!} \exp\left(\frac{w}{2\sqrt{1-\rho^2}}\right) \sum_{i=0}^{m-1} \sum_{l=0}^{m-i-1} \frac{(m+i-1)!\left(1-\rho^2\right)^{i+l+1} 2^{l+1}}{i!(m-i-l-1)!l!} (-w)^{m-i-l-1}, & w < 0 \\ 1 - \frac{(1-\rho^2)^{\frac{m}{2}}}{2^{2m}(m-1)!} \exp\left(-\frac{w}{2\sqrt{1-\rho^2}}\right) \sum_{i=0}^{m-1} \sum_{l=0}^{m-i-1} \frac{(m+i-1)!\left(1-\rho^2\right)^{i+l+1} 2^{l+1}}{i!(m-i-l-1)!l!} (w)^{m-i-l-1}, & w \geq 0 \end{array} \right\}$$

Comparing this cdf with the cdf in (4), we have noticed that there were discrepancies between the two cdfs. For example, in Figure 3 the plots of the cdf $F_W(w)$ and the cdf $P_W(w)$ are displayed for $\rho = 0.7$, where $m = 2$ is used in $P_W(w)$, and $m = 4$ is used in $F_W(w)$ (that is, the degrees of freedom equal 4).

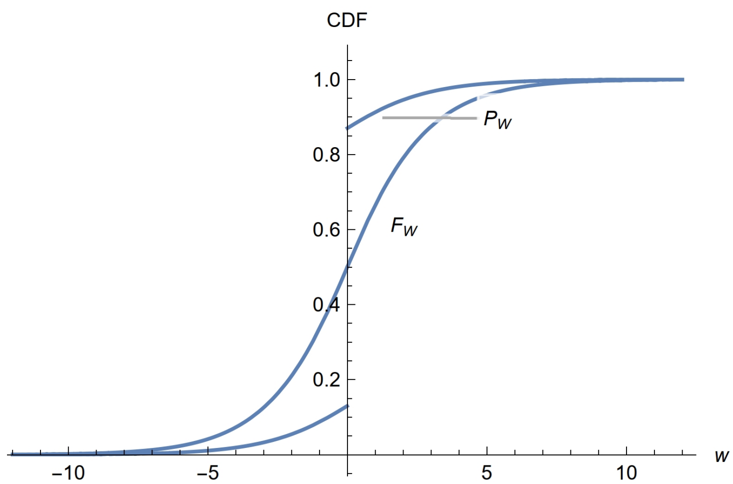

FIGURE 3. Comparison of the cdfs $P_W(w)$ and $F_W(w)$ for $\rho = 0.7$ and degrees of freedom 4.

In order to determine the source of this discrepancy, we examine the derivation of $P_W(w)$ for $w < 0$. This is enough, as for a symmetric density function about the y-axis one has $F(x) = 1 - F(-x)$. This formula was employed in [6] for his piecewise presentation of $P_W(w)$.

To start with, consider the following integral

$$\int_{-\infty}^{w} (-y)^{m-i-1} e^{\frac{y}{a}} dy = a^{m-i}\Gamma(m-i, -\frac{w}{a}), \ Re(w) < 0,$$

where $a = 2\sqrt{1-\rho^2}$, and $\Gamma[n, x]$ is the upper (or complementary) incomplete gamma function. For an integer n, the expansion $\Gamma(n, x) = (n-1)! e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}$ is given in [7]. Therefore, the correction of the CDF in the notation given in [6], for $w < 0$, can be written as

$$P_W(w) = \frac{1}{2^m (m-1)!} \exp\left(\frac{w}{2\sqrt{1-\rho^2}}\right) \sum_{i=0}^{m-1} \sum_{l=0}^{m-i-1} \frac{(m+i-1)!}{2^i i! l!} \left(\frac{-w}{2\sqrt{1-\rho^2}}\right)^l .$$

The results from this formula matches those obtained from our cdf (5). Furthermore, it is worth noting that the pdf and the cdf of a sum of dependent rvs X and Y, say $V = X + Y$, is sometimes derived indirectly from $W = X - Z$ by setting $Y = -Z$ (e.g., see [6], Ch. 5).

For m odd, we note that no expression for the cdf is given in [6], and to our knowelege it is unknown. In our representation of the cdf in (4), or in its regularized version, we encouter the evaluation of the gamma function at negative integers (poles) where it is not defined. It turns out that for odd m the integral J can be expressed in terms of the Meijer G-function. Recall, that the Meijer G-function is defined as a Mellin-Barnes integral (an inverse Mellin transform) (see [1])

$$G_{p,q}^{m,n}\left(\begin{array}{c}\mathbf{a}_p\\ \mathbf{b}_q\end{array}|z,r\right) := \frac{r}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma\left(b_j - rs\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-rs\right)}{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+rs\right)\prod_{j=1}^{p}\Gamma\left(a_j-rs\right)} z^s ds,$$

where $\mathbf{a}_p = (a_1, \cdots, a_p)$ and $\mathbf{b}_q = (b_1, \cdots, b_q)$, and L is a contour in the complex s-plane with certain properties (e.g., see [11] for further conditions for which the definition holds). For brevity, we describe this integral in the following remark in a bit more details in our specific case.

Remark 2. *Returning to the contour integral representation of Meijer G, but now as given in [10], we see in our specific case that* $G_{1,3}^{2,1}\left(\begin{array}{ccc} & 1 & \\ \frac{1}{2} & \frac{m}{2} & 0\end{array}|z\right) = \frac{1}{2\pi i}\int_L \frac{\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{m}{2}+s\right)\Gamma(-s)}{\Gamma(1+s)} z^{-s} ds$, *where L is a contour in the complex s-plane, which exists as $a_1 - b_i - 1 \notin \mathbb{N}$ (see [11]). In [10], s is replaced with $-s$, and hence the poles and L undergo a reflection. More specifically, a contour L is chosen to separate the poles of $\Gamma\left(\frac{1}{2}+s\right)$ and $\Gamma\left(\frac{m}{2}+s\right)$ from those of $\Gamma(-s)$. The contour in our case starts at $-\infty$ encircling the poles of $\Gamma\left(\frac{1}{2}+s\right)$ (at $s = -\frac{1}{2} - n, n = 0, 1, 2, \cdots$) and those of $\Gamma\left(\frac{m}{2}+s\right)$ (at $s = -\frac{m}{2} - n, n = 0, 1, 2, \cdots$) but not those of $\Gamma(-s)$ (at $n = 0, 1, 2, \cdots$) and returning to $-\infty$. Since in our case it holds that $0 \le m < q$ and $0 \le p < q$; and that $1 + \frac{1}{2}, 1 + \frac{m}{2}$ are not integers whenever m is odd, the integral converges for all $z \ne 0$; and the Meijer function is an analytic function except for $z = 0$ (e.g., see [11]).*

Theorem 2. *For odd $m > 0$, the J in (4) is given in trems of the Meijer G-function, and hence, the cdf can be expressed as*

$$F_W(w) = \frac{1}{2} + \frac{1}{2\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)} G_{1,3}^{2,1}\left(\begin{array}{ccc} & 1 & \\ \frac{1}{2} & \frac{m}{2} & 0\end{array}|\frac{w^2}{c}\right), \quad w > 0. \tag{6}$$

Proof. The entry 07.34.03.0605.01 in [10] gives a representation of the Bessel K function as

$$G_{0,2}^{2,0}\left(\begin{array}{c} - \\ b_1, b_2\end{array}|z\right) = 2z^{\frac{b_1+b_2}{2}} K_{b_1-b_2}\left(2\sqrt{z}\right).$$

However, the MeijerReduce command in [5] gave us

$$G_{0,2}^{2,0}\left(\begin{array}{c} - \\ b_1, b_2\end{array}|z, \frac{1}{2}\right) = 2z^{\frac{b_1+b_2}{2}} K_{b_1-b_2}\left(2\sqrt{z}\right),$$

where the $\frac{1}{2}$ equals the r in the contour integral in the definition of the Meijer G-function given above. Consequently, the integral in terms of the Meijer G-function becomes

$$\int_0^w x^{\frac{m-1}{2}} K_{\frac{m-1}{2}}\left(\frac{x}{2\sqrt{1-\rho^2}}\right) dx = \int_0^w \frac{1}{2} x^{\frac{m-1}{2}} G_{0,2}^{2,0}\left(\begin{matrix} - \\ \frac{m-1}{4}, \frac{1-m}{4} \end{matrix} \,\middle|\, \frac{x}{4\sqrt{1-\rho^2}}, \frac{1}{2}\right) dx.$$

This integral evaluates to a Meijer G-function with $r = 1$ when an odd positive numerical value of m is specified. Observing the pattern, we arrive at the expression in (6). By Remark 2, $z \neq 0$, and as the limit of $G_{1,3}^{2,1}(z|\cdot) \to 0$ as $z \to 0$, $F_W(0) := \frac{1}{2}$. Moreover, $\lim_{w\to\infty} F_W(w) = 1$. □

Let $b = 2\sqrt{1-\rho^2}$, and hence $4b^2 = c$.The integral $I = \int_{-\infty}^{w} f_W(w)\, dw$ for $Re(w) < 0$, and odd $m > 2$, evaluates to

$$I = \frac{b^{\frac{m+1}{2}}\left[2\pi(m-2)!! - 2^{\frac{m+1}{2}} G_{1,2}^{3,1}\left(\begin{matrix} 1 \\ \frac{1}{2}, \frac{m}{2}, 0 \end{matrix} \,\middle|\, \frac{w^2}{4b^2}\right)\right]}{2^{m+2}\sqrt{\pi}\,(1-\rho^2)^{\frac{m+1}{4}}\,\Gamma\left(\frac{m}{2}\right)}, \quad m = 4k-1;$$

$$I = \frac{b^{\frac{m+1}{2}}\pi\,(m-2)!! + w^{\frac{m+1}{2}} G_{1,2}^{3,1}\left(\begin{matrix} \frac{3-m}{4} \\ \frac{1-m}{4}, \frac{m-1}{4}, -\frac{m+1}{4} \end{matrix} \,\middle|\, \frac{w^2}{4b^2}\right)}{2^{m+2}\sqrt{\pi}\,(1-\rho^2)^{\frac{m+1}{4}}\,\Gamma\left(\frac{m}{2}\right)}, \quad m = 4k+1,$$

where $k = 1, 2, \cdots$; and $n!!$ is the double factorial, which is the product of all positive odd integers up to n.

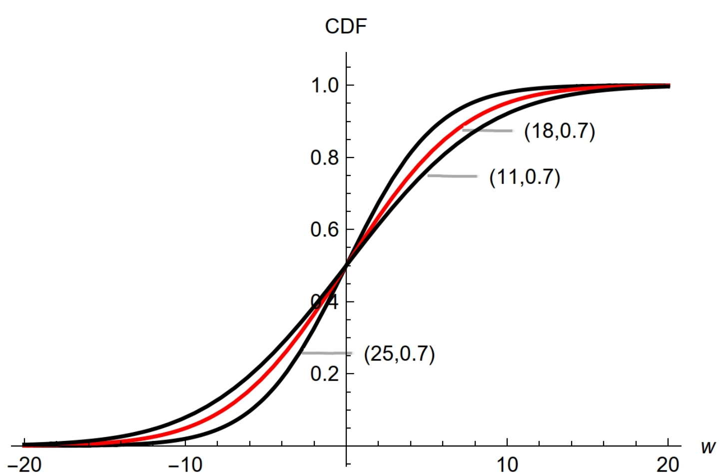

FIGURE 4. The cdf $F_W(w)$ for $m = 11, 18, 25$ and $\rho = 0.7$.

Figure 4 shows plots of the cdf generated using the Meijer function representation of $F_W(w)$.

4. PERCENTILES

In this section, to illustrate the use of the equation $F_W(w) = \frac{1}{2} + J$, we compute the 95^{th} percentile ($\alpha = 0.05$), for $\rho = 0.8, 0.9, 0.95$, and for degrees of freedom $m = 3, \cdots, 30$. For m even, using either (4), or the regularized generalized hypergeometric function representations of J, one obtains the percentiles reported in Table 1.

Remark 3. *Given that the integral representig $G^{2,1}_{1,3}$ converges, L. Slater's Theorem (e.g., see [11] for the general statement, [1, 4, 12]) which gives us an expression of the Meijer G-function in terms of two generalized hypergeometric functions for $b_i - b_j \notin \mathbb{Z}$, $i \neq j$. In our specific case, using this theorem gives us that*

$$
\begin{aligned}
& G^{2,1}_{1,3}\left(\begin{array}{ccc} & a_1 & \\ b_1 & b_2 & b_3 \end{array} \Big| z\right) \\
= & \frac{\prod_{j=1}^{2}\Gamma\left(b_j - b_{h=1}\right)^{*}\prod_{j=1}^{1}\Gamma\left(1+b_1-a_j\right)}{\prod \lim_{j=3}^{3}\Gamma\left(1+b_1-b_j\right)}\,{}_1F_2\left(1+b_1-a_1;1+b_1-b_2,1+b_1-b_3;z\right) \\
& + \frac{\prod_{j=1}^{2}\Gamma\left(b_j - b_{h=2}\right)^{*}\prod_{j=1}^{1}\Gamma\left(1+b_2-a_j\right)}{\prod_{j=3}^{3}\Gamma\left(1+b_2-b_j\right)}\,{}_1F_2\left(1+b_2-a_1;1+b_2-b_1,1+b_2-b_3;z\right),
\end{aligned}
$$

where the * indicates that the term corresponding to $j = h$ is omitted. This equation reduces to the expression $\frac{1}{2\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)}G^{2,1}_{1,3}\left(\begin{array}{ccc} & 1 & \\ \frac{1}{2} & \frac{m}{2} & 0 \end{array} \Big| \frac{w^2}{16(1-\rho^2)}\right) = J$, where J is as in Theorem 1.
A similar conclusion can be obtained from formula 07.34.03.0727.01 in [10]

$$
\begin{aligned}
& G^{2,1}_{1,3}\left(\begin{array}{ccc} & a_1 & \\ b_1 & b_2 & b_3 \end{array} \Big| z\right) \\
= & \pi\csc\left(\left(b_2-b_1\right)\pi\right)\left[\begin{array}{c} \Gamma\left(1-a_1+b_1\right)z^{b_1}\,{}_1\widetilde{F}_2\left(1-a_1+b_1;b_1-b_2+1,b_1-b_3+1;z\right) \\ -\Gamma\left(1-a_1+b_2\right)z^{b_2}\,{}_1\widetilde{F}_2\left(1-a_1+b_2;1-b_1+b_2,b_2-b_3+1;z\right) \end{array}\right],
\end{aligned}
$$

where $b_2 - b_1 \notin \mathbb{Z}$. Solving the linear system

$$
\begin{aligned}
1-a_1+b_1 &= \frac{1}{2},\; b_1-b_2+1=\frac{3-m}{2},\; b_1-b_3+1=\frac{3}{2},\; 1-a_1+b_2=\frac{m}{2}, \\
1-b_1+b_2 &= \frac{m+1}{2},\; b_2-b_3+1=\frac{m+2}{2};
\end{aligned}
$$

we obtain that $a_1 = 1 + b_3$, $b_1 = \frac{1}{2} + b_3$, $b_2 = \frac{m}{2} + b_3$. For simplicity, take $b_3 = 0$, and then we have $G^{2,1}_{1,3}\left(\begin{array}{ccc} & 1 & \\ \frac{1}{2} & \frac{m}{2} & 0 \end{array} \Big| \frac{w^2}{c}\right)$, with $\frac{m-1}{2} \notin \mathbb{Z}$, $w > 0$. Furthermore, evaluating the contour integral described in Remark 2, using formula 07.34.06.0045.01 in [10] for the residues series, resulted in combination of four ${}_1F_2$ generalized hypergeometric functions that also blow up for odd m. Therefore, it appears that in the hypergeometric representation of this Meijer G-function, the restriction $b_2 - b_1 \notin \mathbb{Z}$ for odd m cannot be removed.

Clearly, the representation in (6) holds for even m as the condition $\frac{m-1}{2} \notin \mathbb{Z}$ holds; and the computed percentiles from this expression were identical to those reported in Table 1. It turns out that in Mathematica [5] for the case where $m \geq 1$ is odd, it is possible to solve for $w > 0$ using the FindRoot command, which solves numerically for an initial guess at the root. As it is not clear enough how the regularization has occured for odd m, we implemented Newton's method to find the root using formula 07.34.20.0001.01 in [10] for the derivative of the Meijer G-function

$$F'_W(w) = \frac{2w}{2c\sqrt{w}\Gamma\left(\frac{m}{2}\right)} G^{2,2}_{2,4}\left(\begin{array}{c} -1,0 \\ -\frac{1}{2}, \frac{m}{2}-1, 0, -1 \end{array} \Big| \frac{w^2}{c}\right).$$

Solving (6) by Newton's method, using positive initial guesses, gave the same values as those reported in Table 1. Newton's method worked as an expansion of $G^{2,1}_{1,3}\left(\begin{array}{ccc} & 1 & \\ \frac{1}{2} & \frac{m}{2} & 0 \end{array} \Big| z\right)$ can be written as

$$\pi \sec\left(\frac{m\pi}{2}\right)\left(-\sqrt{\pi z}\ {}_1\widetilde{F}_2\left(\frac{1}{2}; \frac{3-m}{2}, \frac{3}{2}; z\right) + z^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)\ {}_1\widetilde{F}_2\left(\frac{m}{2}; \frac{m+1}{2}, \frac{m+2}{2}; z\right)\right).$$

Then the ratio of this G-function to its derivative is clearly regular for odd m and $w > 0$, as the secant function cancels out.

$m\backslash\rho$	0.8	0.9	0.95	m$\backslash\boldsymbol{\rho}$	0.8	0.9	0.95
4	3.92617	2.85230	2.04325	**3**	3.39639	2.46742	1.76754
6	4.81249	3.49619	2.50450	**5**	4.39180	3.19057	2.28557
8	5.55949	4.03888	2.89325	**7**	5.19934	3.77723	2.70582
10	6.21796	4.51724	3.23593	**9**	5.89785	4.28469	3.06934
12	6.81358	4.94995	3.54590	**11**	6.52251	4.73850	3.39442
14	7.36152	5.34802	3.83106	**13**	7.09280	5.15280	3.69121
16	7.87168	5.71864	4.09655	**15**	7.62084	5.53641	3.96601
18	8.35093	6.06681	4.34596	**17**	8.11482	5.89528	4.22309
20	8.80428	6.39616	4.58189	**19**	8.58058	6.23365	4.46548
22	9.23552	6.70945	4.80631	**21**	9.02246	6.55467	4.69544
24	9.64758	7.00881	5.02076	**23**	9.44379	6.68607	4.91470
26	10.0428	7.29594	5.22645	**25**	9.84718	7.15381	5.12463
28	10.4231	7.57224	5.42437	**27**	10.2347	7.43537	5.32633
30	10.7901	7.83883	5.61535	**29**	10.6082	7.70668	5.52069

Table 1. The 95^{th} Percentile ($\alpha = 0.05$) for $3 \leq m \leq 30$, and $\rho = 0.8, 0.9, 0.95$.

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